M^2-factor of coherent and partially coherent dark hollow beams propagating in turbulent atmosphere

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Abstract: Analytical formula is derived for the M^2-factor of coherent and partially coherent dark hollow beams (DHB) in turbulent atmosphere based on the extended Huygens-Fresnel integral and the second-order moments of the Wigner distribution function. Our numerical results show that the M^2-factor of a DHB in turbulent atmosphere increases on propagation, which is much different from its invariant properties in free-space, and is mainly determined by the parameters of the beam and the atmosphere. The relative M^2-factor of a DHB increases slower than that of Gaussian and flat-topped beams on propagation, which means a DHB is less affected by the atmospheric turbulence than Gaussian and flat-topped beams. Furthermore, the relative M^2-factor of a DHB with lower coherence, longer wavelength and larger dark size is less affected by the atmospheric turbulence. Our results will be useful in long-distance free-space optical communications.

OCIS codes: (010.1300) Atmospheric propagation; (030.0030) Coherence and statistical optics.

References and links

1. A. E. Siegman, "New developments in laser resonators," in Optical Resonators, D. A. Holmes, ed., Proc. SPIE **1224**, 2C14 (1990).
2. R. Martinez-Herrero, and P. M. Mejias, "Second-order spatial characterization of hard-edge diffracted beams," Opt. Lett. **18**(19), 1669–1671 (1993).
3. R. Martinez-Herrero, P. M. Mejias, and M. Arias, "Parametric characterization of coherent, lowest-order Gaussian beams propagation through hard-edged apertures," Opt. Lett. **20**(2), 124–126 (1995).
4. F. Gori, M. Santarsiero, and A. Sona, "The change of width for a partially coherent beam on paraxial propagation," Opt. Commun. **82**(3-4), 197–203 (1991).
5. M. Santarsiero, F. Gori, R. Borghi, G. Cincotti, and P. Vahimaa, "Spreading properties of beams radiated by partially coherent Schell-model sources," J. Opt. Soc. Am. A **16**(1), 106–112 (1999).
6. B. Zhang, X. Chu, and Q. Li, "Generalized beam-propagation factor of partially coherent beams propagating through hard-edged apertures," J. Opt. Soc. Am. A **19**(7), 1370–1375 (2002).
7. X. Chu, B. Zhang, and Q. Wen, "Generalized M2 factor of a partially coherent beam propagating through a circular hard-edged aperture," Appl. Opt. **42**(21), 4280–4284 (2003).
8. P. A. Belanger, "Beam quality factor of the LP_{01} mode of the step-index fiber," Opt. Eng. **32**(9), 2107–2109 (1993).
9. P. A. Belanger, Y. Champagne, and C. Pare, "Beam-propagation factor of diffracted laser-beams," Opt. Commun. **105**(3-4), 233–242 (1994).
10. S. A. Amarande, "Beam propagation factor and the kurtosis parameter of flattened Gaussian beams," Opt. Commun. **129**, 311–317 (1996).
11. S. Saghafi, and C. J. R. Sheppard, "The beam propagation factor for higher order Gaussian beams," Opt. Commun. **153**(4-6), 207–210 (1998).
12. B. Lü, B. Zhang, and H. Ma, "Beam-propagation factor and mode-coherence coefficients of hyperbolic-cosine Gaussian beams," Opt. Lett. **24**(10), 640–642 (1999).
13. Z. Mei, and D. Zhao, "The generalized beam propagation factor of truncated standard and elegant Laguerre-Gaussian beams," J. Opt. A, Pure Appl. Opt. **6**(11), 1005–1011 (2004).

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14. D. Deng, Z. Xia, C. Wei, J. Shao, and Z. Fan, "Far-field intensity distribution, M^2 factor of beams generated by Gaussian mirror resonator," Optik (Stuttg.) **118**(11), 533–536 (2007).
15. G. Zhou, "Beam propagation factors of a Lorentz-Gauss beam," Appl. Phys. B **96**(1), 149–153 (2009).
16. J. Yin, W. Gao, and Y. Zhu, "Generation of dark hollow beams and their applications," in *Progress in Optics,* Vol. **44**, E. Wolf, ed., North-Holland, Amsterdam, 2003, pp.119–204.
17. X. Wang, and M. G. Littman, "Laser cavity for generation of variable-radius rings of light," Opt. Lett. **18**(10), 767–768 (1993).
18. H. S. Lee, B. W. Stewart, K. Choi, and H. Fenichel, "Holographic nondiverging hollow beam," Phys. Rev. A **49**(6), 4922–4927 (1994).
19. J. Yin, H. Noh, K. Lee, K. Kim, Y. Wang, and W. Jhe, "Generation of a dark hollow beam by a small hollow fiber," Opt. Commun. **138**(4-6), 287–292 (1997).
20. N. R. Heckenberg, R. McDuff, C. P. Smith, and A. G. White, "Generation of optical phase singularities by computer generated holograms," Opt. Lett. **17**(3), 221–223 (1992).
21. Z. Liu, H. Zhao, J. Liu, J. Lin, M. A. Ahmad, and S. Liu, "Generation of hollow Gaussian beams by spatial filtering," Opt. Lett. **32**(15), 2076–2078 (2007).
22. Z. Liu, J. Dai, X. Sun, and S. Liu, "Generation of hollow Gaussian beam by phase-only filtering," Opt. Express **16**(24), 19926–19933 (2008).
23. F. Gori, G. Guattari, and C. Padovani, "Bessel-Gauss beams," Opt. Commun. **64**(6), 491–495 (1987).
24. T. Kuga, Y. Torii, N. Shiokawa, T. Hirano, Y. Shimizu, and H. Sasada, "Novel optical trap of atoms with a doughnut beam," Phys. Rev. Lett. **78**(25), 4713–4716 (1997).
25. J. C. Gutiérrez-Vega, M. D. Iturbe-Castillo, and S. Chávez-Cerda, "Alternative formulation for invariant optical fields: Mathieu beams," Opt. Lett. **25**(20), 1493–1495 (2000).
26. Y. Cai, X. Lu, and Q. Lin, "Hollow Gaussian beams and their propagation properties," Opt. Lett. **28**(13), 1084–1086 (2003).
27. Z. Mei, and D. Zhao, "Controllable dark-hollow beams and their propagation characteristics," J. Opt. Soc. Am. A **22**(9), 1898–1902 (2005).
28. Y. Cai, "Model for an anomalous hollow beam and its paraxial propagation," Opt. Lett. **32**(21), 3179–3181 (2007).
29. D. Deng, H. Yu, S. Xu, G. Tian, and Z. Fan, "Nonparaxial propagation of vectorial hollow Gaussian beams," J. Opt. Soc. Am. B **25**(1), 83–87 (2008).
30. Y. Cai, and S. He, "Propagation of hollow Gaussian beams through apertured paraxial optical systems," J. Opt. Soc. Am. A **23**(6), 1410–1418 (2006).
31. G. Wu, Q. Lou, and J. Zhou, "Analytical vectorial structure of hollow Gaussian beams in the far field," Opt. Express **16**(9), 6417–6424 (2008).
32. Y. Cai, and S. He, "Propagation of various dark hollow beams in a turbulent atmosphere," Opt. Express **14**(4), 1353–1367 (2006).
33. Y. Chen, Y. Cai, H. T. Eyyuboğlu, and Y. Baykal, "Scintillation properties of dark hollow beams in a weak turbulent atmosphere," Appl. Phys. B **90**(1), 87–92 (2008).
34. D. Deng, "Generalized M^2-factor of hollow Gaussian beams through a hard-edge circular aperture," Phys. Lett. A **341**(1-4), 352–356 (2005).
35. D. Deng, X. Fu, C. Wei, J. Shao, and Z. Fan, "Far-field intensity distribution and M^2 factor of hollow Gaussian beams," Appl. Opt. **44**(33), 7187–7190 (2005).
36. Z. Mei, and D. Zhao, "Generalized M^2 factor of hard-edged diffracted controllable dark-hollow beams," Opt. Commun. **263**(2), 261–266 (2006).
37. Y. Cai, and L. Zhang, "Coherent and partially coherent dark hollow beams with rectangular symmetry and paraxial propagation," J. Opt. Soc. Am. B **23**(7), 1398–1407 (2006).
38. X. Lu, and Y. Cai, "Partially coherent circular and elliptical dark hollow beams and their paraxial propagations," Phys. Lett. A **369**(1-2), 157–166 (2007).
39. C. Zhao, Y. Cai, F. Wang, X. Lu, and Y. Wang, "Generation of a high-quality partially coherent dark hollow beam with a multimode fiber," Opt. Lett. **33**(12), 1389–1391 (2008).
40. L. C. Andrews, R. L. Phillips, and C. Y. Hopen, *Laser Beam Scintillation with Applications* (SPIE Press, Washington, 2001).
41. L. C. Andrews, and R. L. Phillips, Laser beam propagation in the turbulent atmosphere, 2nd edition, SPIE press, Bellington, 2005.
42. J. C. Ricklin, and F. M. Davidson, "Atmospheric turbulence effects on a partially coherent Gaussian beam: implications for free-space laser communication," J. Opt. Soc. Am. A **19**(9), 1794–1802 (2002).
43. O. Korotkova, "Scintillation index of a stochastic electromagnetic beam propagating in random media," Opt. Commun. **281**, 2342–2348 (2008).
44. Y. Cai, and S. He, "Propagation of a partially coherent twisted anisotropic Gaussian Schell-model beam in a turbulent atmosphere," Appl. Phys. Lett. **89**(4), 041117 (2006).
45. R. J. Noriega-Manez, and J. C. Gutiérrez-Vega, "Rytov theory for Helmholtz-Gauss beams in turbulent atmosphere," Opt. Express **15**(25), 16328–16341 (2007).
46. Y. Cai, "Propagation of various flat-topped beams in a turbulent atmosphere," J. Opt. A, Pure Appl. Opt. **8**(6), 537–545 (2006).
47. C. Y. Young, Y. V. Gilchrest, and B. R. Macon, "Turbulence-induced beam spreading of higher-order mode optical waves," Opt. Eng. **41**, 1097–1103 (2002).
48. M. Alavinejad, B. Ghafary, and F. D. Kashani, "Analysis of the propagation of flat-topped beam with various beam orders through turbulent atmosphere," Opt. Lasers Eng. **46**(1), 1–5 (2008).

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49. H. T. Eyyuboğlu, "Propagation and coherence properties of higher order partially coherent dark hollow beams in turbulence," Opt. Laser Technol. **40**(1), 156–166 (2008).
50. Y. Dan, and B. Zhang, "Beam propagation factor of partially coherent flat-topped beams in a turbulent atmosphere," Opt. Express **16**(20), 15563–15575 (2008).
51. Y. Dan, and B. Zhang, "Second moments of partially coherent beams in atmospheric turbulence," Opt. Lett. **34**(5), 563–565 (2009).
52. M. H. Mahdieh, "Numerical approach to laser beam propagation through turbulent atmosphere and evaluation of beam quality factor," Opt. Commun. **281**(13), 3395–3402 (2008).
53. L. Mandel, and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge U. Press, 1995).
54. A. Erdelyi, W. Magnus, and F. Oberhettinger, Tables of Integral Transforms (McGraw-Hill, 1954).

1. Introduction

The propagation factor (best known as M^2-factor) proposed by Siegman is a particularly important property of an optical laser beam [1] being regarded as a beam quality factor in many practical applications. Martinez-Herrero et al. developed the generalized second moments of hard-edge diffracted coherent laser beam to calculate its M^2-factor [2,3]. Gori et al. extended the definition of M^2-factor from to the partially coherent beam-like fields, and studied the M^2-factor of partially coherent beam in the absence of the aperture [4,5]. Zhang et al. studied the M^2-factor of hard-edge diffracted partially coherent beams [6,7]. In other papers [8–15] various aspects have been related to the M^2-factor of main classes of coherent and partially coherent beams.

Dark hollow beams (DHBs), i.e. the beams with zero on-axis intensity have important applications in laser optics, atomic optics, binary optics, optical trapping of particles and medical sciences [16]. Up to now, various methods such as geometrical optical method, mode conversion, optical holography, transverse-mode selection, hollow-fiber method, computer-generated holography, nonlinear optical method and spatial filtering have been proposed to generate DHBs experimentally [17–22]. Several theoretical models have been used to describe DHBs and their propagation properties in free space or through paraxial optical system [16,23–35]. The M^2**-**factor of various DHBs in free space was studied in [27,34–36]. Recently, Cai and associates extended DHB to the partially coherent case, and studied the M^2-factor and propagation properties of a partially coherent DHB of circular or non-circular symmetry in free space [37,38]. Zhao et al. generated a partially coherent circular **D**HB experimentally with a multimode fiber [39].

Propagation characteristics of different types of laser beams in a turbulent atmosphere are being studied extensively due to their important applications in free-space optical communications, remote sensing of atmosphere and target tracking [40–50]. It is necessary and important to find suitable ways to overcome or reduce the destructive effect of atmospheric turbulence in these applications. One possible way for reducing the effect of atmospheric turbulence is using partially coherent beam or electromagnetic partially coherent beam instead of coherent beam [40–44]. Another possible way is using laser beam with special beam profile, such as Helmholtz-Gauss beams [45], cosh-Gaussian beams [46], higher-order laser beams [47], flat-topped beams [48,49], DHBs [32,33,49] and so on. The average intensity and scintillation index of coherent DHBs in turbulent atmosphere have been studied in [32,33] (see also [41] where intensity, coherence and scintillation of an annular beam are discussed). It was shown in [32,33] that DHBs have advantage over a Gaussian beam and flat-topped beam for overcoming the destructive effect of atmospheric turbulence from the aspect of scintillation, and, hence, have important potential application in free-space optical communications. It was also shown in [49] that partially coherent DHB have advantage over coherent DHB. Up to now, to our knowledge, the M^2**-**factor of coherent and partially coherent DHBs in turbulent atmosphere hasn't been reported. In fact, only few papers have been published on the M^2-factor of laser beams in turbulent atmosphere [50–52].

In this paper, our aim is to investigate the M^2-factor of coherent and partially coherent DHBs in turbulent atmosphere. Analytical formula for the M^2-factor of DHBs on propagation is derived, and some numerical examples are given. Our results clearly show that

DHBs have advantage over a Gaussian beam and flat-topped beam for overcoming the destructive effect of atmospheric turbulence from the aspect of M^2-factor

2. Formulation

The electric field of a DHB with circular symmetry at z = 0 can be expressed as the following finite sum of Gaussian modes [27]

$$E_N(\boldsymbol{\rho};0)=\sum_{n=1}^{N}\frac{(-1)^{n-1}}{N}\binom{N}{n}\left[\exp\left(-\frac{n\boldsymbol{\rho}^2}{w_0^2}\right)-\exp\left(-\frac{n\boldsymbol{\rho}^2}{w_p^2}\right)\right], \tag{1}$$

where $\binom{N}{n}$ denotes a binomial coefficient, N is the beam order of a circular DHB, $\boldsymbol{\rho}\equiv(\rho_x \quad \rho_y)$ is the position vector in the source plane, $w_p = pw_0$ with w_0 being the beam waist size of the fundamental Gaussian mode, p ($0<p<1$) is a scaling factor for controlling the dark size of the DHB. When N = 1 and p = 0, Eq. (1) reduces to the expression for the electric field of a fundamental Gaussian beam. When N>1 and p = 0, Eq. (1) reduces to the expression for the electric field of a flat-topped beam. Figure 1 shows the cross line (y = 0) of the normalized intensity distribution of a circular DHB for several different values of N and p with $w_0 = 1mm$. One sees from Fig. 1 that the central dark size across a DHB increases as N or p increase.

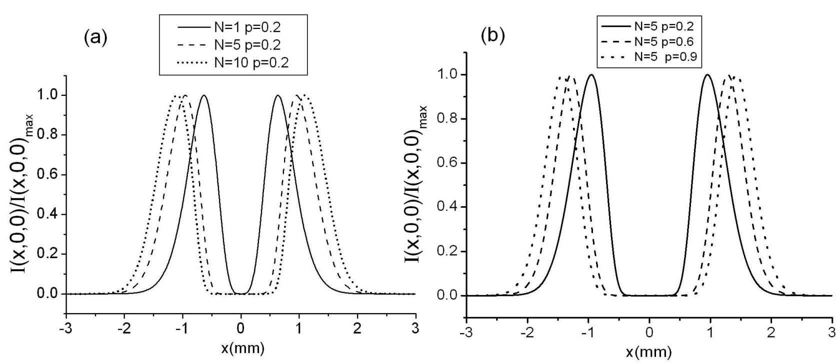

Fig. 1. Cross line (y = 0) of the normalized intensity distribution of a circular DHB for different values of N and p with $w_0 = 1mm$

A partially coherent beam which has a DHB intensity distribution and a Gaussian spatial correlation can be characterized by the cross-spectral density (CSD) [53] of the form [37,38]

$$\begin{aligned}W_N(\boldsymbol{\rho}_1',\boldsymbol{\rho}_2';0)&=\left\langle E_N(\boldsymbol{\rho}_1';0)E_N^*(\boldsymbol{\rho}_2';0)\right\rangle\\&=\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{(-1)^{n+m}}{N^2}\binom{N}{n}\binom{N}{m}\left[\exp\left(-\frac{n\boldsymbol{\rho}_1'^2}{w_0^2}\right)-\exp\left(-\frac{n\boldsymbol{\rho}_1'^2}{w_p^2}\right)\right]\\&\times\left[\exp\left(-\frac{m\boldsymbol{\rho}_2'^2}{w_0^2}\right)-\exp\left(-\frac{m\boldsymbol{\rho}_2'^2}{w_p^2}\right)\right]\exp\left[-\frac{(\boldsymbol{\rho}_1'-\boldsymbol{\rho}_2')^2}{2\sigma_g^2}\right],\end{aligned} \tag{2}$$

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where $\boldsymbol{\rho}_1^{'}$, $\boldsymbol{\rho}_2^{'}$ are two arbitrary points in the source plane, σ_g is the transverse coherence width. Under the condition of $\sigma_g -> \infty$, a partially coherent DHB reduces to a coherent DHB.

Within the validity of the paraxial approximation, the propagation of the cross-spectral density of a partially coherent beam in the turbulent atmosphere can be studied with the help of the following generalized Huygens-Fresnel integral [40,41]

$$W(\boldsymbol{\rho},\boldsymbol{\rho}_d;z)=(\frac{k}{2\pi z})^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(\boldsymbol{\rho}^{'},\boldsymbol{\rho}_d^{'};0)\times\exp\left[\frac{ik}{z}(\boldsymbol{\rho}-\boldsymbol{\rho}^{'})\cdot(\boldsymbol{\rho}_d-\boldsymbol{\rho}_d^{'})-H(\boldsymbol{\rho}_d,\boldsymbol{\rho}_d^{'};z)\right]d^2\boldsymbol{\rho}^{'}d^2\boldsymbol{\rho}_d^{'}, \tag{3}$$

where $k=2\pi/\lambda$ is the wave number with λ being the wavelength. In Eq. (3) we have used the following sum and difference vector notation

$$\boldsymbol{\rho}^{'}=\frac{(\boldsymbol{\rho}_1^{'}+\boldsymbol{\rho}_2^{'})}{2},\ \boldsymbol{\rho}_d^{'}=\boldsymbol{\rho}_1^{'}-\boldsymbol{\rho}_2^{'},\ \boldsymbol{\rho}=\frac{(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2)}{2},\ \boldsymbol{\rho}_d=\boldsymbol{\rho}_1-\boldsymbol{\rho}_2, \tag{4}$$

where $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ are two arbitrary points in the receiver plane, perpendicular to the direction of propagation of the beam. We can express the cross-spectral density in the source plane as follows

$$W(\boldsymbol{\rho}^{'},\boldsymbol{\rho}_d^{'};0)=W(\boldsymbol{\rho}_1^{'},\boldsymbol{\rho}_2^{'};0)=W(\boldsymbol{\rho}^{'}+\frac{\boldsymbol{\rho}_d^{'}}{2},\boldsymbol{\rho}^{'}-\frac{\boldsymbol{\rho}_d^{'}}{2};0). \tag{5}$$

In Eq. (3), the term $H(\boldsymbol{\rho}_d,\boldsymbol{\rho}_d^{'},z)$ is the contribution from the atmospheric turbulence expressed as

$$H(\rho_d,\rho_d^{'};z)=4\pi^2k^2z\int_0^1 d\xi\int_0^{\infty}[1-J_0(\kappa\left|\rho_d^{'}\xi+(1-\xi)\rho_d\right|)]\Phi_n(\kappa)\kappa d\kappa, \tag{6}$$

where J_0 is the Bessel function of zero order, Φ_n represents the one-dimensional power spectrum of the index-of-refraction fluctuations [41].

After some operations as shown in [50], Eq. (3) can be expressed in the following alternative form

$$W(\boldsymbol{\rho},\boldsymbol{\rho}_d;z)=(\frac{1}{2\pi})^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(\boldsymbol{\rho}",\boldsymbol{\rho}_d+\frac{z}{k}\boldsymbol{\kappa}_d;0)\times\exp\{-i\boldsymbol{\rho}\cdot\boldsymbol{\kappa}_d+i\boldsymbol{\rho}"\cdot\boldsymbol{\kappa}_d-H(\boldsymbol{\rho}_d,\boldsymbol{\rho}_d+\frac{z}{k}\boldsymbol{\kappa}_d;z)\}d^2\boldsymbol{\rho}"d^2\boldsymbol{\kappa}_d, \tag{7}$$

where $\boldsymbol{\kappa}_d\equiv\begin{pmatrix}\kappa_{dx} & \kappa_{dy}\end{pmatrix}$ is the position vector in spatial-frequency domain. For a partially coherent circular DHB, using Eq. (2), we can express the CSD $W(\boldsymbol{\rho}",\boldsymbol{\rho}_d+\frac{z}{k}\boldsymbol{\kappa}_d;0)$ as

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$$W(\boldsymbol{\rho}'', \boldsymbol{\rho}_d + \frac{z}{k}\kappa_d ; 0) = \sum_{n=1}^{N}\sum_{m=1}^{N}\frac{(-1)^{n+m}}{N^2}\binom{N}{n}\binom{N}{m}$$
$$\{\exp[-A_1\boldsymbol{\rho}''^2 - A_2\boldsymbol{\rho}''(\boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d) - A_3(\boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d)^2]$$
$$-\exp[-B_1\boldsymbol{\rho}''^2 - B_2\boldsymbol{\rho}''(\boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d) - B_3(\boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d)^2] \qquad (8)$$
$$-\exp[-C_1\boldsymbol{\rho}''^2 - C_2\boldsymbol{\rho}''(\boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d) - C_3(\boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d)^2]$$
$$+\exp[-D_1\boldsymbol{\rho}''^2 - \mathbf{D}_2\boldsymbol{\rho}''(\boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d) - D_3(\boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d)^2]\},$$

where

$$A_1 = \frac{n}{w_0^2} + \frac{m}{w_0^2},\ A_2 = \frac{n}{w_0^2} - \frac{m}{w_0^2},\ A_3 = \frac{A_1}{4} + \frac{1}{2\sigma_g^2},\ B_1 = \frac{n}{w_0^2} + \frac{m}{w_p^2},$$
$$B_2 = \frac{n}{w_0^2} - \frac{m}{w_p^2},\ B_3 = \frac{B_1}{4} + \frac{1}{2\sigma_g^2},\ C_1 = \frac{n}{w_p^2} + \frac{m}{w_0^2},\ C_2 = \frac{n}{w_p^2} - \frac{m}{w_0^2}, \qquad (9)$$
$$C_3 = \frac{C_1}{4} + \frac{1}{2\sigma_g^2},\ D_1 = \frac{n}{w_p^2} + \frac{m}{w_p^2},\ D_2 = \frac{n}{w_p^2} - \frac{m}{w_p^2},\ D_3 = \frac{D_1}{4} + \frac{1}{2\sigma_g^2}.$$

The Wigner distribution of a partially coherent beam on propagation in turbulent atmosphere can be expressed in terms of the cross-spectral density function by the formula [50]

$$h(\boldsymbol{\rho}, \boldsymbol{\theta}; z) = (\frac{k}{2\pi})^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(\boldsymbol{\rho}, \boldsymbol{\rho}_d; z)\exp(-ik\boldsymbol{\theta}\cdot\boldsymbol{\rho}_d)d^2\boldsymbol{\rho}_d, \qquad (10)$$

where $\boldsymbol{\theta} \equiv (\theta_x, \theta_y)$ denotes an angle which the vector of interest makes with the z-direction, $k\theta_x$ and $k\theta_y$ are the wave vector components along the *x*-axis and *y*-axis, respectively. Substituting from Eqs. (7), (8) and (9) into Eq. (10), we obtain (after tedious integration)

$$h(\boldsymbol{\rho}, \boldsymbol{\theta}; z)$$
$$= \frac{k^2}{16\pi^3}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{(-1)^{n+m}}{N^2}\binom{N}{n}\binom{N}{m}\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{A_1}\exp[-a_1\boldsymbol{\rho}_d^2 - b_1\boldsymbol{\kappa}_d^2 + c_1\boldsymbol{\rho}_d\cdot\boldsymbol{\kappa}_d - ik\boldsymbol{\theta}\cdot\boldsymbol{\rho}_d$$
$$-i\boldsymbol{\rho}\cdot\boldsymbol{\kappa}_d - H(\boldsymbol{\rho}_d, \boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d, z)]d^2\boldsymbol{\kappa}_d d^2\boldsymbol{\rho}_d$$
$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{B_1}\exp[-a_2\boldsymbol{\rho}_d^2 - b_2\boldsymbol{\kappa}_d^2 + c_2\boldsymbol{\rho}_d\cdot\boldsymbol{\kappa}_d - ik\boldsymbol{\theta}\cdot\boldsymbol{\rho}_d - i\boldsymbol{\rho}\cdot\boldsymbol{\kappa}_d - H(\boldsymbol{\rho}_d, \boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d, z)]d^2\boldsymbol{\kappa}_d d^2\boldsymbol{\rho}_d$$
$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{C_1}\exp[-a_3\boldsymbol{\rho}_d^2 - b_3\boldsymbol{\kappa}_d^2 + c_3\boldsymbol{\rho}_d\cdot\boldsymbol{\kappa}_d - ik\boldsymbol{\theta}\cdot\boldsymbol{\rho}_d - i\boldsymbol{\rho}\cdot\boldsymbol{\kappa}_d - H(\boldsymbol{\rho}_d, \boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d, z)]d^2\boldsymbol{\kappa}_d d^2\boldsymbol{\rho}_d$$
$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{D_1}\exp[-a_4\boldsymbol{\rho}_d^2 - b_4\boldsymbol{\kappa}_d^2 + c_4\boldsymbol{\rho}_d\cdot\boldsymbol{\kappa}_d - ik\boldsymbol{\theta}\cdot\boldsymbol{\rho}_d - i\boldsymbol{\rho}\cdot\boldsymbol{\kappa}_d - H(\boldsymbol{\rho}_d, \boldsymbol{\rho}_d + \frac{z}{k}\boldsymbol{\kappa}_d, z)]d^2\boldsymbol{\kappa}_d d^2\boldsymbol{\rho}_d\}$$

(**11**)

where

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$$a_1 = A_3 - \frac{A_2^2}{4A_1}, b_1 = \frac{A_3 z^2}{k^2} + \frac{1}{4A_1} - \frac{A_2^2 z^2}{4A_1 k^2} + \frac{iA_2 z}{2kA_1}, c_1 = \frac{zA_2^2}{2kA_1} - \frac{2zA_3}{k} - \frac{iA_2}{2A_1},$$
$$a_2 = B_3 - \frac{B_2^2}{4B_1}, b_2 = \frac{B_3 z^2}{k^2} + \frac{1}{4B_1} - \frac{B_2^2 z^2}{4B_1 k^2} + \frac{iB_2 z}{2kB_1}, c_2 = \frac{zB_2^2}{2kB_1} - \frac{2zB_3}{k} - \frac{iB_2}{2B_1},$$
$$a_3 = C_3 - \frac{C_2^2}{4C_1}, b_3 = \frac{C_3 z^2}{k^2} + \frac{1}{4C_1} - \frac{C_2^2 z^2}{4C_1 k^2} + \frac{iC_2 z}{2kC_1}, c_3 = \frac{zC_2^2}{2kC_1} - \frac{2zC_3}{k} - \frac{iC_2}{2C_1},$$
$$a_4 = D_3 - \frac{D_2^2}{4D_1}, b_4 = \frac{D_3 z^2}{k^2} + \frac{1}{4D_1} - \frac{D_2^2 z^2}{4D_1 k^2} + \frac{iD_2 z}{2kD_1}, c_4 = \frac{zD_2^2}{2kD_1} - \frac{2zD_3}{k} - \frac{iD_2}{2D_1}. \quad (12)$$

In above derivations, we have used the integral formula [54]

$$\int_{-\infty}^{\infty} \exp(-s^2 x^2 \pm qx) dx = \frac{\sqrt{\pi}}{s} \exp(\frac{q^2}{4s^2}), (s > 0) \quad (13)$$

Based on the second-order moments of the Wigner distribution function, the M^2-factor of a partially coherent beam is defined as follows [2–7,50,51]

$$M^2(z) = k\left(\left\langle \rho^2 \right\rangle \left\langle \theta^2 \right\rangle - \left\langle \rho \cdot \theta \right\rangle^2\right)^{1/2}$$
$$= k\left[\left(\left\langle x^2 \right\rangle + \left\langle y^2 \right\rangle\right)\left(\left\langle \theta_x^2 \right\rangle + \left\langle \theta_y^2 \right\rangle\right) - \left(\left\langle x\theta_x \right\rangle + \left\langle y\theta_y \right\rangle\right)^2\right]^{1/2}, \quad (14)$$

where

$$< x^{n_1} y^{n_2} \theta_x^{m_1} \theta_y^{m_2} > = \frac{1}{P} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^{n_1} y^{n_2} \theta_x^{m_1} \theta_y^{m_2} h(\boldsymbol{\rho}, \boldsymbol{\theta}, z) d^2\boldsymbol{\rho} d^2\boldsymbol{\theta}, \quad (15)$$

$$P = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\rho, \theta, z) d^2\rho d^2\theta. \quad (16)$$

Substituting Eq. (11) into Eqs. (15) and (16), we obtain (after integration) the expressions

$$P = \pi \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{(-1)^{n+m}}{N^2} \binom{N}{n}\binom{N}{m} (\frac{1}{A_1} - \frac{1}{B_1} - \frac{1}{C_1} + \frac{1}{D_1}), \quad (17)$$

$$< \rho^2 > = \frac{\pi}{P} \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{(-1)^{n+m}}{N^2} \binom{N}{n}\binom{N}{m} [(\frac{z^2}{k^2} + \frac{1}{A_1^2} - \frac{A_2^2 z^2}{A_1^2 k^2}) - (\frac{z^2}{k^2} + \frac{1}{B_1^2} - \frac{B_2^2 z^2}{B_1^2 k^2})$$
$$- (\frac{z^2}{k^2} + \frac{1}{C_1^2} - \frac{C_2^2 z^2}{C_1^2 k^2}) + (\frac{z^2}{k^2} + \frac{1}{D_1^2} - \frac{D_2^2 z^2}{D_1^2 k^2})] + \frac{4}{3}\pi^2 T z^3, \quad (18)$$

$$< \theta^2 > = \frac{\pi}{P} \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{(-1)^{n+m}}{N^2} \binom{N}{n}\binom{N}{m} \left[(\frac{1}{k^2} - \frac{A_2^2}{A_1^2 k^2}) - (\frac{1}{k^2} - \frac{B_2^2}{B_1^2 k^2}) \right.$$
$$\left. - (\frac{1}{k^2} - \frac{C_2^2}{C_1^2 k^2}) + (\frac{1}{k^2} - \frac{D_2^2}{D_1^2 k^2}) \right] + 4\pi^2 z T, \quad (19)$$

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$$< \rho\cdot\theta > = \frac{\pi}{P}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{(-1)^{n+m}}{N^2}\binom{N}{n}\binom{N}{m}\Bigg[(\frac{zC_2^2}{k^2C_1^2}-\frac{z}{k^2})-(\frac{zD_2^2}{k^2D_1^2}-\frac{z}{k^2}) -(\frac{zA_2^2}{k^2A_1^2}-\frac{z}{k^2})+(\frac{zB_2^2}{k^2B_1^2}-\frac{z}{k^2})\Bigg]+2\pi^2z^2T, \tag{20}$$

where

$$T=\int_0^\infty \Phi_n(\kappa)\kappa^3 d\kappa. \tag{21}$$

In above derivations, we have used the following relations

$$\delta(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-isx)dx, \tag{22}$$

$$\delta^n(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(-ix)^n\exp(-isx)dx,\ (n=0,\ 1,\ 2) \tag{23}$$

$$\int_{-\infty}^{\infty}f(x)\delta^n(x)dx=(-1)^n f^{(n)}(0),\ \ (n=1,\ 2) \tag{24}$$

Substituting from Eqs. (17)-(20) into Eq. (14), we obtain the following expression for the M^2-factor of a partially coherent circular DHB in turbulent atmosphere

$$\begin{aligned}M^2(z)&=k(< \rho^2 >< \theta^2 > - < \rho\cdot\theta >^2)^{1/2}\\ &=k\Big(\{\frac{\pi}{P}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{(-1)^{n+m}}{N^2}\binom{N}{n}\binom{N}{m}[(\frac{z^2}{k^2}+\frac{1}{A_1^2}-\frac{A_2^2z^2}{A_1^2k^2})-(\frac{z^2}{k^2}+\frac{1}{B_1^2}-\frac{B_2^2z^2}{B_1^2k^2})\\ &-(\frac{z^2}{k^2}+\frac{1}{C_1^2}-\frac{C_2^2z^2}{C_1^2k^2})+(\frac{z^2}{k^2}+\frac{1}{D_1^2}-\frac{D_2^2z^2}{D_1^2k^2})]+\frac{4}{3}\pi^2Tz^3\}\cdot\{\frac{\pi}{P}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{(-1)^{n+m}}{N^2}\binom{N}{n}\binom{N}{m}\\ &[(\frac{1}{k^2}-\frac{A_2^2}{A_1^2k^2})-(\frac{1}{k^2}-\frac{B_2^2}{B_1^2k^2})-(\frac{1}{k^2}-\frac{C_2^2}{C_1^2k^2})+(\frac{1}{k^2}-\frac{D_2^2}{D_1^2k^2})]+4\pi^2zT\}\\ &-\{\frac{\pi}{P}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{(-1)^{n+m}}{N^2}\binom{N}{n}\binom{N}{m}[-(\frac{zA_2^2}{k^2A_1^2}-\frac{z}{k^2})+(\frac{zB_2^2}{k^2B_1^2}-\frac{z}{k^2})+(\frac{zC_2^2}{k^2C_1^2}-\frac{z}{k^2})\\ &-(\frac{zD_2^2}{k^2D_1^2}-\frac{z}{k^2})]+2\pi^2z^2T\}^2\Big)^{1/2}.\end{aligned} \tag{25}$$

Equation (25) is the main analytical result of the present paper, which provides with a convenient way for studying the M^2-factor of coherent ($\sigma_g -> \infty$) and partially coherent circular DHBs in free space ($\Phi_n(\kappa)=0$) and turbulent atmosphere.

3. Numerical examples

In this section, we study the M^2-factor of a circular DHB in turbulent atmosphere numerically. In the following numerical examples, we choose the Tatarskii spectrum for the spectral density of the index-of-refraction fluctuations, which is expressed as [41]

$$\Phi_n(\kappa)=0.033C_n^2\kappa^{-11/3}\exp(-\frac{\kappa^2}{\kappa_m^2})� \tag{26}$$

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where C_n^2 is the structure constant of the turbulent atmosphere, $\kappa_m = 5.92 / l_0$ with l_0 being the inner scale of the turbulence. Substituting from Eq. (28) into Eq. (21), we obtain

$$T = \int_0^\infty \Phi_n(\kappa)\kappa^3 d\kappa = 0.1661 C_n^2 l_0^{-1/3}. \qquad (27)$$

Substituting Eq. (26) into Eqs. (25), we calculate the M^2-factor of a partially coherent circular DHB numerically.

To check the validity of our formulae, we calculate in Fig. 2 the M^2-factor of a coherent circular DHB in the source plane (z = 0) versus *N* and *p* using Eq. (25) with $w_0 = 1mm$, $\sigma_g = \infty$, $C_n^2 = 0$ and $\lambda = 632.8nm$. One finds that our results agree well with Fig. 3 of [27]. The M^2-factor increases as *p* or *N* increases (i.e., the central dark size increases).

For the convenience of comparison, we now study the normalized M^2-factor of DHBs defined as $M_r^2(z) = M^2(z)/M^2(0)$ on propagation in turbulent atmosphere. We calculate in Fig. 3 the normalized M^2-factor of a coherent circular DHB on propagation using different values of the structure constant (C_n^2) of the turbulent atmosphere with $w_0 = 20mm$, N=10, $p = 0.9$, $\sigma_g = \infty$, $\lambda = 632.8nm$ and $l_0 = 10mm$. For comparison, the corresponding result in free space ($C_n^2 = 0$) is also shown. One finds from Fig. 3 that the normalized M^2-factor of a coherent circular DHB remain invariant on propagation in free-space as expected, while the normalized M^2-factor increases on propagation in turbulent atmosphere, and its value increases more rapidly as C_n^2 increases, which means that the beam quality of a DHB degrades in turbulence. Figure 4 shows the normalized M^2-factor of a coherent circular DHB on propagation in turbulent atmosphere for different values of inner scale of the turbulence (l_0) with $w_0 = 20mm$, N=5, $p = 0.9$, $\sigma_g = \infty$, $\lambda = 632.8nm$ and $C_n^2 = 10^{-15} m^{-2/3}$. It is clear from Fig. 4 that the normalized M^2-factor increases more rapidly as l_0 decreases.

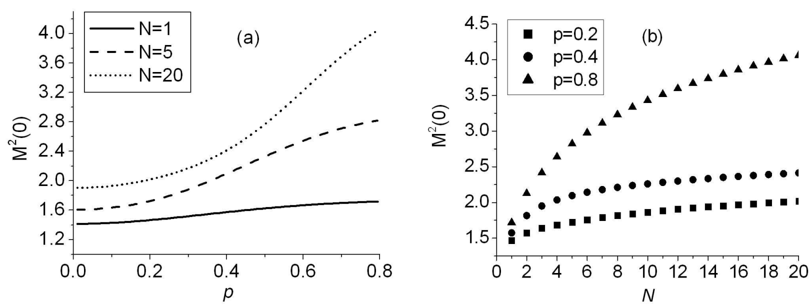

Fig. 2. M^2-factor of a coherent circular DHB in the source plane (z = 0) versus *N* and *p*

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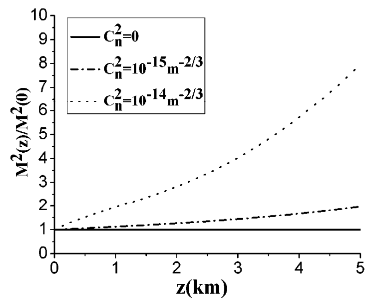

Fig. 3. Normalized M^2 -factor of a coherent circular DHB on propagation in turbulent atmosphere for different values of the structure constant (C_n^2) of the turbulent atmosphere

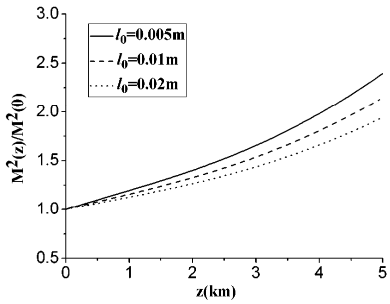

Fig. 4. Normalized M^2 -factor of a coherent circular DHB on propagation in turbulent atmosphere for different values of inner scale of the turbulence (l_0)

To learn about the dependence of M^2 -factor of a circular DHB in turbulent atmosphere on its initial beam parameters, we calculate in Fig. 5 the normalized M^2-factor of a coherent circular DHB on propagation in turbulent atmosphere for different values of beam order *N* with $w_0 = 20mm,\ p = 0.9$, $\sigma_g = \infty$, $\lambda = 632.8nm$, $C_n^2 = 10^{-15} m^{-2/3}$ and $l_0 = 10mm$. Figure 6 shows the normalized M^2-factor of a coherent circular DHB on propagation in turbulent atmosphere for different values of scaling factor p with $w_0 = 20mm,\ N$=10 , $\sigma_g = \infty$, $\lambda = 632.8nm$, $C_n^2 = 10^{-15} m^{-2/3}$ and $l_0 = 10mm$. Figure 7 shows the normalized M^2-factor of a coherent circular DHB on propagation in turbulent atmosphere for different values of wavelength λ with $w_0 = 20mm,\ p = 0.9,\ N$=5 , $\sigma_g = \infty$, $C_n^2 = 10^{-15} m^{-2/3}$ and $l_0 = 10mm$. One finds from Figs. 5-7 that the normalized M^2-factor of a coherent circular DHB are closely related to its initial beam parameters, and its value increases slower on propagation as its beam order *N*, scaling factor and wavelength λ increases, which means a DHB with larger beam order, larger scaling factor and longer wavelength λ is less affected by the atmospheric turbulence.

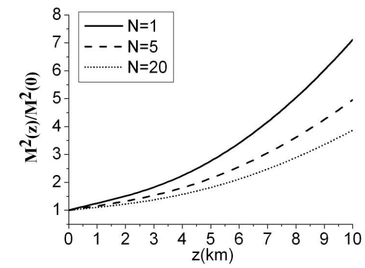

Fig. 5. Normalized M^2 -factor of a coherent circular DHB on propagation in turbulent atmosphere for different values of beam order *N*

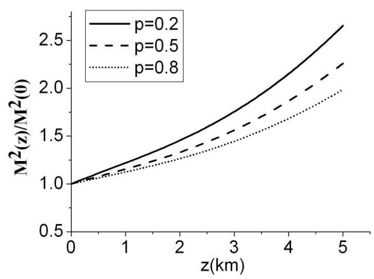

Fig. 6. Normalized M^2 -factor of a coherent circular DHB on propagation in turbulent atmosphere for different values of scaling factor *p*

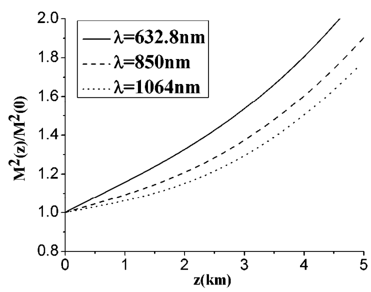

Fig. 7. Normalized M^2 -factor of a coherent circular DHB on propagation in turbulent atmosphere for different values of wavelength λ

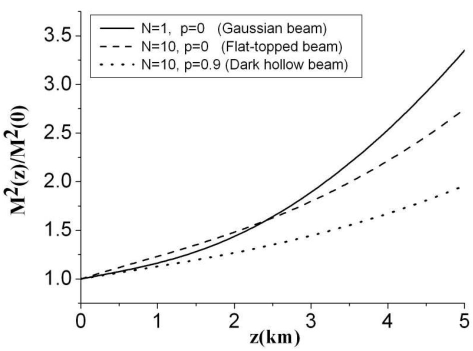

Fig. 8. Normalized M^2-factors of coherent Gaussian beam, circular flat-topped beam and circular DHB on propagation in turbulent atmosphere.

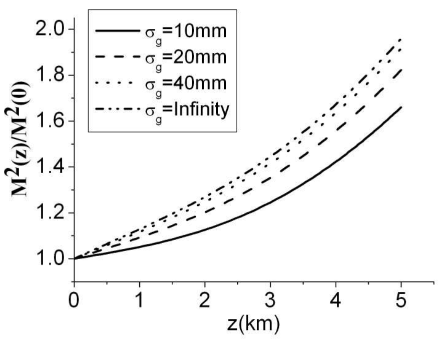

Fig. 9. Normalized M^2-factor of a partially coherent circular DHB on propagation in turbulent atmosphere for different values of the initial transverse coherence width σ_g

Equation (25) can also be used to calculate the M^2-factor of coherent Gaussian (N = 1 and p = 0) or flat-topped beam (N>1 and p = 0). We calculate in Fig. 8 the normalized M^2-factors of coherent Gaussian beam, circular flat-topped beam and circular DHB on propagation in turbulent atmosphere with $C_n^2 = 10^{-15} m^{-2/3}$, $w_0 = 20mm$, $l_0 = 10mm$ and $\lambda = 632.8nm$. One finds from Fig. 8 that the normalized M^2-factor of flat-topped beam is larger than that of a Gaussian beam at short propagation distance (z<2.5km), but is smaller than that of a Gaussian beam at a long propagation distance, which means a flat-topped beam has advantage over a Gaussian beam for long-distance free-space optical communications. We also note that normalized M^2-factor of a DHB is always smaller than that of Gaussian and flat-topped beams at any propagation distance except at z = 0, which means a DHB is less affected by the atmospheric turbulence than Gaussian and flat-topped beams. Our results agree well with those reported in [33], where we found that a DHB has advantage over Gaussian and flat-topped beams for overcoming the destructive effect of atmospheric turbulence from the aspect of scintillation. The results presented in this paper will be useful in long-distance free-space optical communications

We now turn to calculations relating to the M^2-factor of a partially coherent circular DHB on propagation in turbulent atmosphere. Our numerical results (not shown here to save space) show that the dependence of the normalized M^2-factor of a partially coherent circular DHB on the parameters (C_n^2 and l_0) of the turbulent atmosphere and the initial beam parameters (N, p and λ) is similar to that of a coherent circular DHB. We calculate in Fig. 9 the normalized M^2-factor of a partially coherent circular DHB on propagation in turbulent

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atmosphere for different values of the initial transverse coherence width σ_g with $w_0 = 20mm,\ p = 0.9,\ N=10$, $\lambda = 632.8nm$, $C_n^2 = 10^{-15}\, m^{-2/3}$ and $l_0 = 10mm$. One finds from Fig. 9 that the normalized M^2-factor of a partially coherent circular DHB also increases on propagation in turbulent atmosphere, but the increment are slower as its initial coherence width decreases, which means a DHB with lower coherence is less affected by the atmospheric turbulence.

4. Conclusion

We have derived the analytical formula for the M^2-factor of coherent and partially coherent DHBs in turbulent atmosphere by means of the extended Huygens-Fresnel integral and the second-order moments of the Wigner distribution function. We have found that the M^2 factor of a DHB in turbulent atmosphere increases upon propagation, and these increases become accelerated as the structure constant of turbulence increases or as the inner scale decreases, which is very different from its properties in free space, where its value remains invariant on propagation. Our numerical results have shown that a DHB with lower coherence, longer wavelength and larger dark size is less affected by the atmosphere, and a DHB is less affected by the atmospheric turbulence than Gaussian and flat-topped beams, which might be very useful for free-space optical communications.

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