HAMILTON-JACOBI FORMULATION OF INTERACTING FIELDS

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ABSTRACT

HAMILTON-JACOBI FORMULATION OF INTERACTING FIELDS

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Two singular systems are investigated by the canonical method. The total differential equations are obtained for the Proca model. Interactions of quantized fields are investigated and generalized to n-fields. In application to Electromagnetic interactions, it is verified that interpretation of interactions by constraints is convenient with the Feynman diagrams and S-Matrix.
Keywords: Constrained systems, Hamiltonian Formulation, Hamilton-Jacobi partial differential equations, total differential equations, quantized fields, perturbation theory , S-matrix.

ÖZ

ETKİLEŞEN ALANLARIN HAMILTON-JACOBI FORMÜLASYONU

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İki bağıl sistem kanonik yöntem ile incelendi. Toplam diferansiyel denklemler Proca Model için elde edildi. Kuantalaştırılmış alanların etkileşimleri incelendi ve n-alana genelleştirildi. Elektromanyetik alan uygulamasında, etkileşimlerin bağıl ifadesinin Feynman diyagram ve S-Matriks ile uygun olduğu gözlendi.

Anahtar kelimeler: Bağıl sistemler, Hamilton yöntemi, Hamilton-Jacobi Kısmi türevli denklemleri, Kuantalaştırılmış alanlar, S-Matriks.

Dedicated To My Father

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CHAPTER 1

INTRODUCTION

The Euler-Lagrange equations of motion leads us to express the systems described by a Lagrangian $L(q_i, \dot{q}_i)$ in two ways.

In its explicit form,

$$\frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j - \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j = 0 \qquad (1.1)$$

the Hessian matrix A_{ij} appears as,

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \quad , \quad i, j = 1, 2, ...n \qquad (1.2)$$

Firstly, the Lagrangian $L(q_i, \dot{q}_i)$ is called regular if the rank of the Hessian matrix is n and secondly, if the Hessian matrix has rank n-r, where $r < n$, not invertible, the Lagrangian $L(q_i, \dot{q}_i)$ is called singular. The Hamiltonian formulation of constrained systems is initiated by Dirac[1,2] which is the fundamental tool for the study of classical systems of particles and fields. The general covariance of Einstein's theory of gravitation leads Bergman[3] and his collaborators to work the relationship between invariance and constraints treating the field theories as singular systems. Further studies on the systems which have an invariance under a global transformation give prime importance to constrained systems , especially all gauge theories, Einstein's theory of gravitation and string theories.

1.1 Dirac's Method

The Hamiltonian formulation of constrained systems was first examined by Dirac. Starting from the property that the Hessian has rank n-r, the canonical momenta p_i read as

$$p_i = \frac{\partial L}{\partial \dot{q}}, \quad i = 1, ..., n \tag{1.3}$$

and leads to r functionally independent primary constraints of the form

$$\phi_\mu(p_i, q_i) = 0, \quad \mu = 1, ..., r, \quad r < n. \tag{1.4}$$

Using the canonical momenta (1.3) and primary constraints (1.4) one can define the canonical Hamiltonian as follows:

$$H_T = H_0 + v_\mu \phi_\mu. \tag{1.5}$$

Here H_0 is the canonical Hamiltonian

$$H_0 = -L + p_i \dot{q}_i, \tag{1.6}$$

and v_μ are the unknown coefficients. The consistency condition in terms of the vanishing Poisson bracket for arbitrary function which is not depending on time explicitly

$$\dot{f} = \{f, H_T\} = 0 \tag{1.7}$$

is defined as

$$\dot{\phi}_\mu = \{\phi_\mu, H_0\} + v_\mu\{\phi_\mu, \phi_\nu\} \approx 0, \tag{1.8}$$

for primary constraints (1.4) in the extended phase space Γ.

The weak equality $\approx$ denotes equality up to terms vanishing on Γ constructed by the surface of all constraints of the theory and described by 2n canonical variables $z_M = (q_\mu, p_\mu), \quad \mu = 1, ...n$. Due to the weak equality $\approx$, the consistency condition (1.8) for primary constraints leads to new relations called the secondary constraints when $\dot{\phi}_\mu$ does not vanish identically. Primary and secondary constraints are classified as first class constraints which have vanishing Poisson brackets with all other constraints and second class constraints which have non-vanishing Poisson brackets. When there is an even number of second class constraints , one can use them to eliminate a conjugate pairs of the p's and q's from the theory by expressing them as functions of the remaining p's and q's. Then the total Hamiltonian is defined as

$$H_T = H_0(q_a, p_a) + v_a \phi_a, \quad a = 1, ..., \beta, \quad \beta < m \tag{1.9}$$

where ϕ_α are all the independent remaining first class constraints.

1.2 The Canonical Method

Instead of usual variational principle, singular systems are examined by equivalent Lagrangians method [4,5,6]. Using this method [7,8,9], one can obtain the set of Hamilton-Jacobi partial differential equations[HJPDE] as

$$H'_\alpha(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_\alpha}) = 0, \quad \alpha, \beta = 0, n-r+1, ...n \quad a = 1, ..., n-r \tag{1.10}$$

where

$$H'_\alpha = H_\alpha(t_\beta, q_a, p_a) + p_\alpha \tag{1.11}$$

and H_0 is defined as

$$H_0 = -L(t, q_i, \dot{q}_\nu, \dot{q}_a = w_a) + p_a w_a + \dot{q}_\mu p_\mu; \quad \nu = 0, n-r+1, ..., n \tag{1.12}$$

The equations of motion are obtained as total differential equations in many variables as follows:

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha, \quad dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad dp_\mu = -\frac{\partial H'_\alpha}{\partial t_\mu} dt_\alpha, \quad \mu = 1, ..., r \tag{1.13}$$

$$dz = (-H_\alpha + p_\alpha \frac{\partial H'_\alpha}{\partial p_a}) dt_\alpha \tag{1.14}$$

where, $z = S(t_\alpha, q_a)$ is the Hamilton-Jacobi function. For this set of total differential equations , integrability conditions are obtained defining linear operators X_α as

$$X_\alpha f(t_\beta, q_a, p_a) = [f, H'_\alpha] = \frac{\partial f}{\partial q_a}\frac{\partial H'_\alpha}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial H'_\alpha}{\partial q_a} + \frac{\partial f}{\partial t_\alpha}, \tag{1.15}$$

and using these operators one can obtain a complete set of partial differential equations in terms of the bracket relations

$$(X_\alpha, X_\beta) = X_\alpha X_\beta - X_\beta X_\alpha \tag{1.16}$$

Those relations which cannot be expressible in the form

$$(X_{\alpha'}, X_{\beta'}) = C^J_{\alpha'\beta'} X_J f \tag{1.17}$$

may be added as new equations. Thus , one can obtain either a complete system or a trivial solution

$$f = constant \tag{1.18}$$

This treatment leads us to the conclusion that a set of equations(1.13) is integrable iff the following equations

$$[H'_\alpha, H'_\beta] = 0 \quad \forall \alpha, \beta \tag{1.19}$$

hold. Some applications of this formulation are given in the references [9], [13]. Equations of motion reveal the fact that Hamiltonians $H'_\alpha$are considered as the infinitesimal generators of canonical transformations given by parameters t_α. In other words , variation of a function $F(t_\alpha, q_a, p_a)$defined in the phase space is given by

$$dF = [F, H'_\alpha] dt_\alpha \tag{1.20}$$

So, the integrability conditions (1.19) can be interpreted as

$$dH'_\alpha = 0, \quad \forall \alpha \tag{1.21}$$

Quantization of classical systems can be achieved by the canonical quantization method [1]. If we ignore the ordering problems, it consists in replacing the classical Poisson bracket, by quantum commutators when classically all the states on the phase space are accessible. This is no longer correct in the presence of constraints. An approach due to Dirac[2] is widely used for quantizing the constrained Hamiltonian systems [10,11,12,13]

Recently [14], another appraoach is proposed that Lagrangians

$$L(\phi_i, \frac{\partial \phi_i}{\partial x_\mu}), \quad i = 1, ..., n \tag{1.22}$$

can be treated as a singular system with constraints

$$H'_0 = p_0 - L = 0 \tag{1.23}$$

$$H'_\mu = p_\mu + \pi_i \frac{\partial \phi_i}{\partial x^\mu} = constant \tag{1.24}$$

where p_0, p_μ and π_i are generalized momenta corresponding to τ, x_μ and ϕ_i, and $\mu = 0, 1, 2, 3$ respectively. The canonical equations are proposed as,

$$\begin{aligned}
dp_\mu &= -\frac{\partial H'_0}{\partial x_\mu} d\tau - \frac{\partial H'_\nu}{\partial x_\mu} dx^\nu && (1.25) \\
dx_\mu &= \frac{\partial H'_0}{\partial p_\mu} d\tau + \frac{\partial H'_\nu}{\partial p_\mu} dx^\nu && (1.26) \\
d\phi_i &= \frac{\partial H'_0}{\partial \pi_i} d\tau + \frac{\partial H'_\nu}{\partial \pi_i} dx^\nu && (1.27) \\
d\pi_i &= -\frac{\partial H'_0}{\partial \phi_i} d\tau - \frac{\partial H'_\nu}{\partial \phi_i} dx^\nu = \frac{\partial L}{\partial \phi_i} d\tau && (1.28) \\
dp_0 &= -\frac{\partial H'_0}{\partial \tau} d\tau - \frac{\partial H'_\nu}{\partial \tau} dx^\nu = 0 && (1.29)
\end{aligned}$$

The canonical equations are integrable iff all variations of constraints (1.23) and (1.24) vanish. Variation of constraints can be expressed as

$$dH'_\mu = dp_\mu + \frac{\partial\phi_i}{\partial x^\mu} + \pi_i d(\frac{\partial\phi_i}{\partial x^\mu}) \tag{1.30}$$

$$dH'_\mu = -\frac{\partial H'_0}{\partial x_\mu}d\tau - \frac{\partial H'_\nu}{\partial x_\mu}dx^\nu + \frac{\partial L}{\partial\phi_i}\frac{\partial\phi_i}{\partial x^\mu}d\tau + \pi^i\frac{\partial^2\phi_i}{\partial x^\mu\partial x_\nu}dx_\nu \tag{1.31}$$

$$dH'_\mu = \frac{\partial L_0}{\partial x_\mu}d\tau - \pi^i\frac{\partial^2\phi_i}{\partial x^\mu\partial x_\nu}dx_\nu + \frac{\partial L}{\partial\phi_i}\frac{\partial\phi_i}{\partial x^\mu}d\tau + \pi^i\frac{\partial^2\phi_i}{\partial x^\mu\partial x_\nu}dx_\nu \tag{1.32}$$

$$\begin{aligned} dH'_\mu &= (\frac{\partial L}{\partial x_\mu} + \frac{\partial L}{\partial\phi_i}\frac{\partial\phi_i}{\partial x^\mu})d\tau \\ &= F_\mu d\tau = 0 \end{aligned} \tag{1.33}$$

where

$$dH'_\mu = F_\mu = \frac{dL}{dx^\mu} \tag{1.34}$$

When dH'_μ does not vanish identically F_μ is treated as a new constraint and its variation should be considered.

At first glance it seems that constraints (1.24) and eq. (1.25)are irrelevant. Starting from eq.(1.25) and using the additional constraint(1.33), we get [14],

$$dp_\mu = -\frac{\partial L}{\partial\phi_i}\frac{\partial\phi_i}{\partial x^\mu}d\tau - \pi^i\frac{\partial^2\phi_i}{\partial x^\mu\partial x_\nu}dx_\nu \tag{1.35}$$

$$dp_\mu = -\frac{\partial\phi_i}{\partial x^\mu}d\pi^i - \pi^i\frac{\partial^2\phi_i}{\partial x^\mu\partial x_\nu}dx_\nu = -d(\pi^i\frac{\partial\phi_i}{\partial x^\mu}) \tag{1.36}$$

which is equivalent to the usual Lagrangian formalism . Besides, for classical fields,

$$d\pi_i = \frac{\partial L}{\partial\phi_i}d\tau \tag{1.37}$$

completes the equivalence to Euler-Lagrange equations.

These quantization schemes have the properties that by using them one can easily control important properties of quantum theory such as unitarity and positive definiteness of the metric. Besides relativistically covariant formulation of quantum theory is obtained by these quantization schemes , it enables us to examine the interactions of quantized fields.

S-Matrix method is another approach carrying the properties above to examine the interactions. So, we would like to make a brief review of the interaction representation and S-Matrix now.

1.3 The Interaction Representation and S-Matrix

Let us once more draw attention to the differences between the various representations in quantum mechanics. In the Schrödinger representation the states are time dependent. In the Heisenberg representation the state vector is time independent, whereas the operators are time dependent and satisfy the Heisenberg's equations of motion. In the interaction representation the time dependence is shared between operators and the states. Thus , in the interaction representation , the field operators of an interacting nonlinear field theory satisfy the free field equations.

The relation between Schrödinger and Heisenberg operators in the interacting field theory is

$$\psi_H(x,t) = e^{iHt}\psi(x)e^{-iHt}. \tag{1.38}$$

In the interaction representation one obtains

$$\psi_I(x) = e^{iH_0t}\psi(x)e^{-iH_0t} \tag{1.39}$$

by dividing the Lagrangian density and the Hamiltonian into a free and an interaction part, where H_0 is time independent:

$$L = L_0 + L_I \tag{1.40}$$

$$H = H_0 + H_I. \tag{1.41}$$

Then, the definition of the interaction representation reads:

$$|\psi, t\rangle_I = e^{iH_0 t}|\psi, t\rangle \tag{1.42}$$

for state vector and

$$A_I(t) = e^{iH_0 t} A e^{-iH_0 t} \tag{1.43}$$

for an arbitrary operator. In the interaction representation , the states and the operators satisfy the equation of motion

$$i\frac{\partial}{\partial t}|\psi, t\rangle_I = H_I(t)|\psi, t\rangle_I \tag{1.44}$$

$$\frac{d}{dt}A_I(t) = i[H_0, A_I(t)] + \frac{\partial}{\partial t}A_I(t) \tag{1.45}$$

Since the interaction representation arises from the Schrödinger representation ,and also from the Heisenberg representation , through a unitary transformation , the interacting fields obey the same commutation relations as the free fields. Since the equations of motion in the interaction picture are identical to the free equations of motion , the operators have the same simple form, the same time dependence , and the same representation in terms of creation and annihilation operators as free operators.

One finds the time-evolution operator in the interaction picture by starting from the formal solution of the Schrödinger equation, as

$$|\psi, t\rangle = e^{-iH(t-t_0)}|\psi, t_0\rangle \tag{1.46}$$

This leads , in the interaction picture, to

$$|\psi(t)\rangle = e^{iH_0 t} e^{-iH(t-t_0)}|\psi, t_0\rangle \tag{1.47}$$

$$|\psi(t)\rangle = e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t}|\psi, t_0\rangle \tag{1.48}$$

$$|\psi(t)\rangle \equiv U'(t, t_0)|\psi, t_0\rangle \tag{1.49}$$

The equation of motion for this time-evolution operator can be written as

$$i\frac{\partial}{\partial t}U'(t,t_0) = H_I(t)U'(t,t_0) \tag{1.50}$$

The solution for the time evolution operator can be obtained using the initial condition

$$U'(t_0,t_0) = 1 \tag{1.51}$$

as

$$U'(t,t_0) = 1 - i\int_{t_0}^{t} dt_1 H_I(t_1)U'(t_1,t_0) \tag{1.52}$$

and by the iteration of (1.52)

$$U'(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1 H_I(t_1)U'(t_1,t_0) + (i^2)\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2)$$
$$+(-i)^3\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 H_I(t_1)H_I(t_2)H_I(t_3) + ... \tag{1.53}$$

Making use of the time -ordering operator T, this infinite series can be written in the form

$$U'(t,t_0) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 ... \times \int_{t_0}^{t} dt_n T(H_I(t_1)H_I(t_2).....H_I(t_n)) \tag{1.54}$$

which is in compact form

$$U'(t,t_0) = Texp(-i)\int_{t_0}^{t} dt' H_I(t') \tag{1.55}$$

where the times fulfil either the inequality sequence $t_1 \geq t_2 \geq ... \geq t_n$,or a permutation of this inequality sequence.

The time- evolution operator $U'(t,t_0)$, in the interaction picture, gives the state $|\psi(t)\rangle$ from a specified state $|\psi(t_0)\rangle$. If the system is in the state $|i\rangle$, then the probability of finding the system at a later time t in the state $|f\rangle$ is given by

$$|\langle f|U'(t,t_0)|i\rangle|^2 \tag{1.56}$$

Thus, one obtains the transition rate, i.e., the probability per unit time of a transition from an initial state $|i\rangle$, to a final state $|f\rangle$ differing from the initial state ($\langle i|f\rangle = 0$) as,

$$w_{i\to f} = \frac{1}{t-t_0}|\langle f|U'(t,t_0)|i\rangle|^2 \tag{1.57}$$

A wavepacket representing some desired state $|\phi\rangle$ can be expressed as

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\phi(k)|k\rangle \tag{1.58}$$

where $\phi(k)$ is the Fourier transform of the spatial wavefunction, and $|k\rangle$ is a one-particle state of momentum $\overrightarrow{k}$ in the interacting theory.In the free theory, we would have $|k\rangle = \sqrt{2E_k}a_k^\dagger|0\rangle$. The factor $\sqrt{2E_k}$ converts our relativistic normalization of $|k\rangle$ to the conventional normalization in which the sum of all probabilities adds up to 1:

$$\langle\phi|\phi\rangle = 1 \quad if \quad \int \frac{d^3k}{(2\pi)^3}|\phi(k|^2) = 1 \tag{1.59}$$

The probability we wish to find is

$$P = |\underbrace{\langle\phi_1\phi_2.....|}_{future}\underbrace{\phi_A\phi_B\rangle}_{past}|^2 \tag{1.60}$$

where $|\phi_A\phi_B\rangle$ is a state of two wavepackets constructed in the far past and $\langle\phi_1\phi_2.....|$ is a state of several wavepackets constructed in the far future.

We can write the initial state as

$$|\phi_A\phi_B\rangle_{in} = \int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\frac{\phi_A(k_A)\phi_B(k_B)}{\sqrt{2E_A}\sqrt{2E_B}}|k_Ak_B\rangle \tag{1.61}$$

and the out states can be written as

$$\langle\phi_1\phi_2....| = (\prod_f \int \frac{d^3p_f}{(2\pi)^3}\frac{\phi_f(p_f)}{\sqrt{2E_f}})_{out}\langle p_1p_2...| \tag{1.62}$$

The overlap of in states with out states are related by time translation:

$$_{out}\langle p_1p_2...|k_Ak_B\rangle_{in} = \lim_{T\to\infty}\underbrace{\langle p_1p_2..}_{T}|\underbrace{k_Ak_B\rangle}_{-T} \tag{1.63}$$

$$= \lim_{T\to\infty}\langle p_1p_2...|e^{-iH(2T)}|k_Ak_B\rangle. \tag{1.64}$$

The in and out states are related by the limit of a sequence of unitary operators. This limiting unitary operator is called S-matrix:

$$_{out}\langle p_1p_2...|k_Ak_B\rangle_{in} = \langle p_1p_2...|S|k_Ak_B\rangle. \tag{1.65}$$

If there is no interaction, S is simply the identity operator.For the interactions we define T-matrix by

$$S = 1 + iT. \tag{1.66}$$

The matrix elements of S should reflect 4-momentum conservation. Thus, S or T should always contain a factor

$$\delta^4(k_A + k_B - \sum p_f) \tag{1.67}$$

Extracting this factor, we define the invariant matrix element M, by

$$\langle p_1p_2...|iT|k_Ak_B = (2\pi)^4\delta^4(k_A + k_B - \sum p_f).iM(k_A, k_B \to p_f) \tag{1.68}$$

where all 4-momenta are on mass shell: $p^0 = E_p, k^0 = E_k$.
From (1.64), S matrix is simply the time evaluation operator , $e^{(-iHt)}$. To compute this quantity we can replace the external plane-wave states in (1.64), which are eigenstates of H, with their counterparts in the unperturbed theory, which are eigenstates of H_0.

For the vacuum state, interactions in the form of the single-particle states is written as

$$\lim_{T\to\infty(1-i\epsilon)} \langle p_1 p_2 ... | e^{-iH(2T)} | p_A p_B \rangle \propto \lim_{T\to\infty(1-i\epsilon)} \langle p_1 p_2 ... p_n | T(e[-i \int_{-T}^{T} dt H_I(t)]) | p_A p_B \rangle \tag{1.69}$$

The formula for the nontrivial part of the S-matrix can be written in the form:

$$\langle p_1 ... p_n | iT | p_A p_B \rangle = \lim_{T\to\infty(1-i\epsilon)} (\langle p_1 p_2 ... p_n | T(e[-i \int_{-T}^{T} dt H_I(t)]) | p_A p_B \rangle) \tag{1.70}$$

which is restricted on the class of possible Feynman diagrams.

This thesis is arranged as follows: In chapter two, first, the Proca Model is investigated by the canonical method. The integrability conditions are obtained considering the variation of constraints. Second, the interactions of fermions with electromagnetic field is investigated in terms of interactions diagrams and integrability conditions is generalized for n-field interaction by using the canonical method. In chapter three, following the results obtained from previous chapters, S-Matrix formulation of constrained systems are discussed.

Throughout this thesis, Einstein summation rule is used and metric $g_{\mu\nu}$ has signature -1.

CHAPTER 2

MASSIVE AND INTERACTING FIELDS

2.1 The Proca Model

A complete theory of the weak interactions includes the equations of motion of the boson fields due to the exchange of very heavy vector bosons, the analogue of Maxwell's equations. Finding the correct form of these equations of motion was not straightforward, because Maxwell's equations prohibit the generation of mass for the vector particle.
Now, we examine these interactions using the Canonical method.

Massive spin-1 particles (e.g.the intermediate bosons $W^{\pm}$ of weak interactions) are described by the Proca equations

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \tag{2.1}$$

$$(\partial^\mu \partial_\mu + m^2) A^\mu = 0 \tag{2.2}$$

which generalise Maxwell's equations. The Lagrangian for the Proca model is given by,

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \tag{2.3}$$

where the antisymmetric matrix $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \tag{2.4}$$

We determine the fields as

$$A_\mu(x_\nu) = a_\mu^+(k) e^{ik_\nu x^\nu} + a_\mu^-(k) e^{-ik_\nu x^\nu}, \tag{2.5}$$

$$= A_\nu^+(x_\mu) + A_\nu^-(x_\mu)$$

with the conditions

$$k_\mu k^\mu = -m^2 \tag{2.6}$$

Here,

$$[a_\mu^\pm]^* = a_\mu^\pm \tag{2.7}$$

since the fields should be real quantities. This choice of the field expansion (2.5) enables us to examine the singular treatment of the massive vector field by obtaining the generalized momenta p_0, p_μ and π_i. Expressing the Lagrangian (2.3) as

$$L = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2}m^2 A_\mu A^\mu \tag{2.8}$$

and inserting the field expansion (2.5) into it, we obtain

$$\begin{aligned} p_0 - L = p_0 - \frac{1}{2}(ik_\mu)(A_\nu^+ - A_\nu^-)(ik^\mu)(A^{\nu+} - A^{\nu-}) \\ \frac{1}{2}(ik_\mu)(A_\nu^+ - A_\nu^-)(ik^\nu)(A^{\mu+} - A^{\mu-}) + \frac{m^2}{2}(A_\mu^+ + A_\mu^-)(A^{\mu+} + A^{\mu-}) \\ = p_0 - \frac{1}{2}(k_\mu k^\mu + m^2)(A_\nu^+ A^{\nu+} + A_\nu^- A^{\nu-}) \\ +\frac{1}{2}(\partial_\mu A^\mu)(\partial^\nu A_\nu) \\ +\frac{1}{2}(-k_\mu k^\mu + m^2)(A_\nu^+ A^{\nu-} + A_\nu^- A^{\nu+}) \\ = 0 \end{aligned} \tag{2.9}$$

Using the condition (2.6) and Lorentz condition

$$\partial_\mu A^\mu = ik_\mu(A^{\mu+} - A^{\mu-}) = 0 \tag{2.10}$$

we get

$$p_0 = L = -m^2(a_\mu^+ a^{\mu-} + a_\mu^- a^{\mu+}) = constant \tag{2.11}$$

We can check the consistency condition $dH_0' = 0$ considering the variation of Lagrangian. Explicitly,

$$\frac{dL}{dx^\rho} = \frac{ik_\rho}{2}(k_\mu k^\mu + m^2)(A_\nu^+ + A_\nu^-)(A^{\nu+} - A^{\nu-})$$

$$+\frac{ik_\rho}{2}(k_\mu k^\mu + m^2)(A_\nu^+ - A_\nu^-)(A^{\nu+} + A^{\nu-}) \quad (2.12)$$

and using the condition (2.6)it becomes

$$\frac{dL}{dx^\rho} = 0. \quad (2.13)$$

Since the higher order variations of Lagrangian

$$\frac{d^n L}{dx^n} = \frac{d^{n-1}}{dx^{n-1}}(\frac{dL}{dx^\rho}) \quad (2.14)$$

contains the condition (2.6), it vanishes for all variations. For the same choice of the field expansion (2.5) , the constraints

$$H'_\mu = p_\mu + \pi_i \frac{\partial \phi_i}{\partial x^\mu} = 0 \quad (2.15)$$

should be checked first obtaining π_i and p_μ from the equations of motion. Indeed,using the conjugate momentum of the field given by eq(1.28) we get

$$d\pi_\rho = \frac{\partial L}{\partial A_\rho} dt = m^2 A_\rho dt. \quad (2.16)$$

Integrating with respect to t

$$\pi^\rho = -\frac{im^2}{k_0}(A^{\rho+} - A^{\rho-}) \quad (2.17)$$

Since the Lagrangian does not contain the parameter τ explicitly one may identify

$$\tau = ct \quad (2.18)$$

In this case the constraints read as

$$H'_\mu = p_\mu + \pi_i \frac{\partial \phi_i}{\partial x^\mu} + K_\mu = 0 \quad (2.19)$$

where K_μ are constants.

Canonical momentum of the field can be obtained by (1.36)

$$dp_\mu = -ik_\mu m^2(A^{\rho+}A^+_\rho - A^{\rho-}A^-_\rho)dt - i\frac{k_\mu k^\nu}{k_0}m^2(A^{\rho+}A^+_\rho - A^{\rho-}A^-_\rho)dx_\nu \quad (2.20)$$

Identifying $dx_0 = dt$ we get

$$dp_\mu = -2ik_\mu m^2(A^{\rho+}A^+_\rho - A^{\rho-}A^-_\rho)dt - i\frac{k_\mu k^\varrho}{k_0}(A^{\rho+}A^+_\rho - A^{\rho-}A^-_\rho)dx_\varrho \quad (2.21)$$

where $\varrho = 1,2,3$ correspond to spatial components in spacetime. Partial differential equation corresponding to time component is

$$\frac{\partial p_\mu}{\partial t} = -2ik_\mu m^2(A^{\rho+}A^+_\rho - A^{\rho-}A^-_\rho) \quad (2.22)$$

Integration of (2.22) gives

$$p_\mu = -k_\mu \frac{m^2}{k_0}(A^{\rho+}A^+_\rho - A^{\rho-}A^-_\rho) + F_\mu(x,y,z) \quad (2.23)$$

Partial differential equation corresponding to x component is

$$\frac{\partial p_\mu}{\partial x} = -2ik_1 k_\mu \frac{m^2}{ck_0}(A^{\rho+}A^+_\rho - A^{\rho-}A^-_\rho) + \frac{\partial F_\mu(x,y,z)}{\partial x} \quad (2.24)$$

$$= -ik_\mu k^1 \frac{m^2}{k_0}(A^{\rho+}A^+_\rho - A^{\rho-}A^-_\rho) \quad (2.25)$$

Here (2.25) is obtained from (2.21) by taking $\varrho = 1$ as x-component, and

$$F_\mu(x,y,z) = k_\mu \frac{m^2}{2k_0}(A^{\rho+}A^+_\rho - A^{\rho-}A^-_\rho) + G_\mu(y,z). \quad (2.26)$$

Substituting (2.26) into (2.23) one gets

$$p_\mu = -k_\mu \frac{m^2}{2k_0}(A^{\rho+}A^+_\rho - A^{\rho-}A^-_\rho) + G_\mu(y,z). \quad (2.27)$$

Partial differential equation corresponding to y-component is

$$\frac{\partial p_\mu}{\partial y} = -ik^2 k_\mu \frac{m^2}{k_0}(A^{\rho+}A^+_\rho - A^{\rho-}A^-_\rho) + \frac{\partial G_\mu(y,z)}{\partial y} \quad (2.28)$$

$$= -ik_\mu k^2 \frac{m^2}{k_0}(A^{\rho+}A_\rho^+ - A^{\rho-}A_\rho^-) \tag{2.29}$$

Then one can get

$$G_\mu(y,z) = Constant \tag{2.30}$$

and

$$p_\mu = -k_\mu \frac{m^2}{2k_0}(A^{\rho+}A_\rho^+ + A^{\rho-}A_\rho^-) + Constant \tag{2.31}$$

Thus , we get

$$H'_\mu = k_\mu \frac{m^2}{k_0}(a^{\rho+}a_\rho^-) + Constant \tag{2.32}$$

2.2 Electromagnetic Interactions

Unlike the Lagrangian of free fields, the fields ϕ_i appears nonlinear in interacting terms up to n-th order. As an example, for the interactions of fermions with electromagnetic field, the Lagrangian density

$$L = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\nu) - e\bar{\psi}\gamma^\mu A_\mu\psi$$

$$= L_0 + L_I \tag{2.33}$$

leads the equation of motions of the field operators in the Heisenberg picture as:

$$(i\gamma^\mu - \partial_\mu)\psi = eA_\mu\gamma^\mu\psi \tag{2.34}$$

$$\partial_\mu\partial^\mu A^\mu = e\bar{\psi}\gamma^\mu\psi. \tag{2.35}$$

These are nonlinear field equations which, in general , cannot be solved exactly. An expansion occurs for simplified case of one space and one time dimension:

a few, such (1 + 1) -dimensional field theories, can be solved exactly. An interesting example is Thirring model [15]. Equations of motion

$$(i\gamma^\mu \partial_\mu - m)\psi = g\gamma_\mu \bar{\psi}\gamma^\mu \psi\psi \tag{2.36}$$

can be also obtained as a limiting case of (2.35) with a massive radiation field i.e.,

$$(\partial_\mu \partial^\mu + M)A^\mu = e\bar{\psi}\gamma^\mu \psi \tag{2.37}$$

in the limit of infinite mass M [15].

Investigation of interaction Lagrangian becomes possible when there is an association of field expansions in terms of the creation and annihilation operators corresponding to the particle antiparticle interpretation which seemed basically in the so called ϕ^4-theory. This choice of field expansions enables us to examine the singular treatment of the fields in canonical method and in perturbation theory or more generally S-matrix formulation.

The bilinear form of field expansion in terms of creation and annihilation operators enables us to explain the physical processes of Electromagnetic interactions using singular field theory approach.
The Lagrangian for this well known model is given as,

$$\begin{aligned} L &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\nu) - e\bar{\psi}\gamma^\mu A_\mu \psi \\ &= L_0 + L_I \end{aligned} \tag{2.38}$$

The first term

$$L_0 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\nu) \tag{2.39}$$

denotes free spinor and electromagnetic fields. They are investigated in [14]. The interaction Lagrangian

$$L_I = -e\bar{\psi}\gamma^\mu A_\mu \psi \tag{2.40}$$

describes the interaction of charged spin $\frac{1}{2}$ particles with electromagnetic field where A^{μ} and ψ appear as the lowest nonlinear term. The determination of the fields ψ and $\bar{\psi}$ as

$$\psi(x) = \sum_{r=1,2} (b_r(k)u_r(k^1)e^{-ik^1x} + d_r^{\dagger}(k)w_r(k^1)e^{ik^1x} \tag{2.41}$$

$$= \psi^{+}(x) + \psi^{-}(x)$$

$$\bar{\psi}(x) = \sum_{r=1,2} (b_r^{\dagger}(k)\bar{u}_r(k^2)e^{ik^2x} + d_r(k)\bar{w}_r(k^2)e^{-ik^2x} \tag{2.42}$$

$$= \bar{\psi}^{-}(x) + \bar{\psi}^{+}(x)$$

is the starting point to apply the H-J formulation. Here, r=1,2 and $\bar{\psi} = \psi^{\dagger}\gamma^{0}$ is the adjoint spinor, and k^1, k^2 shows the 4-momentum of particles and antiparticles in the processes respectively

In the same way electromagnetic field is expressed as

$$A^{\mu}(x) = \sum_{\lambda} \epsilon_{\lambda}^{\mu} a_{\lambda}(k)e^{-ik^3x} + \epsilon_{\lambda}^{\mu} a_{\lambda}^{\dagger}(k)e^{ik^3x} \tag{2.43}$$

$$= A^{\mu+}(x) + A^{\mu-}(x)$$

where ϵ_{λ}^{μ} is the 4-polarisation vector which takes the values $\lambda = 0, 1, 2, 3$ and k^3 denotes the 4-momentum of photons.

These forms of fields $A^{\mu}(x_{\mu})$, $\psi(x_{\mu})$,$\bar{\psi}(x_{\mu})$ lead us to 8-following terms in the interaction Lagrangian

$$L_I = -e\bar{\psi}\gamma^{\mu}A_{\mu}\psi$$

In fact

$$L_I = -e[(\bar{\psi}^{-} + \bar{\psi}^{+})(A_{\mu}^{+} + A_{\mu}^{-})(\psi^{+} + \psi^{-})] \tag{2.44}$$

$$= -e[\bar{\psi}^{-}A_{\mu}^{+}\psi^{+} + \bar{\psi}^{-}A_{\mu}^{+}\psi^{-} + \bar{\psi}^{+}A_{\mu}^{-}\psi^{+} + \bar{\psi}^{-}A_{\mu}^{-}\psi^{-}$$

$$+\bar{\psi}^{+}A_{\mu}^{+}\psi^{+} + \bar{\psi}^{+}A_{\mu}^{+}\psi^{-} + \bar{\psi}^{+}A_{\mu}^{-}\psi^{+} + \bar{\psi}^{+}A_{\mu}^{-}\psi^{-}] \tag{2.45}$$

Besides the consistency condition $dH_0' = 0$ and $H_0' = constant$ requires $\overrightarrow{k_{tot}^{\mu}} = 0$ on the vertices, In fact, the choice of field expansion (2.41), (2.42), (2.43) lead us to interpret the physical processes of Electromagnetic interactions in accordance with its Feynman diagram explanation.

One can assign for each determination of fields a particle-antiparticle interpretation to explain these processes as;

$$\bar{\psi}^{+} : annihilation \quad of \quad e^{+} \quad with \quad k^{2}$$

$$\bar{\psi}^{-} : creation \quad of \quad e^{-} \quad with \quad k^{2}$$

$$\psi^{+} : annihilation \quad of \quad e^{-} \quad with \quad k^{1}$$

$$\psi^{-} : creation \quad of \quad e^{+} \quad with \quad k^{1}$$

$$A^{+} : annihilation \quad of \quad \gamma \quad with \quad k^{3}$$

$$A^{-} : creation \quad of \quad \gamma \quad with \quad k^{3}$$

These 8-terms correspond to the one of the processes of particle-antiparticle interaction with photons. We consider below 3 of them as an illustration

1. e^{-}- Scattered by photon emission is described by the interaction Lagrangian,

$$L_1 = \bar{\psi}^{-} A^{-} \psi^{+}$$

$$= \Sigma(b_r^{\dagger}(k)\bar{u}_r(k^2))(\Sigma\epsilon_{\lambda}^{\mu}a^{\dagger}(k))(\Sigma b_r(k)u_r(k^1))e^{i(k^2+k^3-k^1)x} \qquad (2.46)$$

with its physical interpretation

$$e^{-} \rightarrow e^{-} + \gamma \qquad (2.47)$$

and conservation of 4-momentum corresponding to (2.47)on the vertices

$$k^1 = k^2 + k^3 \qquad (2.48)$$

leads to

$$p_0 = L_I = \Sigma(b^\dagger_{rk}(k)\bar{u}_r(k^2))(\Sigma\epsilon^\mu_\lambda a^\dagger(k))(\Sigma b_r(k)u_r(k^1)) = constant. \tag{2.49}$$

2. e^-- Scattered by a photon absorbtion is described by the interaction Lagrangian

$$L_2 = \bar{\psi}^- A^+ \psi^+$$
$$= \Sigma(b^\dagger_r(k)\bar{u}_r(k^2))(\Sigma\epsilon^\mu_\lambda a_\lambda(k))(\Sigma(b_r(k)u_r(k^1))e^{i(k^2-k^3-k^1)x} \tag{2.50}$$

with its physical interpretation

$$e^- + \gamma \to e^-. \tag{2.51}$$

On the vertices, corresponding 4-momentum conservation is written as

$$k^1 + k^3 = k^2. \tag{2.52}$$

And the constraint p_0 can be obtained as

$$p_0 = L_I = \Sigma(b^\dagger_r(k)\bar{u}_r(k^2))(\Sigma\epsilon^\mu_\lambda a_\lambda(k))(\Sigma(b_r(k)u_r(k^1)) = constant \tag{2.53}$$

3. Pair annihilation by a photon absorption is described by the interaction Lagrangian

$$L_3 = \bar{\psi}^+ A^+ \psi^+$$
$$= \Sigma(d_r(k)\bar{w}_r(k^2))(\Sigma\epsilon^\mu_\lambda a_\lambda(k))(\Sigma(b_r(k)u_r(k^1))e^{i(-k^2-k^3-k^1)x} \tag{2.54}$$

where its physical correspondence is given as

$$e^- + \gamma \to e^+ \tag{2.55}$$

and conservation of 4-momentum on the vertices

$$k^1 + k^3 + k^2 = 0. \tag{2.56}$$

leads us to express

$$p_0 = L_I = \Sigma(d_r(k)\bar{w}_r(k^2))(\Sigma\epsilon^\mu_\lambda a_\lambda(k))(\Sigma(b_r(k)u_r(k^1)) = constant. \quad (2.57)$$

One can express other 5-processes in the same way.

Investigation of new constraints in the vicinity of vertexes is examined by the variation of L as,

$$dL = -e[d\bar{\psi}_\mu A_\mu \psi_\mu + \bar{\psi}_\mu dA_\mu \psi_\mu + \bar{\psi}_\mu A_\mu d\psi_\mu]. \quad (2.58)$$

Introducing the bilinear forms,

$$d\bar{\psi}_\mu = ik^2_\nu(\bar{\psi}^- - \bar{\psi}^+)dx^\nu \quad (2.59)$$

$$d\psi_\mu = ik^1_\nu(-\psi_\mu + \psi^-_\mu)dx^\nu \quad (2.60)$$

$$dA_\mu = ik^3_\nu(-A^+_\mu + A^-_\mu)dx^\nu \quad (2.61)$$

it becomes,

$$dL = -e[\bar{\psi}^- A^+ \psi^+(k^2 - k^3 - k^1) + \bar{\psi}^- A^+ \psi^-(k^2 - k^3 + k^1)$$

$$+\bar{\psi}^- A^- \psi^+(k^2 + k^3 - k^1) + \bar{\psi}^- A^- \psi^-(k^2 + k^3 + k^1)$$

$$+\bar{\psi}^+ A^+ \psi^+(-k^2 - k^3 - k^1) + \bar{\psi}^+ A^+ \psi^-(-k^2 - k^3 + k^1)$$

$$+\bar{\psi}^+ A^- \psi^+(k^2 + k^3 - k^1) + \bar{\psi}^+ A^- \psi^-(-k^2 + k^3 + k^1)]dx^\nu \quad (2.62)$$

since in each vertex $\overrightarrow{k^\mu_{tot}} = 0$ then

$$dH'_\mu = \frac{dL}{dx^\mu} = 0 \quad (2.63)$$

Besides , one can show that singular treatment of the interacting fields is valid not only pointwise in vertexes but also around vertexes by using

$$\frac{dL}{dx_\mu} = \frac{d^2L}{dx_\nu dx_\sigma} = ... = 0. \quad (2.64)$$

Based on this description of the singular treatment of the interacting fields, one can modify it to the many particle systems by interpreting the each particle in bilinear field form where it appears as ϕ^n nonlinear term corresponding to n-particles in the interaction term of the Lagrangian.

The constraint $p_0 = L_I$ can be generalized to ϕ^n by the conservation of momentum in the vertices as

$$\overrightarrow{k_\mu^{tot}} = \overrightarrow{k_\mu^1} + \overrightarrow{k_\mu^2} + ... + \overrightarrow{k_\mu^n} \tag{2.65}$$

where each momentum takes its 4-component form making the exponential terms constant.

Since the interaction term ϕ^n decompose into the 2^n term due to the bilinear form of the field $\frac{d\phi}{dx^\mu} = ik_\mu(\phi^+ - \phi^-)$ the variation of Lagrangian can be expressed as

$$\frac{dL}{dx^\mu} = \sum_{j=1}^{n} \phi^1\phi^2 ... \frac{d\phi^j}{dx^\mu} ......\phi^n = i\sum_{j=1}^{n} \phi^1 ....\phi^{j_{1,2}} ...\phi^n \overrightarrow{k_\mu^j} = 0 \tag{2.66}$$

where $d\phi_j = ik_\mu(\phi^{j+} - \phi^{j-}) = ik_\mu(\phi^{j_1} - \phi^{j_2})dx^\mu$ corresponds to the variation of j-th field for $j_1 \neq j_2$.

CHAPTER 3

S-MATRIX FORMULATION OF CONSTRAINED SYSTEMS

The invariant matrix element M appears in S-matrix formulation as

$$\langle p_1 p_2 ... | iT | k_A k_B \rangle = (2\pi)^4 \delta^4 (k_A + k_B - \sum p_f) . iM(k_A, k_B \rightarrow p_f) \tag{3.1}$$

where

$$\delta^4 (k_A + k_B - \sum p_f) \tag{3.2}$$

implies the conservation of 4-momentum in vertices and A,B denotes the incident fields.

Expressing the field expansion

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} \phi(k) |k\rangle \tag{3.3}$$

as

$$|\phi\rangle_I = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_k}} (a(k) e^{-ikx} + a^\dagger(k) e^{ikx}) \tag{3.4}$$

in terms of the creation and annihilation operators for interaction description, the invariant matrix element M appears as operators in (3.1).

In constrained systems, the invariant matrix element M can be interpreted by the constraint

$$p_0 = L_I \tag{3.5}$$

with the choice of the field expansion

$$|\phi\rangle_I = a(k) e^{-ikx} + a^\dagger(k) e^{ikx} \tag{3.6}$$

For n-field interaction, the conservation of 4-momentum

$$\overrightarrow{k_\mu} = 0 \tag{3.7}$$

leads to $\langle 0|p_0|0\rangle = M$.

Then S-matrix formulation can be written as

$$\langle p_1 p_2 ... |iT| k_A k_B\rangle = (2\pi)^4 \delta^4 (k_A + k_B - \sum p_f).ip_0(k_A, k_B \to p_f) \tag{3.8}$$

for constrained systems.

As an example let us consider the emission of a photon γ . The initial state in terms of the annihilation operator is

$$|i\rangle = b_r(k|0\rangle) \tag{3.9}$$

and the final state in terms of the creation operators of electron and photon is

$$|f\rangle = b_r^\dagger(k) a_\lambda^\dagger(k)|0\rangle. \tag{3.10}$$

Introducing the field expansions as

$$\psi(x) = \sum_{r=1,2} (\frac{m}{VE_k})^{\frac{1}{2}} (b_r(k) u_r(k^1) e^{-ik^1 x} + d_r^\dagger(k) w_r(k^1) e^{ik^1 x} \tag{3.11}$$

$$\bar{\psi}(x) = \sum_{r=1,2} (\frac{m}{VE_k})^{\frac{1}{2}} (b_r^\dagger(k) \bar{u}_r(k^2) e^{ik^2 x} + d_r(k) \bar{w}_r(k^2) e^{-ik^2 x} \tag{3.12}$$

$$A^\mu(x) = \sum_\lambda \epsilon_\lambda^\mu (\frac{m}{V|k|})^{\frac{1}{2}} a_\lambda(k) e^{-ik^3 x} + \epsilon_\lambda^\mu a^\dagger(k) e^{ik^3 x} \tag{3.13}$$

S-matrix element can be written as

$$\langle f|S|i\rangle = -ie \int d^4x [(\frac{m}{VE_k})^{\frac{1}{2}} \bar{u}_r(k^2) e^{ik^2 x}] \gamma^\mu \times [(\frac{m}{V|k'|})^{\frac{1}{2}} \epsilon_{\lambda\mu}(k) e^{ik^3 x}]$$

$$[(\frac{m}{VE_k})^{\frac{1}{2}}u_r(k^1)e^{-ik^1x}] \tag{3.14}$$

$$= -ie(\frac{m}{V|k'|})^{\frac{1}{2}}(\frac{m}{2VE_k})^{\frac{1}{2}}(\frac{m}{VE_k})^{\frac{1}{2}} \times M(2\pi)^4\delta(k^2-k^1). \tag{3.15}$$

where

$$M = \bar{u}_r(k^2)\gamma^\mu\epsilon_{\lambda\mu}u_r(k^1) \tag{3.16}$$

is the spinor matrix element.

To examine this process as constrained system the fields are introduced as

$$\psi(x) = \sum_{r=1,2}(b_r(k)u_r(k^1)e^{-ik^1x} + d_r^\dagger(k)w_r(k^1)e^{ik^1x} \tag{3.17}$$

$$= \psi^+(x) + \psi^-(x)$$

$$\bar{\psi}(x) = \sum_{r=1,2}(b_r^\dagger(k)\bar{u}_r(k^2)e^{ik^2x} + d_r(k)\bar{w}_r(k^2)e^{-ik^2x} \tag{3.18}$$

$$= \bar{\psi}^-(x) + \bar{\psi}^+(x)$$

for fermionic particles and

$$A^\mu(x) = \sum_\lambda \epsilon_\lambda^\mu a_\lambda(k)e^{-ik^3x} + \epsilon_\lambda^\mu a^\dagger(k)e^{ik^3x} \tag{3.19}$$

$$= A^{\mu+}(x) + A^{\mu-}(x)$$

for electromagnetic field [16]. Then the interaction field is written as

$$L_I = \psi^+(x)\bar{\psi}^-(x)A^{\mu-}(x) \tag{3.20}$$

and using $p_0 = L_I$, we get

$$p_0 = \sum_{r=1,2}(b_r(k)u_r(k^1)(b_r^\dagger(k)\bar{u}_r(k^2)e^{i(k^2-k^1+k^3)x} \tag{3.21}$$

and using the descriptions of initial and final states (3.14),(3.15), and conservation of 4-momentum on the vertices

$$M = \langle 0|p_0|0\rangle = \bar{u}_r(k^2)\gamma^\mu\epsilon_{\lambda\mu}u_r(k^1) \tag{3.22}$$

CHAPTER 4

CONCLUSIONS

Recently the canonical method [7,8,9], based on variational principle, is introduced for singular systems. In this formalism equations of motion appear total differential equations in many variables. The momentum p_0 is considered as the interaction term. To have an integrable system the variation of constraints which may lead to a set of constraints should be considered.

In the first section of chapter 2, the Proca model is investigated. The constraints are obtained using the Proca equations and field expansions. Since the integrability conditions are not satisfied identically the condition (2.18) is used. In the second section, Electromagnetic interactions are investigated with the field expansions carrying creation and annihilation operators corresponding to particles and antiparticles. Interpreting the interaction in terms of the constraints (2.49),(2.53), (2.57) and satisfying the integrability conditions, we reach the Feynman rules. Using the bilinear form of the field expansions we confirm the validity of canonical method to express the n-field interactions corresponding to the n-particle interactions.

In chapter four, since S-Matrix carries basic properties of quantum mechanics, we examine the emission of a phpton and show that the constraint (3.5) corresponds to the scattering amplitude M.

The advantages of our approach are twofold: First, we don't need to express the field expansions in terms of the normalized Fourier transforms. Second, only the conservation of momentum and integrability condition (1.34) is sufficient to express the interaction up to n-fields.

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