B. BENLİ

ÇANKAYA UNIVERSITY

DISCRETE MITTAG–LEFFLER FUNCTIONS IN FRACTIONAL CALCULUS

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DISCRETE MITTAG–LEFFLER FUNCTIONS IN FRACTIONAL CALCULUS

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Submitted by **Betül Benli**

Approval of the Graduate School of Natural and Applied Sciences, Çankaya University

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Prof. Dr. Taner Altunok
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science

Assist. Prof. Dr. Emre Sermutlu
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

DISCRETE MITTAG–LEFFLER FUNCTIONS IN FRACTIONAL CALCULUS

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M.S.c, Department of Mathematics and Computer Science

Supervisor: Assoc. Prof. Dr. Thabet Abdeljawad

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A specific type of Caputo q–fractional differential equations has been solved. The solutions have been expressed via the generalized q- Mittag Leffler functions which were comprised a while ago. The method of successive approximation has been used to reach the solutions. Mittag-Leffler function makes q-analog, which was proposed by Kilbas and Saigo before, generalized. In addition to this, discrete Mittag-Leffler functions and q-analog Mittag- Leffler functions, concluded by T. Abdeljawad and D.Baleanu recently, have been discussed in the content of this thesis.

Keywords: Fractional Sums, Fractional Differences, Mittag–Leffler Function, Q–Mittag-Leffler Function, Discrete Mittag–Leffler Function, Caputo Q–Fractional Integral, Caputo Q–Fractional Derivatives, Time Scale, Euler Lagrange Equation.

ÖZ

KESİRLİ KALKULÜS DE AYRIK MITTAG–LEFFLER FONKSİYONLARI

BENLİ, Betül

M.S.c, Matematik–Bilgisayar Bölümü

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Temmuz 2011, 82 sayfa

Belirli bir türdeki Caputo q–kesirli diferansiyel denklemler çözülmüştür. Çözümler kısa bir süre önce oluşturulan genelleştirilmiş Mittag–Leffler fonksiyonları ile ifade edilmiştir. Bunlara ulaşmak için ardışık yaklaşım yöntemi kullanılmıştır. Ortaya çıkan Mittag–Leffler fonksiyonu daha önceden Kilbas ve Saigo'nun öne sürdüğü q–analogu genelleştirmektedir. Ayrıca, bu tezdeki içerikde çok yakın zamanlarda T.Abdeljawad ve D.Baleanu tarafından sonuçlandırılmış ayrık Mittag–Leffler fonksiyonları ve q–analog Mittag–Leffler fonksiyonları ele alınmaktadır.

Anahtar Kelimeler: Kesirli Toplamlar, Kesirli Farklar, Mittag–Leffler Fonksiyonu, Q–Mittag–Leffler Fonksiyonu, Ayrık Mittag–Leffler Fonksiyonu, Caputo Q–Kesirli İntegral, Caputo Q–kesirli Türev, Zaman ölçeği, Euler Lagrange Denklemi

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Contents

CHAPTER 1

INTRODUCTION

The concept of fractional calculus is not new. However, it has gained its popularity and importance during the last three decades. This is due to its distinguished applications in numerous diverse fields of science and engineering ([1], [2], [3]). The q-calculus is also not of recent appearance. It was initiated in twenties of the last century. For the basic concepts in q–calculus we refer the reader to [4]. Starting from the q–analogue of Cauchy formula [5], Al–Salam started the fitting of the concept of q-fractional calculus. After that he ([6], [7]) and Agarwal R. [8] continued on by studying certain q–fractional integrals and derivatives, where they proved the semigroup properties for left and right (Riemann) type fractional integrals but without variable lower limit and variable upper limit, respectively. Recently, the authors in [9] have generalized the notion of the (left) fractional q–integral and q–derivative by introducing variable lower limit and proved the semigroup properties.

Very recently and after the appearance of time scale calculus (see for example [10]), some authors started to pay attention and apply the techniques of time scale to discrete fractional calculus ([11], [12], [13], [14]) benefitting from the results announced before in [15]. All of these results are mainly about fractional calculus on the time scales $T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $h\mathbb{Z}$ [16]. As a new contribution in this direction and being motivated by all the above, in this article we introduce the q–analogue of a generalized type Mittag–Leffler function used before by Kilbas and Saigo in [17]. Such functions are obtained by solving linear q–Caputo initial value problems. The results obtained in this article generalize also the results of [18].For the theory of q–calculus we refer the reader to the survey article [4] and for the basic definitions and results for the q–fractional calculus we refer to [13].

This thesis is organized as follows: Chapter 2 contains some preliminary results about the fractional calculus and Mittag–Leffler functions. Chapter 3 is a survey about fractional difference equations and discrete Mittag–Leffler functions studied before in ([11, 12, 15, 19]). Chapter 4 is a survey of q–fractional calculus and q–Mittag–Leffler functions. Chapter 5 contains the original contribution about the q–analogue of the generalized Mittag–Leffler functions introduced before by Kilbas and Saigo by solving a certain Caputo q–initial value problem. Finally, in Chapter 6 our conclusion is presented.

CHAPTER 2

PRELIMINARIES ON FRACTIONAL CALCULUS

Fractional calculus deals with the study of fractional order integrals and derivatives and their diverse applications [1, 2, 3, 20, 21]. Riemann–Liouville and Caputo are kinds of fractional derivatives. They all generalize the ordinary integral and differential operators. However, the fractional derivatives have less properties than the corresponding classical ones. As a result, it makes these derivatives very useful to describe the anomalous phenomena [23, 24, 25, 26, 27, 35].

Very recently some solutions of the equations containing left and right fractional derivatives were investigated in [28, 29, 30] whereas some exact solutions were found in [28, 31]. It has been found that the left and the right derivatives have interesting applications in fractional variational principles, fractional control theory as well as in fractional Lagrangian and Hamiltonian dynamics ([32], [33]).

2.1 Classical Mittag–Leffler Functions

In this section we present the definitions and some properties of two classical *Mittag–Leffler functions* [3].

The function $E_\alpha(z)$ is called Mittag–Leffler function and is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad (z \in \mathbb{C};\ \Re(\alpha)\ >\ 0). \tag{2.1}$$

In particular, when $\alpha = 1$ and $\alpha = 2$ we have

$$E_1(z) = e^z \qquad and \qquad E_2(z) = \cos(\sqrt{z}). \tag{2.2}$$

Let $\alpha = n \in \mathbb{N}$, then the following differentiation formulas hold for the function $E_n(\lambda z^n)$:

$$\left(\frac{d}{dz}\right)^n E_n(\lambda z^n) = \lambda E_n(\lambda z^n), \qquad (n \in \mathbb{N};\ \lambda \in \mathbb{C}) \tag{2.3}$$

and

$$\left(\frac{d}{dz}\right)^n \left[z^{n-1} E_n\left(\frac{\lambda}{z^n}\right)\right] = \frac{(-1)^n \lambda}{z^{n+1}} E_n\left(\frac{\lambda}{z^n}\right),\ (z \neq 0;\ n \in \mathbb{N};\ \lambda \in \mathbb{C}). \tag{2.4}$$

Let $\alpha = 1/n$ $(n \in \mathbb{N}\backslash\{1\})$, then the function $E_{1/n}(z)$ has the following representation:

$$E_{1/n}(z) = e^{z^n} \left[1 + n \int_0^z e^{-t^n} \left(\sum_{k=1}^{n-1} \frac{t^{k-1}}{\Gamma(k/n)}\right) dt\right] \qquad (n \in \mathbb{N} \setminus \{1\}). \tag{2.5}$$

In particular, for $n = 2$, equation 2.5 becomes

$$E_{1/2}(z) = e^{z^2} \left[1 + \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt\right], \tag{2.6}$$

which yields the asymptotic estimate

$$E_{1/2}(z) \sim 2e^{z^2} \qquad (\mid z \mid \to \infty;\ \mid \arg(z) \mid < \frac{\pi}{4}).$$

The asymptotic behavior of $E_\alpha(z)$ for any α is more complicated. It is based on the integral representation of $E_\alpha(z)$ in the form

$$E_\alpha(z) = \frac{1}{2\pi}\int_{\mathcal{C}} \frac{t^{\alpha-1}e^t}{t^\alpha - z}dt. \tag{2.7}$$

Here the path of integration $\mathcal{C}$ is a loop which starts and ends at $-\infty$ and encircles the circular disk $\mid t \mid \leqq \mid z \mid^{1/\alpha}$ in the positive sense: $\mid \arg(t) \mid \leqq \pi$ on $\mathcal{C}$. The integrand in (2.7) has a branch point at $t = 0$. The complex $t - plane$ is cut along the negative real axis, and in the cut plane the integrand is single-valued the principal branch of t^α is taken in the cut plane.

When $\alpha > 0$, $E_\alpha(z)$ has different asymptotic behavior at infinity for $0 < \alpha < 2$ and $\alpha \geqq 2$. If $0 < \alpha < 2$ and μ is a real number such that

$$\frac{\pi\alpha}{2} < \mu < min[\pi, \pi\alpha], \tag{2.8}$$

then, the following asymptotic expansions are valid:

$$E_\alpha(z) = \frac{1}{\alpha}z^{(1-\beta)/\alpha}exp(z^{1/\alpha}) - \sum_{k=1}^{N}\frac{1}{\Gamma(1-\alpha k)}\frac{1}{z^k} + O\left(\frac{1}{z^{N+1}}\right),\ for\ N \in \mathbb{N}\backslash\{1\}, \tag{2.9}$$

with $\mid z \mid \rightarrow \infty$, $\mid \arg(z) \mid \leqq \mu$, and

$$E_\alpha(z) = -\sum_{k=1}^{N}\frac{1}{\Gamma(1-\alpha k)}\frac{1}{z^k} + O\left(\frac{1}{z^{N+1}}\right) \tag{2.10}$$

with $\mid z \mid \rightarrow \infty$, $\mu \leqq \mid \arg(z) \mid \leqq \pi$.

If $\alpha \geqq 2$, then the following asymptotic estimate holds :

$$E_\alpha(z) = \frac{1}{\alpha}\sum_n z^{1/\alpha}\exp\left[\exp\left(\frac{2n\pi i}{\alpha}\right)z^{1/\alpha}\right] - \sum_{k=1}^{N}\frac{1}{\Gamma(1-\alpha k)}\frac{1}{z^k} + O\left(\frac{1}{z^{N+1}}\right) \quad (2.11)$$

(for $N \in \mathbb{N}\setminus\{1\}$) with $\mid z \mid \rightarrow \infty$, $\mid \arg(z) \mid \leqq \frac{\alpha\pi}{2}$, where the first sum is taken over all integers n such that

$$\mid \arg(z) + 2\pi n \mid \leqq \frac{\alpha\pi}{2}. \quad (2.12)$$

For $\alpha > 0$, $E_\alpha(z)$ can be represented in the form of Mellin–Barnes contour integral.

Lemma 1. *[3] For $\alpha > 0$ and $z \in \mathbb{C}$ ($\mid \arg(z) \mid < \pi$), the following relation holds:*

$$E_\alpha(z) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)}(-z)^{-s}ds, \quad (2.13)$$

where the path of integration separates all the poles at $s = -k$ ($k \in \mathbb{N}_0$) to the left and all the poles at $s = n+1$ ($n \in \mathbb{N}_0$) to the right.

We define the Mellin transform ([3],pages 18) of the Mittag–Leffler function (2.1):

$$\mathcal{M}[E_\alpha(-t)](s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)}, \qquad (0 < \Re(s) < 1). \quad (2.14)$$

with $s = 2$ and $q = 1$ as follows:

$$\mathcal{L}[E_\alpha(t)](s) = \frac{1}{s}\, {}_2\Psi_1\left[\begin{array}{cc|c} (1,1), & (1,1) & \\ & & \frac{1}{s} \\ & (\alpha,1) & \end{array}\right] (\Re(s) > 0). \quad (2.15)$$

The Mittag–Leffler function $E_{\alpha,\beta}(z)$, generalizing the one in (2.1), is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (z, \beta \in \mathbb{C};\ \Re(\alpha) > 0). \tag{2.16}$$

For $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the Mittag–Leffler function (2.1), i.e,

$$E_{\alpha,1}(z) = E_\alpha(z), \qquad (z \in \mathbb{C};\ \Re(\alpha) > 0). \tag{2.17}$$

We also recall two other particular cases of (2.16):

$$E_{1,2}(z) = \frac{e^z - 1}{z} \qquad \textit{and} \qquad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}. \tag{2.18}$$

Similar to (2.1)$E_\alpha(z)$, $E_{\alpha,\beta}(z)$ is an entire function of z with order $[\Re(\alpha)]^{-1}$ and type 1, and satisfies the following differentiation formulas generalizing those in (2.3) and (2.4):

$$\left(\frac{d}{dz}^{n}\right)[z^{\beta-1}E_{n,\beta}(\lambda z^n)] = z^{\beta-n-1}E_{n,\beta-n}(\lambda z^n), \qquad (n \in \mathbb{N};\ \lambda \in \mathbb{C}) \tag{2.19}$$

and

$$\left(\frac{d}{dz}^{n}\right)\left[z^{n-\beta}E_{n,\beta}\left(\frac{\lambda}{z^n}\right)\right] = \frac{(-1)^n\lambda}{z^{n+\beta}}E_{n,\beta}\left(\frac{\lambda}{z^n}\right), \qquad (z \neq 0;\ n \in \mathbb{N};\ \lambda \in \mathbb{C}). \tag{2.20}$$

One can easily prove that the usual derivatives of $E_{\alpha,\beta}(z)$ are expressed in terms of the generalized Mittag-Leffler function (2.32), that is ,

$$\left(\frac{d}{dz}^{n}\right)[E_{\alpha,\beta}(z)] = n!E^{n+1}_{\alpha,\beta+\alpha n}(z), \qquad (n \in \mathbb{N}). \tag{2.21}$$

In particular, we have

$$\left(\frac{d}{dz}^{n}\right)[E_\alpha(z)] = n!E^{n+1}_{\alpha,1+\alpha n}(z), \qquad (n \in \mathbb{N}). \tag{2.22}$$

For $\alpha = 1/n$ $(n \in \mathbb{N})$, the function $E_{1/n,\beta}(z)$ has more general representation than the one in (2.5):

$$E_{1/n,\beta}(z) = z^{(1-\beta)n} e^{z^n}\left[z_0^{(\beta-1)n} e^{-z_0^n} E_{1/n,\beta}(z_0)\right] + n\int_0^z e^{-t^n}\left(\sum_{k=1}^{n}\frac{t^{\beta n-k-1}}{\Gamma(\beta-[k/n])}\right)dt$$

for any $z_0 \in \mathbb{C}\setminus\{0\}$.

The function $E_{\alpha,\beta}(z)$ has the integral representation

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi}\int_{\mathcal{C}}\frac{t^{\alpha-\beta}e^t}{t^\alpha - z}dt, \tag{2.23}$$

which generalizes the one in (2.7) on the same path $\mathcal{C}$.

The representation (2.23) can be applied utilized to obtain the asymptotic behavior of $E_{\alpha,\beta}(z)$ at infinity, which is different for the cases $0 < \alpha < 2$ and $\alpha \geqq 2$. For $0 < \alpha < 2$ and $\mu \in \mathbb{R}$ satisfying (2.8), then

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}z^{(1-\beta)/\alpha}exp(z^{1/\alpha}) - \sum_{k=1}^{N}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k} + O\left(\frac{1}{z^{N+1}}\right), \ (for\ N \in \mathbb{N}) \tag{2.24}$$

with $\mid z \mid \to \infty$, $\mid arg(z) \mid \leqq \mu$; and

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{N}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k} + O\left(\frac{1}{z^{N+1}}\right) \tag{2.25}$$

with $\mid z \mid \to \infty$ and $\mu \leqq \mid \arg(z) \mid \leqq \pi$

If $\alpha \leqq 2$, then

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_{n}\left(z^{1/\alpha}\exp\left[\frac{2n\pi i}{\alpha}\right]\right)^{1-\beta}\exp\left[\exp\left(\frac{2n\pi i}{\alpha}\right)z^{1/\alpha}\right]$$

$$-\sum_{k=1}^{N}\frac{1}{\Gamma(\beta-\alpha k)}\frac{1}{z^k}+O\left(\frac{1}{z^{N+1}}\right), \qquad (\mid z \mid \to \infty;\ \mid \arg(z) \mid \leqq \frac{\alpha\pi}{2}), \tag{2.26}$$

where the first sum is taken over all integers n satisfying the condition (2.12).

In particular, when $\alpha = 2$, we have

$$E_{2,\beta}(z) = \frac{1}{2}z^{(1-\beta)/2}\left[e^{\sqrt{z}}+e^{-\pi i(1-\beta)sign(\arg z)}e^{-\sqrt{z}}\right]$$

$$-\sum_{k=1}^{N}\frac{1}{\Gamma(\beta-2k)}\frac{1}{z^k}+O\left(\frac{1}{z^{N+1}}\right), \qquad (\mid z \mid \to \infty;\ \mid \arg(z) \mid \leqq \pi) \tag{2.27}$$

and, for $z = x > 0$, we have

$$E_{2,\beta}(-x) = x^{(1-\beta)/2}\cos\left(\sqrt{x}+\frac{\pi(1-\beta)}{2}\right)$$

$$-\sum_{k=1}^{N}\frac{(-1)^k}{\Gamma(\beta-2k)}\frac{1}{x^k}+O\left(\frac{1}{x^{N+1}}\right), \qquad (x>0;\ x\to\infty). \tag{2.28}$$

Lemma 2. *[3] For $\alpha > 0$ and $z \in \mathbb{C}$ ($\mid \arg(z) \mid < \pi$), the following relation holds:*

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}ds, \tag{2.29}$$

where the path of integration separates all the poles at $s = -k$ ($k \in \mathbb{N}_0$) to the left and all the poles at $s = n+1$ ($n \in \mathbb{N}_0$) to the right.

In view of (2.29), one can write the Mellin transform of the Mittag–Leffler function

$E_{\alpha,\beta}(Z)$ in the form

$$\mathcal{M}[E_{\alpha}(-t)](s) = \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}, \qquad (0 < \Re(s) < 1) \tag{2.30}$$

and

$$\mathcal{L}[E_{\alpha,\beta}(t)](s) = \frac{1}{s}\ {}_2\Psi_1\left[\begin{array}{cc|c} (1,1), & (1,1) & \\ & & \frac{1}{s} \\ & (\alpha,\beta) & \end{array}\right] (\Re(s) > 0). \tag{2.31}$$

It is to be noted that for $\beta = 1$, the relations (2.30) and (2.31) coincide with the ones in (2.14) and (2.15).

2.2 Generalized Mittag–Leffler Functions

We consider the generalized Mittag–Leffler function defined for complex $z \in \mathbb{C}$, α , β, ρ, $\in \mathbb{C}$, and $\Re(\alpha) > 0$ by

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!} = \frac{1}{\Gamma(\rho)}\ {}_1\Psi_1\left[\begin{array}{c|c} (\rho,1) & \\ & z \\ (\beta,\alpha) & \end{array}\right]. \tag{2.32}$$

For $\rho = 1$, in particular, (2.32) coincides with the Mittag–Leffler function (2.16). That is,

$$E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z), \qquad (z \in \mathbb{C}). \tag{2.33}$$

For the definition of hypergeometric functions ${}_1\Psi_1$ we refer to ([3], pages 27). For $\alpha = 1$, $E^{\rho}_{1,\beta}(z)$ coincides with the Kummer confluent hypergeometric function $\Phi(\rho;\beta;z)$:

$$E^{\rho}_{1,\beta}(z) = \frac{1}{\Gamma(\beta)}\Phi(\rho;\beta;z).Thatis, \tag{2.34}$$

For $\alpha = m \in \mathbb{N}$ is a positive integer, $E^{\rho}_{m,\beta}(z)$ coincides with the generalized hypergeometric function with $p = 1$ and $q = m$, apart from a constant multiplier factor:

$$E^{\rho}_{m,\beta}(z) = \frac{1}{\Gamma(\beta)} \qquad {}_1F_m\left(\rho; \frac{\beta}{m}, \frac{\beta+1}{m}, \cdots, \frac{\beta+m-1}{m}; \frac{z}{m^m}\right). \tag{2.35}$$

For the generalized Mittag–Leffler function (2.32) the following differentiation formulas hold:

$$\left(\frac{d}{dz}\right)^n \left[E^{\rho}_{\alpha,\beta}(z)\right] = (\rho)_n E^{\rho+n}_{\alpha,\beta+\alpha n}(z), \qquad (n \in \mathbb{N}) \tag{2.36}$$

and

$$\left(\frac{d}{dz}\right)^n \left[z^{\beta-1}E^{\rho}_{\alpha,\beta}(\lambda z^{\alpha})\right] = z^{\beta-n-1}E^{\rho}_{\alpha,\beta-n}(\lambda z^{\alpha}), \qquad (\lambda \in \mathbb{C}) \qquad (n \in \mathbb{N}). \tag{2.37}$$

The following relation is valid for the δ–derivative of (2.32):

$$\left(\prod_{j=1}^{n} \delta z^{\beta-j}\right) E^{\rho}_{\alpha,\beta}(z) = z^{\beta-n-1}E^{\rho}_{\alpha,\beta-n}(z), \qquad \left(\delta = z\frac{d}{dz}; \qquad n \in \mathbb{N}\right). \tag{2.38}$$

For $\rho = 1$, (2.38) becomes

$$\left(\prod_{j=1}^{n} (\delta z^{\beta-j})\right) E_{\alpha,\beta}(z) = z^{\beta-n-1}E_{\alpha,\beta-n}(z) \tag{2.39}$$

and

$$\left(\prod_{n}^{j=1}(\delta z^{1-j})\right) E_{\alpha}(z) = z^{-n} E_{\alpha,\beta-n}(z). \tag{2.40}$$

where $\delta = z\frac{d}{dz}$ and $n \in \mathbb{N}$.

For $\alpha > 0$, the integral representation for $E^{\rho}_{\alpha,\beta}(z)$ can be expressed in terms of a Mellin–Barnes contour integral.

Lemma 3. *[3] For $\alpha_j > 0$ $(j = 1, \cdots, m)$, then, for $z \in \mathbb{C}$ ($|\arg(-z)| < \pi$), the following relation holds:*

$$E^{\rho}_{\alpha,\beta} = \frac{1}{2\pi i \Gamma(\rho)} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(\rho - s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} ds, \tag{2.41}$$

where the path of integration separates all the poles at $s = -k$ $(k \in \mathbb{N}_0)$ to the left and all the poles at $s = n + \rho$ $(n \in \mathbb{N}_0)$ to the right.

The Mellin transform of the generalized Mittag–Leffler function is defined as follows:

$$\mathcal{M}[E^{\rho}_{\alpha,\beta}(-t)](s) = \frac{1}{\Gamma(\rho)} \frac{\Gamma(s)\Gamma(\rho - s)}{\Gamma(\beta - \alpha s)}, \qquad (0 < \Re(s) < \Re(\rho)). \tag{2.42}$$

with $p = 2$ and $q = 1$ as follows:

$$\mathcal{L}[E^{\rho}_{\alpha,\beta}(t)](s) = \frac{1}{s}\, {}_2\Psi_1 \left[\begin{array}{cc|c} (\rho, 1), & (1, 1) & \\ & & \frac{1}{s} \\ & (\alpha, \beta) & \end{array} \right] (\Re(s) > 0). \tag{2.43}$$

The following formula also holds for the Laplace transform of the function $t^{\beta-1}E^{\rho}_{\alpha,\beta}(\lambda t^{\alpha})$:

$$\mathcal{L}\left[t^{\beta-1}E^{\rho}_{\alpha,\beta}(\lambda t^{\alpha})\right](s) = \frac{s^{\alpha\rho-\beta}}{(s^{\alpha}-\lambda)^{\rho}}, \tag{2.44}$$

where $\Re(s) > 0$, $\Re(\beta) > 0$, $\lambda \in \mathbb{C}$, and $|\lambda s^{-\alpha}| < 1$.

$$E_{\rho}((\alpha_j, \beta_j)_{1,m}; z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Pi_{j=1}^{m}\Gamma(\alpha_j k + \beta_j)} \frac{z^k}{k!} = \tag{2.45}$$

$$= \frac{1}{\Gamma(\rho)}\, {}_1\Psi_m \left[\begin{array}{ccc|c} & (\rho, 1) & & \\ (\beta_1, \alpha_1), & \cdots & (\beta_m, \alpha_m) & z \end{array} \right] \tag{2.46}$$

where $z, \rho, \beta_j \in \mathbb{C}$, $\Re(\alpha_j) > 0$, $j = 1, \cdots, n$, and $m \in \mathbb{N}$. For $m = 1$, we obtain, $E_{\rho}((\alpha_1, \beta_1); z) = E^{\rho}_{\alpha,\beta}(z)$.

For the differentiation formula is given by

$$\prod_{j=1}^{m}\left[\left(\frac{d}{dz}\right)^{n_j} z^{\beta_j-1}\right][E_{\rho}((\alpha_j, \beta_j)_{1,m}; z)]$$

$$= z^{\beta_1+\cdots+\beta_m-n_1-\cdots-n_m-1}E_{\rho}((\alpha_j, \beta_j - n)_{1,m}; z), \qquad (n_j \in \mathbb{N}; j = 1, \cdots, n). \tag{2.47}$$

The following lemma presents the integral representation of $E_{\rho}((\alpha, \beta)_n; z)$ with $\alpha_j > 0$ $(j = 1, \cdots, m)$ via the Mellin–Barnes contour integral.

Lemma 4. *[3] If $\alpha_j > 0$ $(j = 1, ..., m)$, then, for $z \in \mathbb{C}$ ($|\arg(-z)| < \pi$), the following relation holds:*

$$E_{\rho}((\alpha_j, \beta_j)_{1,m;}z) = \frac{1}{2\pi i\Gamma(\rho)} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(\rho-s)}{\Pi_{j=1}^{m}\Gamma(\beta_j - \alpha_j s)}(-z)^{-s} ds, \tag{2.48}$$

where the path of integration separates all the poles at $s = -k$ ($k \in \mathbb{N}_0$) to the left and all the poles at $s = n + \rho$ ($n \in \mathbb{N}_0$) to the right.

In virtue of (2.48) we derive the Mellin transform of the generalized Mittag–Leffler function (2.2) as follows:

$$\mathcal{M}[E_\rho((\alpha_j, \beta_j)_{1,m}; -t](s) = \frac{1}{\Gamma(\rho)} \frac{\Gamma(s)\Gamma(\rho - s)}{\Pi_{j=1}^{m} \Gamma(\beta_j - \alpha_j s))}, \qquad (0 < \Re(s) < \Re(\rho)). \quad (2.49)$$

The Laplace transform of $E_\rho((\alpha_j, \beta_j)_{1,m}; t)$ is given in terms of the Wright function with $p = 2$ and $q = m$ by

$$\mathcal{L}[E_\rho((\alpha_j, \beta_j)_{1,m}; t)](s) = \frac{1}{s}\, {}_2\Psi_1 \left[\begin{array}{cc|c} (\rho, 1), & (1, 1) & \\ & (\alpha_j, \beta_j)_{1,m} & \frac{1}{s} \end{array} \right], \qquad (\Re(s) > 0). \quad (2.50)$$

Functions (2.32) and (2.2), generalize the classical Mittag–Leffler function $E_\alpha(z)$ and extend some properties of $E_{\alpha,\beta}(z)$ such as the differentiation formula of the form (2.19) and the integral representation (2.29) via Mellin–Barnes contour integrals.

Now we consider the generalized Mittag–Leffler function which extends the differentiation properties (2.19) and (2.20). Such a function $E_{\alpha,m,l}$, which is introduced by [3] and [17], is defined by the following series:

$$E_{\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k \qquad (2.51)$$

with

$$c_0 = 1 \quad and \quad c_k = \prod_{j=0}^{k-1} \frac{\Gamma[\alpha(jm+l)+1]}{\Gamma[\alpha(jm+l+1)+1]}, \qquad (k \in \mathbb{N}). \tag{2.52}$$

Here an empty product is taken to be one, and α, $l \in \mathbb{C}$ are complex numbers, and $m \in \mathbb{R}$ such that

$$\Re(\alpha) > 0, \qquad m > 0, \qquad and \qquad \alpha(jm+l) \notin Z^-, \qquad (j \in \mathbb{N}_0). \tag{2.53}$$

If $m = 0$, the conditions in (2.53)

$$\Re(\alpha) > 0, \qquad and \qquad \alpha(j+l) \notin Z^-, \qquad (j \in \mathbb{N}_0), \tag{2.54}$$

Moreover (2.51) is reduced to the Mittag–Leffler type function given in (2.16), a part from a constant $\Gamma(\alpha l + 1)$:

$$E_{\alpha,1,l}(z) = \Gamma(\alpha l + 1) E_{\alpha,\alpha l+1}(z). \tag{2.55}$$

For $\alpha \in \mathbb{N}$, we obtain

$$E_{n,m,l}(z) = 1 + \sum_{k=1}^{\infty} \left(\prod_{q=0}^{k-1} \prod_{j=1}^{n} \frac{1}{n(qm+l)+j} \right) z^k \tag{2.56}$$

$(n \in \mathbb{N};\ m > 0;\ n(qm+l) \notin \mathbb{Z}^-\ )(q \in \mathbb{N}_0)$.

The last function has the following differentiation properties:

$$\frac{d^n}{dz^n} \left[z^{n(l-m+1)} E_{n,m,l}(\lambda z^{nm}) \right] = \lambda z^{nl} E_{n,m,l}(\lambda z^{nm}) \tag{2.57}$$

and

$$\frac{d^n}{dz^n}\left[z^{n(m-l)-1}E_{n,m,l}\left(\frac{\lambda}{z^{nm}}\right)\right]=\frac{(-1)^n\lambda}{z^{n(l+1)+1}}E_{n,m,l}\left(\frac{\lambda}{z^{nm}}\right)\qquad(z\neq 0),\qquad(2.58)$$

with $n\in\mathbb{N}$ and $\lambda\in\mathbb{C}$. For $m=1$ and $nl+1=\beta$, then in accordance with (2.55), the relations (2.57) and (2.58) coincide with (2.19) and (2.20), respectively.

The so–called multivariate Mittag–Leffler function $E_{(a_1,\cdots,a_n),b}(z_1,\cdots,z_n)$ of n complex variables $z_1,\cdots,z_n\in\mathbb{C}$ with complex parameters $a_1,\cdots,a_n$, $b\in\mathbb{C}$ is defined by

$$E_{(a_1,\cdots,a_n),b}\left(z_1,\cdots,z_n\right)=\sum_{k=0}^{\infty}\sum_{l_1,\cdots,l_n\geqq 0}^{l_1+\cdots+l_n=k}\begin{pmatrix} k \\ \\ l_1,\quad\cdots\quad,l_n\end{pmatrix}\frac{\prod_{j=1}^{n}z_j^{l_j}}{\Gamma(b+\sum_{j=1}^{n}a_jl_j)},\qquad(2.59)$$

in terms of the multinomial coefficients:

$$\begin{pmatrix} k \\ \\ l_1,\quad\cdots\quad,l_n\end{pmatrix}=\frac{k!}{l_1!\cdots l_n!},\qquad(k,l_1,\cdots,l_n\in\mathbb{N}_0).\qquad(2.60)$$

2.3 Riemann-Liouville Fractional Integrals and Fractional Derivatives

Let $\Omega=[a,b]$ $(-\infty<a<b<\infty)$ be a finite interval on the real axis $\mathbb{R}$. The *Riemann–Liouville fractional integrals* $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ of order $\alpha\in\mathbb{C}$ $(\Re(\alpha)>0)$ are defined by

$$(I_{a+}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}, \qquad (x > a;\ \Re(\alpha) > 0) \tag{2.61}$$

and

$$(I_{b-}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}}, \qquad (x < b;\ \Re(\alpha) > 0), \tag{2.62}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the *left–sided* and *right–sided fractional integrals.* For $\alpha = n \in \mathbb{N}$, the definitions in (2.61) and (2.62) coincide with the $n--th$ integrals of the form

$$(I_{a+}^{n}f)(x) = \int_a^x dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} f(t_n)\,dt_n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(dt),$$

$(n \in \mathbb{N})$ and

$$(I_{b-}^{n}f)(x) = \int_x^b dt_1 \int_{t_1}^b dt_2 \cdots \int_{t_{n-1}}^b f(t_n)\,dt_n = \frac{1}{(n-1)!} \int_x^b (t-x)^{n-1} f(dt),$$

$(n \in \mathbb{N})$. The *Riemann–Liouville fractional derivatives* $D_{a+}^{\alpha}y$ and $D_{b-}^{\alpha}y$ of order $\alpha \in \mathbb{C}\,(\Re(\alpha) \geqq 0)$ are defined by

$$(D_{a+}^{\alpha}y)(x) =$$

$$= \left(\frac{d}{dx}\right)^n (I_{a+}^{n-\alpha}y)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{y(t)\,dt}{(x-t)^{1+\alpha-n}} \tag{2.63}$$

$(n = [\Re(\alpha)] + 1; x > a)$ and

$$(D_{b-}^{\alpha})(x) =$$

$$= \left(-\frac{d}{dx}\right)^n (I_{b-}^{n-\alpha} y)(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b \frac{y(t)\,dt}{(t-x)^{1+\alpha-n}} \tag{2.64}$$

$(n = [\Re(\alpha)] + 1; x < b)$, respectively, where $[\Re(\alpha)]$ means the integer part of $\Re(\alpha)$. In particular, when $\alpha = n \in \mathbb{N}_0$, then

$$(D_{a+}^0 y)(x) = (D_{b-}^0 y)(x) = y(x), \quad (D_{a+}^n y)(x) = y^{(n)}(x),$$

and

$$(D_{b-}^n y)(x) = (-1)^n y^{(n)}(x), \qquad (n \in \mathbb{N}), \tag{2.65}$$

where $y^{(n)}(x)$ is the usual derivative of $y(x)$ of order n. If $0 < \Re(\alpha) < 1$, then

$$(D_{a+}^\alpha y)(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dx}\right) \int_a^x \frac{y(t)\,dt}{(x-t)^{\alpha-[\Re(\alpha)]}}, \quad (0 < \Re(\alpha) < 1;\ x > a), \tag{2.66}$$

and

$$(D_{b-}^\alpha y)(x) = -\frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dx}\right) \int_x^b \frac{y(t)\,dt}{(t-x)^{\alpha-[\Re(\alpha)]}}, \quad (0 = [\Re(\alpha)] < 1; x < b). \tag{2.67}$$

For $\alpha \in \mathbb{R}^+$, then (2.63) and (2.64) take the following forms:

$$(D_{a+}^\alpha y)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{y(t)\,dt}{(x-t)^{1+\alpha-n}}, \quad (n = [\alpha] + 1\,;\ x > a) \tag{2.68}$$

and

$$(D_{b-}^\alpha y)(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b \frac{y(t)\,dt}{(t-x)^{1+\alpha-n}}, \quad (n = [\alpha] + 1\,;\ x < b), \tag{2.69}$$

while (2.66) and (2.67) are given by

$$(D_{a+}^{\alpha}y)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{d}{dx}\right)\int_a^x \frac{y(t)dt}{(x-t)^{\alpha}}, \quad (0 < \alpha < 1\,;\, x > a) \tag{2.70}$$

and

$$(D_{b-}^{\alpha}y)(x) = -\frac{1}{\Gamma(1-\alpha)}\left(\frac{d}{dx}\right)\int_x^b \frac{y(t)dt}{(t-x)^{\alpha}}, \quad (0 < \alpha < 1;\, x < b), \tag{2.71}$$

respectively. If $\Re(\alpha) = 0$ $(\alpha \neq 0)$, then (2.1.5) and (2.1.6) yield fractional derivatives of a purely imaginary order:

$$(D_{a+}^{i\theta}y)(x) = \frac{1}{\Gamma(1-i\theta)}\frac{d}{dx}\int_a^x \frac{y(t)dt}{(x-t)^{i\theta}}, \qquad (\theta \in \mathbb{R}\setminus\{0\}\,;\, x > a), \tag{2.72}$$

and

$$(D_{b-}^{i\theta}y)(x) = -\frac{1}{\Gamma(1-i\theta)}\frac{d}{dx}\int_x^b \frac{y(t)dt}{(t-x)^{i\theta}}, \qquad (\theta \in \mathbb{R}\setminus\{0\}\,;\, x < b). \tag{2.73}$$

It can be directly verified that the Riemann–Liouville fractional integration and fractional and fractional differentiation operators (2.61), (2.63) and (2.62), (2.64) of the power functions $(x-a)^{\beta-1}$ and $(b-x)^{\beta-1}$ yield power functions of the same form.

Property 5. [3] *If* $\Re(\alpha) \geqq 0$ *and* $\beta \in \mathbb{C}$ $(\Re(\beta)) > 0$ *then*

$$(I_{a+}^{\alpha}(t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1} \qquad (\Re(\alpha) > 0), \tag{2.74}$$

$$(D_{a+}^{\alpha}(t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1} \qquad (\Re(\alpha) \geqq 0). \tag{2.75}$$

and

$$(I_{b-}^{\alpha}(b-t)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1} \qquad (\Re(\alpha) > 0), \tag{2.76}$$

$$(D_{b-}^{\alpha}(b-t)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta+-\alpha-1} \qquad (\Re(\alpha) \geqq 0). \tag{2.77}$$

In particular, if $\beta = 1$ *and* $\Re(\alpha) \geqq 0)$*, then the Riemann–Liouville fractional derivatives of a constant are, in general, not equal to zero. For instance,*

$$(D_{a+}^{\alpha}1)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)},\ (D_{b-}^{\alpha}1)(x) = \frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)} \qquad (0 < \Re(\alpha) < 1). \tag{2.78}$$

On the other hand, for $j = 1, 2, \cdots, [\Re(\alpha)] + 1$*, we have*

$$(D_{a+}^{\alpha}(t-a)^{\alpha-j})(x) = 0,\ (D_{b-}^{\alpha}(b-t)^{\alpha-j})(x) = 0. \tag{2.79}$$

Corollary 6. [3] *Let* $\Re(\alpha) > 0$ *and* $n = [\Re(\alpha)] + 1$*.*

- *The equality* $(D_{a+}^{\alpha}y)\ (x) = 0$ *is valid if, and only if,*

$$y(x) = \sum_{j=1}^{n} c_j (x-a)^{\alpha-j}, \tag{2.80}$$

where $c_j \in \mathbb{R}$ $(j = 1, \cdots, n)$ *are arbitrary constants. In particular, when* $0 < \Re(\alpha) \leqq 1$*, the relation* $(D_{a+}^{\alpha}y)\ (x) = 0$ *holds if, and only if,* $y(x) = c(x-a)^{\alpha-1}$ *with any* $c \in \mathbb{R}$*.*

- *The equality* $(D_{b-}^{\alpha}y)\ (x) = 0$ *is valid if, and only if,*

$$y(x) = \sum_{j=1}^{n} d_j (b-x)^{\alpha-j}, \tag{2.81}$$

where $d_j \in \mathbb{R}$ $(j = 1, \cdots, n)$ *are arbitrary constants.*

In particular, when $0 < \Re(\alpha) \leqq 1$*, the relation* $(D^{\alpha}_{b-}y)$ $(x) = 0$ *holds if, and only if,* $y(x) = d(b-x)^{\alpha-1}$ *with any* $d \in \mathbb{R}$*.*

Lemma 7. [3] ***(a)*** *The fractional integration operators* I^{α}_{a+} *and* I^{α}_{b-} *with* $\Re(\alpha) > 0$ *are bounded in* $L_p(a,b)$ $(1 \leqq p \leqq \infty)$*:*

$$\| I^{\alpha}_{a+}f \|_p \leqq K \| f \|_p, \text{ where } \| I^{\alpha}_{b-}f \| \leqq K \| f \|_p \quad \left(K = \frac{(b-a)^{\Re(\alpha)}}{\Re(\alpha) \mid \Gamma(\alpha) \mid}\right). \tag{2.82}$$

(b) *If* $0 < \alpha < 1$ *and* $1 < p < 1/\alpha$*, then the operators* I^{α}_{a+} *and* I^{α}_{b-} *are bounded from* $L_p(a,b)$ *into* $L_q(a,b)$*, where* $q = p/(1-\alpha p)$*.*

Lemma 8. [3] *Let* $\Re(\alpha) \leqq 0$*, and* $n = [\Re(\alpha)] + 1$*. If* $y(x) \in AC^n[a,b]$*, then the fractional derivatives* D^{α}_{a+} *and* $D^{\alpha}_{b-}y$ *exist almost everywhere on* $[a,b]$ *and can be represented in the forms*

$$(D^{\alpha}_{a+}y)(x) = \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} \tag{2.83}$$

and

$$(D^{\alpha}_{b-}y)(x) = \sum_{k=0}^{n-1} \frac{(-1)^k y^{(k)}(b)}{\Gamma(1+k-\alpha)}(b-x)^{k-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{y^{(n)}(t)dt}{(t-x)^{\alpha-n+1}}, \tag{2.84}$$

respectively.

Corollary 9. [3] *If* $0 \leqq \Re(\alpha) < 1$ $(\alpha \neq 0)$ *and* $y(x) \in AC[a,b]$*, then*

$$(D^{\alpha}_{a+}y)(x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{y(a)}{(x-a)^{\alpha}} + \int_a^x \frac{y^{'}(t)dt}{(x-t)^{\alpha}}\right] \tag{2.85}$$

and

$$(D_{b-}^{\alpha}y)(x) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{y(b)}{(b-x)^{\alpha}} - \int_x^b \frac{y^{'}(t)dt}{(t-x)^{\alpha}}\right]. \tag{2.86}$$

Lemma 10. *[3] If $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, then the equations*

$$(I_{a+}^{\alpha}I_{a+}^{\beta}f)(x) = (I_{a+}^{\alpha+\beta}f)(x), \text{ and } (I_{b-}^{\alpha}I_{b-}^{\beta}f)(x) = (I_{b-}^{\alpha+\beta}f)(x) \tag{2.87}$$

are satisfied at almost every point $x \in [a,b]$ for $f(x) \in L_p(a,b)$ $(1 \leqq p \leqq \infty)$. If $\alpha+\beta > 1$, then the relations in (2.87) hold at any point of $[a,b]$.

Lemma 11. [3] *If $\Re(\alpha) > 0$ and $f(x) \in L_p(a,b)$ $(1 \leqq p \leqq \infty)$, then the following equalities*

$$(D_{a+}^{\alpha}I_{a+}^{\alpha}f)(x) = f(x) \text{ and } (D_{b-}^{\alpha}I_{b-}^{\alpha}f)(x) = f(x) \qquad (\Re(\alpha) > 0) \tag{2.88}$$

hold almost everywhere on $[a,b]$.

Property 12. [3] *If $\Re(\alpha) > \Re(\beta) > 0$, then, for $f(x) \in L_p(a,b)$ $(1 \leqq p \leqq \infty)$, the relations*

$$(D_{a+}^{\beta}I_{a+}^{\alpha}f)(x) = I_{a+}^{\alpha-\beta}f(x) \text{ and } (D_{b-}^{\beta}I_{b-}^{\alpha}f)(x) = I_{b-}^{\alpha-\beta}f(x) \tag{2.89}$$

hold almost everywhere on [a,b].

$$(D^{k}I_{a+}^{\alpha}f)(x) = I_{a+}^{\alpha-k}f(x) \text{ and } (D^{k}I_{b-}^{\alpha}f)(x) = (-1)^{k}I_{b-}^{\alpha-\beta}f(x) \tag{2.90}$$

Property 13. [3] *Let $\Re(\alpha) \geqq 0$, $m \in \mathbb{N}$ and $D = d/dx$.*

- *If the fractional derivatives* $(D^{\alpha}_{a+}y)(x)$ *and* $(D^{\alpha+m}_{a+}y)(x)$ *exist, then*

$$(D^m D^{\alpha}_{a+}y)(x) = (D^{\alpha+m}_{a+}y)(x). \tag{2.91}$$

- *If the fractional derivatives* $(D^{\alpha}_{b-}y)(x)$ *and* $(D^{\alpha+m}_{b-}y)(x)$ *exist, then*

$$(D^m D^{\alpha}_{b-}y)(x) = (-1)^m (D^{\alpha+m}_{b-}y)(x). \tag{2.92}$$

To present the next property, we use the spaces of functions $I^{\alpha}_{a+}(L_p)$ and $I^{\alpha}_{b-}(L_p)$ defined for $\Re(\alpha) > 0$ and $1 \leqq p \leqq \infty$ by

$$I^{\alpha}_{a^+}(L_p) = \left\{f : f = I^{\alpha}_{a+}\varphi,\ \varphi \in L_p(a,b)\right\} \tag{2.93}$$

and

$$I^{\alpha}_{b-}(L_p) = \left\{f : f = I^{\alpha}_{b-}\phi,\ \phi \in L_p(a,b)\right\}, \tag{2.94}$$

respectively. The composition of the fractional integration operator $I^{\alpha}_{a^+}$ with the fractional differantation operator $D^{\alpha}_{a^+}$ is given by the following result:

Lemma 14. [3] *Let* $\Re(\alpha) > 0$, $n = [\Re(\alpha)] + 1$ *and let* $f_{n-\alpha}(x) = (I^{n-\alpha}_{a+}f)(x)$ *be the fractional integral (2.1.1) of order* $n - \alpha$.

- *If* $1 \leqq p \leqq \infty$ *and* $f(x) \in I^{\alpha}_{a+}f(L_p)$, *then*

$$(I^{\alpha}_{a+}D^{\alpha}_{a+}f)(x) = f(x). \tag{2.95}$$

- *If* $f(x) \in L_1(a,b)$ *and* $f_{n-\alpha}\ (x) \in AC^n[a,b]$, *then the equality*

$$(I^{\alpha}_{a+}D^{\alpha}_{a+}f)(x) = f(x) - \sum_{j=1}^{n} \frac{f^{(n-j)}_{n-\alpha}(a)}{\Gamma(\alpha - j + 1)}(x-a)^{\alpha-j} \tag{2.96}$$

holds almost everywhere on [a,b].

In particular, if $0 < \Re(\alpha) < 1$*, then*

$$(I_{a+}^{\alpha} D_{a+}^{\alpha} f)(x) = f(x) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1}, \tag{2.97}$$

where $f_{1-\alpha}(x) = (I_{a+}^{1-\alpha} f)(x)$*, while for* $\alpha = n \in \mathbb{N}$*, the following equality holds:*

$$(I_{a+}^{n} D_{a+}^{n} f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^{k}. \tag{2.98}$$

Property 15. [3] *Let* $\alpha > 0$ *and* $\beta > 0$ *be such that* $n-1 < \alpha \leq n$*,* $m-1 < \beta \leq m (n, m, \in \mathbb{N})$ *and* $\alpha + \beta < n$*, and let* $f \in L_1(a,b)$ *and* $f_{m-\alpha} \in AC^m([a,b])$*. Then we have the following index rule:*

$$(D_{a+}^{\alpha} D_{a+}^{\beta} f)(x) = (D_{a+}^{\alpha+\beta} f)(x) - \sum_{j=1}^{m} (D_{a+}^{\beta-j} f)(a+) \frac{(x-a)^{-j-\alpha}}{\Gamma(1-j-\alpha)}. \tag{2.99}$$

Lemma 16. [3] *Let* $\Re(\alpha) > 0$ *and* $n = [\Re(\alpha)] + 1$*. Also let* $g_{n-\alpha}(x) = (I_{b-}^{n-\alpha} g)\ (x)$ *be the fractional integral (2.62) of order* $n - \alpha$*.*

- If $1 \leqq p \leqq \infty$ and $g(x) \in I_{b-}^{\alpha}\ (L_p)$, then

$$(I_{b-}^{\alpha} D_{b-}^{\alpha} g)(x) = g(x). \tag{2.100}$$

- If $g(x) \in L_1(a,b)$ and $g_{n-\alpha}(x) \in AC^n$[a,b], then the formula

$$(I_{b-}^{\alpha} D_{b-}^{\alpha} g)(x) = g(x) - \sum_{j=1}^{n} \frac{(-1)^{n-j} g_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha-j+1)}(b-x)^{\alpha-j} \tag{2.101}$$

  holds almost everywhere on [a,b].

In particular, if $0 < \Re(\alpha) < 1$, then

$$(I_{b^-}^{\alpha} D_{b^-}^{\alpha} g)(x) = g(x) - \frac{g_{1-\alpha}(a)}{\Gamma(\alpha)}(b-x)^{\alpha-1}, \tag{2.102}$$

where $g_{1-\alpha}\ (x) = (I_{b+}^{1-\alpha} g)\ (x)$, while for $\alpha = n \in \mathbb{N}$, the following equality holds:

$$(I_{b-}^{n} D_{b-}^{n} g)(x) = g(x) - \sum_{k=0}^{n-1} \frac{(-1)^k g^{(k)}(b)}{k!}(b-x)^k. \tag{2.103}$$

Lemma 17. [3] *Let $\alpha > 0, p \geqq 1$, $q \geqq 1$, and $(1/p) + (1/q) \leqq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$).*

- *If $\varphi(x) \in L_p(a,b)$ and $\psi(x) \in L_q(a,b)$, then*

$$\int_a^b \varphi(x)(I_{a+}^{\alpha}\psi)(x)(dx) = \int_a^b \psi(x)(I_{b^-}^{\alpha}\varphi)(x)dx. \tag{2.104}$$

- *If $f(x) \in I_{b^-}^{\alpha}(L_p)$ and $g(x) \in I_{a^+}^{\alpha}(L_q)$, then*

$$\int_a^b f(x)(D_{a+}^{\alpha} g)(x)(dx) = \int_a^b g(x)(D_{b-}^{\alpha} f)(x)(dx) \tag{2.105}$$

Now we consider the properties of (2.61) and (2.62) and fractional derivatives (2.63) and (2.64) in the spaces $C_\gamma[a,b]$ and $C_\gamma^n[a,b]$ defined in ([3]), respectively. The existence of the fractional integrals $(I_{a+}^{\alpha} f)$ and $(I_{b-}^{\alpha} f)$ in the space $C_\gamma^n[a,b]$ and the fractional derivatives $(D_{a+}^{\alpha} y)$ and $(D_{b-}^{\alpha} y)$ in the space $C_\gamma^n[a,b]$ are given by the following lemma:

Lemma 18. [3] *Let $\Re(\alpha) \geqq 0$ and $\gamma \in \mathbb{C}$.*

- *Let $\Re(\alpha) > 0$ and $0 \leqq \Re(\gamma) < 1$. If $\Re(\gamma) > \Re(\alpha)$, then the fractional integration operators I_{a+}^{α} and I_{b-}^{α} are bounded from $C_\gamma[a,b]$ into $C_{\gamma-\alpha}[a,b]$:*

$$\| I_{a^+}^{\alpha} f \|_{C_{\gamma-\alpha}} \leqq k_1 \| f \|_{C_\gamma} \quad \text{and} \quad \| I_{b^-}^{\alpha} f \|_{C_{\gamma-\alpha}} \leqq k_1 \| f \|_{C_\gamma}, \tag{2.106}$$

where

$$k_1 = \frac{\Gamma[\Re(\alpha)] \mid \Gamma(1 - \Re(\gamma)) \mid}{\mid \Gamma(\alpha) \mid \Gamma[1 + \Re(\alpha - \gamma)]}. \tag{2.107}$$

In particular, I^{α}_{a+} *and* I^{α}_{b-} *are bounded in* $C_{\gamma}[a,b]$*. If* $\Re(\gamma) \leqq \Re(\alpha)$*, then the fractional integration operators* I^{α}_{a+} *and* I^{α}_{b-} *are bounded from* $C_{\gamma}[a,b]$ *into* $C[a,b]$*:*

$$\| I^{\alpha}_{a+} f \|_C \leqq k_2 \| f \|_{C_\gamma} \qquad \textit{and} \qquad \| I^{\alpha}_{b-} f \|_C \leqq k_2 \| f \|_{C_\gamma}, \tag{2.108}$$

where

$$k_2 = (b-a)^{\Re(\alpha-\gamma)} \frac{\Gamma[\Re(\alpha)] \mid \Gamma(1 - \Re(\gamma)) \mid}{\mid \Gamma(\alpha) \mid \Gamma[1 + \Re(\alpha - \gamma)]}. \tag{2.109}$$

In particular, I^{α}_{a+} *and* I^{α}_{b-} *are bounded in* $C_{\gamma}[a,b]$*.*

- *If* $\Re(\alpha) \geqq 0$*,* $n = [\Re(\alpha)] + 1$ *and* $y(x) \in C^{n}_{\gamma}[a,b]$*, then the fractional derivatives* $D^{\alpha}_{a+}y$ *and* $D^{\alpha}_{b-}y$ *exist on* $(a,b]$*, when* $0 \geqq \Re(\alpha) < 1 (\alpha \neq 0)$ *and* $y(x) \in C_{\gamma}[a,b]$*.*

2.4 Caputo Fractional Integrals and Derivatives

In this section we present the definitions and some properties of the *Caputo fractional derivatives.* Let [a,b] be a finite interval of the real line $\mathbb{R}$, and let $(D^{\alpha}_{a+}[y(t)])\,(x) \equiv (D^{\alpha}_{a+}y)(x)$ and $(D^{\alpha}_{b-}[y(t)])(x) \equiv (D^{\alpha}_{b-}y)(x)$ be the Riemann–Liouville fractional derivatives of order $\alpha \in \mathbb{C}(\Re(\alpha) \geq 0)$ defined by (2.63) and (2.64), respectively. The fractional derivatives $({}^{C}D^{\alpha}_{a+}y)(x)$ and $({}^{C}D^{\alpha}_{b-}y)(x)$ of order $\alpha \in \mathbb{C}(\Re(\alpha) \geq 0)$ on [a,b] are defined by the above Riemann–Liouville fractional derivatives as follows

$$({}^{C}D^{\alpha}_{a+}y)(x) = D^{\alpha}_{a+}\left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k \right](x) \tag{2.110}$$

and

$$({}^C D^{\alpha}_{b-} y)(x) = D^{\alpha}_{b-} \left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(b)}{k!} (b-t)^k \right] (x), \tag{2.111}$$

respectively, where

$$n = [\Re(\alpha)] + 1 \ for \ \alpha \notin \mathbb{N}_0, \ for \ \alpha \in \mathbb{N}_0. \tag{2.112}$$

These derivatives are called *left–sided and right–sided Caputo fractional derivatives* of order α.

In particular, when $0 < \Re(\alpha) < 1$, the relations (2.110) and (2.111) take the following forms:

$$({}^C D^{\alpha}_{a+} y)(x) = (D^{\alpha}_{a+} [y(t) - y(a)])(x) \tag{2.113}$$

and

$$({}^C D^{\alpha}_{b-} y)(x) = (D^{\alpha}_{b-} [y(t) - y(b)])(x). \tag{2.114}$$

If $\alpha \in \mathbb{N}$ and $y(x)$ is a function for which the Caputo fractional derivatives $({}^C D^{\alpha}_{a+} y)(x)$ and $({}^C D^{\alpha}_{b-}\ y)\ (x)$ of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) > 0$) exist together with the Riemann–Liouville fractional derivatives $(D^{\alpha}_{a+}\ y)\ (x)$ and $(D^{\alpha}_{b-} y)\ (x)$, then in accordance with (2.65) and (2.69), they are connected with each other by the following relations :

$$({}^C D^{\alpha}_{a+} y)(x) = (D^{\alpha}_{a+} y)(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha} \ (n = [\Re(\alpha)] + 1) \tag{2.115}$$

and

$$({}^C D_{b-}^{\alpha} y)(x) = (D_{b-}^{\alpha} y)(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha} \ (n = [\Re(\alpha)]+1). \tag{2.116}$$

In particular, when $0 < \Re(\alpha) < 1$, we have

$$({}^C D_{a+}^{\alpha} y)(x) = (D_{a+}^{\alpha} y)(x) - \frac{y(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha} \tag{2.117}$$

and

$$({}^C D_{b-}^{\alpha} y)(x) = (D_{b-}^{\alpha} y)(x) - \frac{y(b)}{\Gamma(1-\alpha)}(b-x)^{-\alpha}. \tag{2.118}$$

If $\alpha \notin \mathbb{N}$, then the Caputo fractional derivatives (2.110) and (2.111) coincide with the Riemann–Liouville fractional derivatives (2.63) and (2.64) in the following cases:

$$({}^C D_{a+}^{\alpha} y)(x) = (D_{a+}^{\alpha} y)(x), \tag{2.119}$$

where $y(a) = y^{'}(a) = \cdots = y^{(n-1)}(a) = 0$ $(n = [\Re(\alpha)]+1)$; and

$$({}^C D_{b-}^{\alpha} y)(x) = (D_{b-}^{\alpha} y)(x), \tag{2.120}$$

where $y(b) = y^{'}(b) = \cdots = y^{(n-1)}(b) = 0$ $(n = [\Re(\alpha)]+1)$. In particular, when $0 < \Re(\alpha) < 1$, we have

$$({}^C D_{a+}^{\alpha} y)(x) = (D_{a+}^{\alpha} y)(x), \ \textit{when}\ y(a) = 0, \tag{2.121}$$

and

$$({}^C D^{\alpha}_{b-}y)(x) = (D^{\alpha}_{b-}y)(x), \; when \; y(b) = 0. \tag{2.122}$$

If $\alpha = n \in \mathbb{N}_0$ and the usual derivative $y^n(x)$ of order n exist, then $({}^C D^{n}_{a+}y)(x)$ coincides with $y^{(n)}$, while $({}^C D^{n}_{b-}y)(x)$ coincides with $y^{(n)}(x)$ with exactness to the constant multiplier $(-1)^n$, that is,

$$({}^C D^{n}_{a+}y)(x) = y^{(n)}(x) \; and \; ({}^C D^{n}_{b-}y)(x) = (-1)^n y^{(n)}(x) \; (n \in \mathbb{N}). \tag{2.123}$$

The Caputo fractional derivatives $({}^C D^{n}_{a+}y)(x)$ and $({}^C D^{n}_{b-}y)(x)$ are defined for functions $y(x)$ for which the Riemann–Liouville fractional derivatives of the right-hand sides of (2.110) and (2.111) exist.

Theorem 19. [3] *Let $\Re(\alpha) \geqq 0$ and let n be given by (2.112). If $y(x) \in AC^n[a,b]$, then the Caputo fractional derivatives $({}^C D^{\alpha}_{a+}y)(x)$ and $({}^C D^{\alpha}_{b-}y)(x)$ exist almost everywhere on $[a,b]$.*

- *If $\alpha \notin \mathbb{N}_0$, $({}^C D^{\alpha}_{a+}y)(x)$ and $({}^C D^{\alpha}_{b-}y)(x)$ are represented by*

$$({}^C D^{\alpha}_{a+}y)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} = (I^{n-\alpha}_{a^+} D^n y)(x) \tag{2.124}$$

and

$$({}^C D^{\alpha}_{b-}y)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{y^{(n)}(t)dt}{(t-x)^{\alpha-n+1}} = (-1)^n (I^{n-\alpha}_{b-} D^n y)(x), \tag{2.125}$$

respectively, where $D^n = d^n/dx^n$ *and* $n = [\Re(\alpha)]+1$*. In particular, when* $0 < \Re(\alpha) < 1$ *and* $y(x) \in AC[a,b]$*,*

$$({}^C D^{\alpha}_{a+}y)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{y^{'}(t)dt}{(x-t)^{\alpha}} = (I^{1-\alpha}_{a+} Dy)(x) \tag{2.126}$$

and

$$({}^C D^{\alpha}_{b-}y)(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^b \frac{y^{'}(t)dt}{(t-x)^{\alpha}} = -(I^{1-\alpha}_{b-} Dy)(x). \tag{2.127}$$

- *If* $\alpha = n \in \mathbb{N}_0$*, then* $({}^C D^{\alpha}_{a+}y)(x)$ *and* $({}^C D^{\alpha}_{b-}y)(x)$ *are represented by in particular,*

$$({}^C D^{0}_{a+}y)(x) = ({}^C D^{0}_{b-}y)(x) = y(x). \tag{2.128}$$

Theorem 20. [3] *Let* $\Re(\alpha) \geqq 0$ *and let* n *be given by (2.112). Also let* $y(x) \in C^n[a,b]$*. Then the Caputo fractional derivatives* $({}^C D^{\alpha}_{a+}y)(x)$ *and* $({}^C D^{\alpha}_{b-}y)(x)$ *are continuous on* $[a,b]$*:*

$$({}^C D^{\alpha}_{a+}y)(x) \in C[a,b] \quad \text{and} \quad ({}^C D^{\alpha}_{b-}y)(x) \in C[a,b]. \tag{2.129}$$

- *If* $\alpha \notin \mathbb{N}_0$*, then* $({}^C D^{\alpha}_{a+}y)(x)$ *and* $({}^C D^{\alpha}_{b-}y)(x)$ *are represented by (2.124) and (2.125), respectively. Moreover,*

$$({}^C D^{\alpha}_{a+}y)(a) = ({}^C D^{\alpha}_{b-}y)(b) = 0. \tag{2.130}$$

- *If* $\alpha = n \in \mathbb{N}_0$*, then the fractional derivatives* $({}^C D^{\alpha}_{a+}y)(x)$ *and* $({}^C D^{\alpha}_{b-}y)(x)$ *have representations given in (2.123).*

Corollary 21. [3] *Let $\Re(\alpha) \geqq 0$ and let n be given by (2.112)*

- *If $\alpha \notin \mathbb{N}_0$, then the Caputo fractional differentiation operators ${}^C D^{\alpha}_{a+}$ and ${}^C D^{\alpha}_{b-}$ are bounded from the space $C^n[a,b]$ to the spaces $C_a[a,b]$ and $C_b[a,b]$ respectively,*

$$||^C D^{\alpha}_{a+} y ||_{C_a} \leqq k_\alpha || y ||_{C^n} \quad \text{and} \quad ||^C D^{\alpha}_{b-} y ||_{C_b} \leqq k_\alpha || y ||_{C^n}, \tag{2.131}$$

where

$$k_\alpha = \frac{(b-a)^{n-\Re(\alpha)}}{| \Gamma(n-\alpha) | \, [n-\Re(\alpha)+1]}. \tag{2.132}$$

- *If $\alpha = n \in \mathbb{N}_0$, then the operators ${}^C D^{n}_{a+}$ and ${}^C D^{n}_{b-}$ are bounded from $C^n, [a,b]$ to $C[a,b]$. Moreover,*

$$||^C D^{n}_{a+} y ||_{C} = || y ||_{C^n} \ \ \text{and} \ \ ||^C D^{n}_{b-} y ||_{C} = k_\alpha || y ||_{C^n} \, . \tag{2.133}$$

Property 22. [3] *Let $\Re(\alpha) > 0$ and let n be given by (2.112). Also let $\Re(\beta) > 0$. Then the following relations hold:*

$$({}^C D^{\alpha}_{a+}(t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-1} \qquad (\Re(\beta) > n), \tag{2.134}$$

$$({}^C D^{\alpha}_{b-}(b-t)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-1} \qquad (\Re(\beta) > n) \tag{2.135}$$

and

$$({}^C D^{\alpha}_{a+}(t-a)^{k})(x) = 0 \quad \text{and} \quad ({}^C D^{\alpha}_{b-}(t-a)^{k})(x) = 0 \quad (k = 0, 1, \cdots, n-1). \tag{2.136}$$

In particular,

$$({}^C D_{a+}^{\alpha} 1)(x) = 0 \quad \text{and} \quad ({}^C D_{b-}^{\alpha} 1)(x) = 0. \tag{2.137}$$

For $\Re(\alpha) \notin \mathbb{N}_0$ and when $\alpha \in \mathbb{N}$, the Caputo fractional differentation operators ${}^C D_{a+}^{\alpha}$ and ${}^C D_{b-}^{\alpha}$ provide operators inverse to the Riemann–Liouville fractional integration operators I_{a+}^{α} and I_{b-}^{α}, from the left. But they do not have such a property when $\Re(\alpha) \in \mathbb{N}$ and $\Re(\alpha) \neq 0$.

Lemma 23. [3] *Let $\Re(\alpha) > 0$ and let $y(x) \in L_{\infty}(a, b)$ or $y(x) \in C[a, b]$.*

- *If $\Re(\alpha) \notin \mathbb{N}$ or $\alpha \in \mathbb{N}$, then*

$$({}^C D_{a+}^{\alpha} I_{a+}^{\alpha} y)(x) = y(x) \text{ and } ({}^C D_{b-}^{\alpha} I_{b-}^{\alpha} y)(x) = y(x) \tag{2.138}$$

- *If $\Re(\alpha) \in \mathbb{N}$ and $\Re(\alpha) \neq 0$, then*

$$({}^C D_{a+}^{\alpha} I_{a+}^{\alpha} y)(x) = y(x) - \frac{(I_{a+}^{\alpha+1-n} y)(a+)}{\Gamma(n-\alpha)} (x-a)^{n-\alpha} \tag{2.139}$$

and

$$({}^C D_{b-}^{\alpha} I_{b-}^{\alpha} y)(x) = y(x) - \frac{(I_{b-}^{\alpha+1-n} y)(b-)}{\Gamma(n-\alpha)} (b-x)^{n-\alpha}. \tag{2.140}$$

Lemma 24. [3] *Let $\Re(\alpha) > 0$ and let n be given by (2.112). If $y(x) \in AC^n[a, b]$ or $y(x) \in C^n[a, b]$, then*

$$(I_{a+}^{\alpha}\ {}^C D_{a+}^{\alpha} y)(x) = y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!} (x-a)^k \tag{2.141}$$

and

$$(I^{\alpha}_{b-}\ {}^{C}D^{\alpha}_{b-}y)(x) = y(x) - \sum_{k=0}^{n-1} \frac{(-1)^k y^{(k)}(b)}{k!}(b-x)^k. \tag{2.142}$$

In particular, if $0 < \Re(\alpha) \leqq 1$ *and* $y(x) \in AC[a,b]$ *or* $y(x) \in C[a,b]$*, then*

$$(I^{\alpha}_{a+}\ {}^{C}D^{\alpha}_{a+}y)(x) = y(x) - y(a), \quad \textit{and} \quad (I^{\alpha}_{b-}\ {}^{C}D^{\alpha}_{b-}y)(x) = y(x) - y(b). \tag{2.143}$$

We have defined the Caputo derivatives on a finite interval $[a,b]$ by (2.110) and (2.111). Moreover, it has been shown in, theorem 19 and 20, that they can be represented in forms (2.123) or (2.124) and (2.125), provided that $f(x) \in AC^n[a,b]$ and $f(x) \in C^n[a,b]$. Formulas (2.124) and (2.125) can be used for the definition of the Caputo fractional derivatives on the half axis $\mathbb{R}^+$ and on the whole axis $\mathbb{R}$. Thus the corresponding *Caputo fractional derivative* of order $\alpha \in \mathbb{C}$ (with $\Re(\alpha) > 0$ and $\alpha \in \mathbb{N}$) on the half axis $\mathbb{R}^+$ and on the whole axis $\mathbb{R}$ can be defined as follows:

$$({}^{C}D^{\alpha}_{0+}y)(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{y^{(n)}(t)dt}{(x-t)^{\alpha+1-n}}, \qquad (x \in \mathbb{R}^+), \tag{2.144}$$

$$({}^{C}D^{\alpha}_{-}y)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^{\infty} \frac{y^{(n)}(t)dt}{(t-x)^{\alpha+1-n}}, \qquad (x \in \mathbb{R}^+) \tag{2.145}$$

and

$$({}^{C}D^{\alpha}_{+}y)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^x \frac{y^{(n)}(t)dt}{(x-t)^{\alpha+1-n}}, \qquad (x \in \mathbb{R}), \tag{2.146}$$

$$({}^{C}D^{\alpha}_{-}y)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^{\infty} \frac{y^{(n)}(t)dt}{(t-x)^{\alpha+1-n}}, \qquad (x \in \mathbb{R}), \tag{2.147}$$

respectively.

For $0 < \Re(\alpha) < 1$, the relations (2.144)–(2.145) and (2.146)–(2.147) take the following forms:

$$({}^C D^{\alpha}_{0+} y)(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{y^{'}(t)dt}{(x-t)^{\alpha}}, \qquad (x \in \mathbb{R}^+), \tag{2.148}$$

$$({}^C D^{\alpha}_{-} y)(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^{\infty} \frac{y^{'}(t)dt}{(t-x)^{\alpha}}, \qquad (x \in \mathbb{R}^+), \tag{2.149}$$

and

$$({}^C D^{\alpha}_{+} y)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^x \frac{y^{'}(t)dt}{(x-t)^{\alpha}}, \qquad (x \in \mathbb{R}), \tag{2.150}$$

$$({}^C D^{\alpha}_{b-} y)(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^{\infty} \frac{y^{'}(t)dt}{(t-x)^{\alpha}}, \qquad (x \in \mathbb{R}). \tag{2.151}$$

For $\alpha = n \in \mathbb{N}$, the Caputo derivatives $({}^C D^{n}_{0+} y)(x)$, $({}^C D^{n}_{-} y)(x)$ and $({}^C D^{n}_{+} y)(x)$, $({}^C D^{n}_{-} y)(x)$ are defined by

$$({}^C D^{n}_{+} y)(x) = y^{(n)}(x), ({}^C D^{n}_{-} y)(x) = (-1)^n y^{(n)}(x), \qquad (x \in \mathbb{R}^+). \tag{2.152}$$

and

$$({}^C D^{\alpha}_{+} y)(x) = y^{(n)}(x), ({}^C D^{n}_{-} y)(x) = (-1)^n y^{(n)}(x), \qquad (x \in \mathbb{R}). \tag{2.153}$$

Property 25. [3] *If $\Re(\alpha) > 0$ and $\lambda > 0$, then*

$$({}^C D^{\alpha}_{+} e^{\lambda t})(t) = \lambda^{\alpha} e^{\lambda x} \textit{ and } ({}^C D^{\alpha}_{-} e^{-\lambda t})(x) = \lambda^{\alpha} e^{-\lambda x}. \tag{2.154}$$

The Mittag–Leffler function $E_\alpha[\lambda(x-a)^\alpha]$ is invariant with respect to the Caputo derivative ${}^C D^\alpha_{a+}$, but it is not the case for the Caputo derivative ${}^C D^\alpha_{-}$. Indeed, we have the following assertion:

Lemma 26. [3] *If $\alpha > 0$, $a \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, then*

$$({}^C D^\alpha_{a+} E_\alpha[\lambda(t-a)^\alpha])(x) = \lambda E_\alpha[\lambda(x-a)^\alpha] \tag{2.155}$$

and

$$({}^C D^\alpha_{-} t^{\alpha-1} E_\alpha(\lambda t^{-\alpha}))(x) = \frac{1}{x} E_{\alpha,1-\alpha}(\lambda x^{-\alpha}). \tag{2.156}$$

In particular, when $\alpha = n \in \mathbb{N}$, we have

$$D^n E_n[\lambda(x-a)^n] = E_n[\lambda(x-a)^n] \tag{2.157}$$

and

$$D^n(t^{n-1} E_n(\lambda t^{-n}))(x) = \frac{1}{x} E_{n,1-n}(\lambda x^{-n}) = \frac{\lambda}{x^{n+1}} E_n(\lambda x^{-n}). \tag{2.158}$$

Lemma 27. [3] *Let $\alpha > 0$, $n-1 < \alpha \leqq n$ $(n \in \mathbb{N})$ be such that $y(x) \in C^n(\mathbb{R}^+)$, $y^{(n)}(x) \in L_1(0,b)$ for any $b > 0$, the Laplace transforms $(\mathcal{L}y)(p)$ and $\mathcal{L}[D^n y(t)]$ exist, and $\lim_{x\to+\infty} (D^k y)(x) = 0$ for $k = 0, 1, \cdots, n-1$. Then the following relation holds:*

$$(\mathcal{L}{}^C D^\alpha_{0+} y)(s) = s^\alpha (\mathcal{L}y)(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} (D^k y)(0). \tag{2.159}$$

In particular, if $0 < \alpha \leqq 1$, then

$$(\mathcal{L}{}^C D^\alpha_{0+} y)(s) = s^\alpha (\mathcal{L}y)(s) - s^{\alpha-1} y(0). \tag{2.160}$$

2.5 Linear Caputo Fractional Differential Equations

In this section we give the construction of the explicit solutions to linear fractional differential equations with the Caputo fractional derivative $({}^C D_{a+}^{\alpha} y)(x)$ of order $\alpha > 0$ ($\alpha \notin \mathbb{N}$) in the space $C_{\gamma}^{\alpha,n-1}[a,b]$ ($n = [\alpha]+1$). Consider the Linear fractional differential equations:

$$({}^C D_{a+}^{\alpha} y)(x) - \lambda y(x) = f(x) \ (a \leqq x \leqq b;\ n-1 < \alpha < n;\ n \in \mathbb{N};\ \lambda \in \mathbb{R}), \quad (2.161)$$

$$y^{(k)}(a) = b_k \ (b_k \in \mathbb{R}; k = 0, \cdots, n-1). \quad (2.162)$$

We suppose that $f(x) \in C_{\gamma}[a,b]$ with $0 \leqq \gamma < 1$ and $\gamma \leqq \alpha$.

$$y(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!}(x-a)^j + \frac{\lambda}{\Gamma(\alpha)} \int_a^x \frac{y(t)dt}{(x-t)^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (2.163)$$

we apply the method of successive approximations by setting

$$y_0(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!}(x-a)^j \quad (2.164)$$

and

$$y_m(x) = y_0(x) + \frac{\lambda}{\Gamma(\alpha)} \int_a^x \frac{y_{m-1}(t)dt}{(x-t)^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (m \in \mathbb{N}). \quad (2.165)$$

for $y_m(x)$:

$$y_m(x) = \sum_{j=0}^{n-1} b_j \sum_{k=0}^{m} \frac{\lambda^k (x-a)^{\alpha k + j}}{\Gamma(\alpha k + j + 1)} + \int_a^m \left[\sum_{k=1}^{m} \frac{\lambda^{k-1}}{\Gamma(\alpha k)} (x-t)^{\alpha k - 1} \right] f(t)dt. \quad (2.166)$$

Taking the limit as $m \to \infty$, we obtain

$$y(x) = \sum_{j=0}^{n-1} b_j(x-a)^j E_{\alpha,j+1}[\lambda(x-a)^\alpha] + \int_a^x (x-t)^{\alpha-1} E_{\alpha,\alpha}[\lambda(x-t)^\alpha] f(t)dt. \quad (2.167)$$

Theorem 28. [3] *Let $n-1 < \alpha < n \quad (n \in \mathbb{N})$ and let $0 \leqq \gamma < 1$ be such that $\gamma \leqq \alpha$. Also let $\lambda \in \mathbb{R}$. If $f(x) \in C_\gamma[a,b]$.*

In particular, if but $f = 0in$ (2.168), then the solution $y(x)$ belongs to the space $\boldsymbol{C}^{\alpha,n-1}[a,b]$. Indeed, the equation.

$$({}^C D_{a+}^\alpha y)(x) - \lambda y(x) = 0, \qquad (a \leqq x \leqq b; n-1 < \alpha < n; n \in \mathbb{N}; \lambda \in \mathbb{R}) \quad (2.168)$$

has a unique solution $y(x) \in \boldsymbol{C}_\gamma^{\alpha,n-1}[a,b]$ of the form

$$y(x) = \sum_{j=0}^{n-1} b_j(x-a)^j E_{\alpha,j+1}[\lambda(x-a)^\alpha]. \quad (2.169)$$

Example 29. [3] *The solution of the Cauchy problem*

$$({}^C D_{a+}^\alpha y)(x) - \lambda y(x) = f(x), \; y(a) = b \; (b \in \mathbb{R}) \quad (2.170)$$

with $0 < \alpha < 1$ and $\lambda \in \mathbb{R}$ has the form

$$y(x) = bE_\alpha[\lambda(x-a)^\alpha] + \int_a^x (x-t)^{\alpha-1} E_{\alpha,\alpha}[\lambda(x-t)^\alpha] f(t)dt, \quad (2.171)$$

while the solution of the problem

$$({}^C D_{a+}^\alpha y)(x) - \lambda y(x) = 0) \; y(a) = b \; (b \in \mathbb{R}) \quad (2.172)$$

is given by

$$y(x) = bE_\alpha[\lambda(x-a)^\alpha]. \tag{2.173}$$

In particular, the Cauchy problem

$$({}^C D_{a+}^{1/2} y)(x) - \lambda y(x) = f(x),\ y(a) = b\ (b \in \mathbb{R}) \tag{2.174}$$

has the solution given by

$$y(x) = bE_{1/2}[\lambda(x-a)^{1/2}] + \int_a^x (x-t)^{-1/2} E_{1/2,1/2}[\lambda(x-t)^{1/2}] f(t) dt, \tag{2.175}$$

and the solution to the problem

$$({}^C D_{a+}^{1/2} y)(x) - \lambda y(x) = 0,\ y(a) = b\ (b \in \mathbb{R}) \tag{2.176}$$

is given by

$$y(x) = bE_{1/2}[\lambda(x-a)^{1/2}]. \tag{2.177}$$

Example 30. [3] *The solution of the Cauchy problem*

$$({}^C D_{a+}^{\alpha} y)(x) - \lambda y(x) = f(x),\ y(a) = b,\ y^{'}(a) = d, \qquad (b, d \in \mathbb{R}) \tag{2.178}$$

with $1 < \alpha < 2$ and $\alpha \in \mathbb{R}$ has the form

$$y(x) = bE_\alpha[\lambda(x-a)^\alpha] + d(x-a)E_{\alpha,2}[\lambda(x-a)^\alpha]$$

$$+\int_a^x (x-t)^{\alpha-1} E_{\alpha,\alpha}[\lambda(x-t)^\alpha] f(t)dt. \qquad (2.179)$$

In particular, the solution of the problem

$$({}^C D_{a+}^\alpha y)(x) - \lambda y(x) = 0 \ (1 < \alpha < 2), \ y(a) = b, y^{'}(a) = d \qquad (b, d \in \mathbb{R}) \quad (2.180)$$

is given by

$$y(x) = bE_\alpha[\lambda(x-a)^\alpha] + d(x-a)E_{\alpha,2}[\lambda(x-a)^\alpha]. \qquad (2.181)$$

In what follows, we study the Cauchy problem for the following generalized homogeneous fractional differential equation.

$$({}^C D_{a+}^\alpha y)(x) - \lambda(x-a)^\beta y(x) = 0, \quad (a \leqq x \leqq b; n-1 < \alpha < n; n \in \mathbb{R}), \quad (2.182)$$

$$y^{(k)}(0) = b_k (b_k \in \mathbb{R}; k = 0, \cdots, n-1), \qquad (2.183)$$

with $\beta > -\alpha$. *In the space* $C^{n-1}[a, b]$ *[3]to the following Volterra integral equation of the second kind:*

$$y(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!}(x-a)^j + \frac{\lambda}{\Gamma(\alpha)} \int_a^x \frac{(t-a)^\beta y(t)dt}{(x-t)^{1-\alpha}}. \qquad (2.184)$$

We again apply the method of successive approximations to solve this integral equation. We use the notation $y_0(x)$ *and set*

$$y_m(x) = y_0(x) + \frac{\lambda}{\Gamma(\alpha)} \int_a^x \frac{(t-a)^\beta y_{m-1}(t)dt}{(x-t)^{1-\alpha}} \qquad (m \in \mathbb{N}). \qquad (2.185)$$

for $y_m(x)$:

$$y_m(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!}(x-a)^j \left[1 + \sum_{k=1}^{m} d_k(\lambda(x-a)^{\alpha+\beta})^k\right], \tag{2.186}$$

where

$$d_k = \prod_{r=1}^{k} \frac{\Gamma(r\alpha + r\beta + \beta + j - \alpha + 1)}{\Gamma(r\alpha + r\beta\alpha + \beta + 1)} \qquad (k \in \mathbb{N}) \tag{2.187}$$

Taking the limit as $m \to \infty$,

$$y(x) = \sum_{j=0}^{n-1} \frac{b_j}{j!}(x-a)^j E_{\alpha,1+\beta/\alpha,(\beta+j)/\alpha}[\lambda(x-a)^{\alpha+\beta}]. \tag{2.188}$$

If $\beta \leqq 0$, then $f[x,y] = \lambda(x-a)^\beta y$ satisfies the Lipschitz condition.

Theorem 31. [3] *Let $n-1 < \alpha < n$ $(n \in \mathbb{N})$ and let $0 \leqq \gamma < 1$ be such that $\gamma \leqq \alpha$. Also let $\lambda \in \mathbb{R}$ and $\beta \leqq 0$. If $f \in C_\gamma[a,b]$, then the Cauchy problem has a unique solution $y(x) \in \boldsymbol{C}_\gamma^{\alpha,n-1}[a,b]$.*

Example 32. *[3] The solution to the Cauchy problem*

$$({}^C D_{a+}^{\alpha} y)(x) - \lambda(x-a)^\beta y(x) = 0, y(a) = b \qquad (b \in \mathbb{R}) \tag{2.189}$$

with $0 < \alpha < 1$, $\beta > -\alpha$ and $\lambda \in \mathbb{R}$ has the form

$$y(x) = bE_{\alpha,1+\beta/\alpha,\beta/\alpha}[\lambda(x-a)^{\alpha+\beta}].$$

In particular, the solution to the Cauchy problem

$$({}^C D_{a+}^{1/2} y)(x) - \lambda(x-a)^\beta y(x) = 0, \qquad y(a) = b \qquad (b \in \mathbb{R}) \tag{2.190}$$

with $\beta > -1/2$ *is given by*

$$y(x) = bE_{1/2,2\beta+1,2\beta}[\lambda(x-a)^{\beta+1/2}]. \tag{2.191}$$

Example 33. [3] *The solution to the Cauchy problem*

$$({}^C D^{\alpha}_{a+}y)(x) - \lambda(x-a)^{\beta}y(x) = 0, y(a) = b, y^{'}(a) = d \qquad (b, d \in \mathbb{R}) \tag{2.192}$$

with $1 < \alpha < 2$, $\beta > -\alpha$ *and* $\lambda \in \mathbb{R}$ *has the form:*

$$y(x) = bE_{\alpha,1+\beta/\alpha,\beta/\alpha}[\lambda(x-a)^{\alpha+\beta}] + d(x-a)E_{\alpha,1+\beta/\alpha,(\beta+1)/\alpha}[\lambda(x-a)^{\alpha+\beta}] \tag{2.193}$$

In above if we let $\beta = 0$, then we obtain $E_{\alpha,1,j/\alpha}(z) = \Gamma(j+1)E_{\alpha,j+1}(z)$.

CHAPTER 3

FRACTIONAL DIFFERENCE EQUATIONS AND DISCRETE MITTAG–LEFFLER FUNCTIONS

This chapter is organized as follows: The first section is devoted to the discretization of fractional derivatives, integrals and fractional initial value problems. These discrete counterparts have been recently discretized by many authors [11, 12, 19]. The discrete versions of left and right Caputo fractional differences defined very recently in [19] are discussed in section 2. Morover, a nonhomogenous fractional linear initial value problem is solved, where the solution is expressed by a new introduced discrete type of Mittag–Leffler functions. Section 3 is devoted to study the by parts formulas for fractional sums and differences [14]. Finally, section 4 contains an application to fractional differences in variational calculus.

3.1 Fractional Sums and Differences

For a natural number n, the fractional polynomial is defined by

$$t^{(n)} = \prod_{j=0}^{n-1}(t-j) = \frac{\Gamma(t+1)}{\Gamma(t+1-n)} \tag{3.1}$$

where Γ denotes the special gamma function and the product is zero when $t+1-j=0$ for some j. More generally, for arbitrary α, define

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)} \tag{3.2}$$

where the convention that division at pole yields zero holds. Given that the forward and backward difference operators are defined by

$$\Delta f(t) = f(t+1) - f(t), \ \nabla f(t) = f(t) - f(t-1) \tag{3.3}$$

respectively, we define iteratively the operators $\Delta^m = \Delta(\Delta^{m-1})$ and $\nabla^m = \nabla(\nabla^{m-1})$, where m is a natural number.

Here are some of the properties of the above fractional function

Lemma 34. ([11]) *Assume the following fractorial functions are well defined.*

- $\Delta t^{(\alpha)} = \alpha t^{(\alpha-1)}$
- $(t-\mu)t^{(\mu)} = t^{(\mu+1)}$, *where* $\mu \in \mathbb{R}$.
- $\mu^{(\mu)} = \Gamma(\mu+1)$.
- *If* $t \leq r$, *then* $t^{(\alpha)} \leq r^{(\alpha)}$ *for any* $\alpha > r$
- *If* $0 < \alpha < 1$, *then* $t^{(\alpha\nu)} \geq (t^{(\nu)})^{\alpha}$.
- $t^{(\alpha+\beta)} = (t-\beta)^{(\alpha)} t^{(\beta)}$

Also, for our purposes we list down the following two properties, which can be easily proved

$$\nabla_s(s-t)^{(\alpha-1)} = (\alpha-1)(\rho(s)-t)^{(\alpha-2)} \tag{3.4}$$

$$\nabla_t(\rho(s)-t)^{(\alpha-1)} = -(\alpha-1)(\rho(s)-t)^{(\alpha-2)} \tag{3.5}$$

For two real numbers a and b, we write $N_a = \{a, a+1, a+2, ...\}$ and $_bN = \{b, b-1, b-2, ...\}$.

If $\alpha > 0$ and $\sigma(s) = s+1$. Then, the $\alpha - th$ fractional sum of f is defined (as done in [15] and used in [11], [12] and [14]) by

$$\Delta^{-\alpha} f(t) \triangleq \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} f(s) \tag{3.6}$$

Note that $\Delta^{-\alpha}$ maps functions defined on N_a to functions defined on $N_{a+\alpha}$. Also to be noted that

- $u(t) = \Delta^{-n} f(t)$, $n \in \mathbb{N}$, satisfies the initial value problem

$$\Delta^n u(t) = f(t), \quad t \in N_a, \; u(a+j-1) = 0, \; j = 1, 2, ..., n \tag{3.7}$$

- the Cauchy function $\frac{(t-\sigma(s))^{(n-1)}}{(n-1)!}$ vanishes at $s = t-(n-1), ..., t-1$.

  If $\alpha > 0$ and $\rho(s) = s-1$. Then, we define the $\alpha - th$ (right) fractional sum of f by

$$\nabla^{-\alpha} f(t) \triangleq \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (\rho(s)-t)^{(\alpha-1)} f(s) \tag{3.8}$$

Note that $\nabla^{-\alpha}$ maps functions defined on ${}_bN$ to functions defined on ${}_{b-\alpha}\mathbb{N}$. Also to be noted that

- $u(t) = \nabla^{-n} f(t)$, $n \in \mathbb{N}$, satisfies the initial value problem

$$\nabla^n u(t) = (-1)^n f(t), \quad t \in \ {}_bN, \ u(b - j + 1) = 0, \ j = 1, 2, ..., n \tag{3.9}$$

- the Cauchy function $\frac{(\rho(s)-t)^{(n-1)}}{(n-1)!}$ vanishes at $s = t+1, t+2, ..., t+(n-1)$.

As used to be done in usual fractional calculus, the Riemann left and the right fractional differences are to be, respectively, defined by

$$\Delta^\alpha f(t) \triangleq \Delta^n \Delta^{-(n-\alpha)} f(s) \ \ and \ \nabla^\alpha f(t) \triangleq (-1)^n \nabla^n \nabla^{-(n-\alpha)} f(s) \tag{3.10}$$

where $n = [\alpha] + 1$. It is clear that, the fractional left difference operator Δ^α maps functions defined on N_a to functions defined on $N_{a+n-\alpha}$, while the fractional right difference operator ∇^α maps functions defined on ${}_bN$ to functions defined on ${}_{b-(n-\alpha)}N$.

Throughout this article, for simplicity we write Δ^α and ∇^α in place of Δ_a^α and ∇_b^α, respectively, where $\alpha \in \mathbb{R}$. Otherwise, we point to the end points up to which we take the fractional sum or difference. However, one has to note that if $\alpha = n \in \mathbb{N}$, then

$$\Delta_a^n f(t) = \Delta^n f(t) \ \texttt{and} \ \nabla_b^n f(t) = (-1)^n \nabla^n f(t). \tag{3.11}$$

The $\nu - th$ left fractional sum behaves well in composition. In fact, Theorem 2.2 in [11] states

Lemma 35. *Let f be a real–valued function, and let $\mu, \nu > 0$. Then, for all t such $t = a + \mu + \nu$, $mod(1)$,we have*

$$\Delta^{-\nu}[\Delta^{-\mu} f(t)] = \Delta^{-(\mu+\nu)} f(t) = \Delta^{-\mu}[\Delta^{\nu} f(t)]. \tag{3.12}$$

Theorem 2.1 and Remark 2.2 in [12] are to be summarized in the following lemma:

Lemma 36. *For any $\nu \in \mathbb{R}$, we have*

$$\Delta^{-\nu} \Delta f(t) = \Delta \Delta^{-\nu} f(t) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} f(a), \tag{3.13}$$

where f is defined on $\mathbb{N}_a i$.

Following inductively, Theorem 2.2 in [12] states

Lemma 37. *For any $\alpha \in \mathbb{R}$ and any positive integer p, the following equality holds:*

$$\Delta^{-\alpha} \Delta^{p} f(t) = \Delta^{p} \Delta^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{(\alpha-p+k)}}{\Gamma(\alpha+k-p+1)} \Delta^{k} f(a) \tag{3.14}$$

where f is defined on $\mathbb{N}_a$

In parallel to Lemma 37 above, the authors in [14] proved:

Lemma 38. [14] *For any $\alpha > 0$, the following equality holds:*

$$\nabla^{-\alpha} \nabla_b f(t) = \nabla_b \nabla^{-\alpha} f(t) - \frac{(b-t)^{(\alpha-1)}}{\Gamma(\alpha)} f(b). \tag{3.15}$$

where f is defined on ${}_bN$.

Remark 39. [14] *Let $\alpha > 0$ and $n = [\alpha] + 1$. Then, by the help of Lemma 38 we can have*

$$\nabla_b \nabla^{\alpha} f(t) = \nabla_b \nabla_b^n (\nabla^{-(n-\alpha)} f(t)) = \nabla_b^n (\nabla_b \nabla^{-(n-\alpha)} f(t)) \tag{3.16}$$

or

$$\nabla_b \nabla^{\alpha} f(t) = \nabla_b^n [\nabla^{-(n-\alpha)} \nabla_b f(t) + \frac{(b-t)^{(n-\alpha-1)}}{\Gamma(n-\alpha)} f(b)] \tag{3.17}$$

Then, using the identity

$$\nabla_b^n \frac{(b-t)^{(n-\alpha-1)}}{\Gamma(n-\alpha)} = \frac{(b-t)^{(-\alpha-1)}}{\Gamma(-\alpha)} \tag{3.18}$$

we infer that (3.15) is valid for any real α.

Using Lemma 38 and Remark 39 the authors in [14] arrived at the following

Theorem 40. [14] *For any real number α and any positive integer p, the following equality holds:*

$$\nabla^{-\alpha} \nabla_b^p f(t) = \nabla_b^p \nabla^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(b-t)^{(\alpha-p+k)}}{\Gamma(\alpha+k-p+1)} \nabla_b^k f(b) \tag{3.19}$$

where f is defined on ${}_bN$ and we remind that $\nabla_b^k f(t) = (-1)^k \nabla^k f(t)$.

In order to prove the commutative property for the right fractional sums, the following power rule obtained in [14] is needed. Here we shall give its proof in detail as the proof in [14] was sketched shortly.

Lemma 41. *Let $\alpha > 0,\ \mu > 0$. Then,*

$$\nabla_{b-\mu}^{-\alpha}(b-t)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(b-t)^{(\mu+\alpha)} \tag{3.20}$$

Proof. The proof can be achieved by checking that both sides of the identity (3.20) verify the difference equation

$$((b-(\mu+\alpha))-t+1)\nabla_b g(t) = (\mu+\alpha)g(t),\quad g(b-(\mu+\alpha)) = \Gamma(\mu+1) \tag{3.21}$$

Let $l(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(b-t)^{(\mu+\alpha)}$ and $h(t) = \nabla_{b-\mu}^{-\alpha}(b-t)^{(\mu)}$.

First, consider $l(t)$ and apply (iii) of Lemma 34 to see that

$$l(b-(\mu+\alpha)) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(\mu+\alpha)^{(\mu+\alpha)} = \Gamma(\mu+1). \tag{3.22}$$

Also apply (3.5) and a similar identity to (ii) of Lemma 34 to see that

$$(b-(\mu+\alpha)-t+1)\nabla_b l(t) = \tag{3.23}$$

$$= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(b-(\mu+\alpha)-t+1)(\mu+\alpha)(b-t)^{(\mu+\alpha-1)} = (\mu+\alpha)l(t). \tag{3.24}$$

Second, consider $h(t)$. Noting that

$$h(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=t+\alpha}^{b-\mu}(\rho(s)-t)^{(\alpha-1)}(b-s)^{(\mu)}, \tag{3.25}$$

we see that

$$h(b-(\mu+\alpha)) = \tag{3.26}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{s=b-\mu}^{b-\mu}(\rho(s)-b+\mu+\alpha)^{(\alpha-1)}(b-s)^{(\mu)} = \frac{1}{\Gamma(\alpha)}(\alpha-1)^{(\alpha-1)}(\mu)^{(\mu)} = \Gamma(\mu+1). \tag{3.27}$$

Finally, we show that $h(t)$ satisfies the desired difference equation. Using (ii) of Lemma 34, adding and subtracting μ and adding and subtracting b, we see that

$$h(t) = \frac{(b-(\mu+\alpha)-t+1)}{\Gamma(\alpha)}\sum_{s=t+\alpha}^{b-\mu}(\rho(s)-t)^{(\alpha-2)}(b-s)^{(\mu)} \tag{3.28}$$

$$-\frac{1}{\Gamma(\alpha)}\sum_{s=t+\alpha}^{b-\mu}(b-s-\mu)(\rho(s)-t)^{(\alpha-2)}(b-s)^{(\mu)}. \tag{3.29}$$

The rest of verification is direct and similar to that in Lemma 2.3 in [11]. □

Actually, formula (3.20) is the analogous of

$$\Delta_{a+\mu}^{-\alpha}(t-a)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{(\mu+\alpha)} \tag{3.30}$$

Theorem 42. [14] *Let $\alpha > 0,\ \mu > 0$. Then, for all t such that $t \equiv b-(\mu+\alpha)$ (mod 1), we have*

$$\nabla^{-\nu}[\nabla^{-\mu}f(t)] = \nabla^{-(\mu+\nu)}f(t) = \nabla^{-\mu}[\nabla^{-\nu}f(t)] \tag{3.31}$$

where f is defined on ${}_bN$. As a consequence of Lemma 37, Lemma 35, (3.7) and that $\Delta^{-(n-\alpha)}f(a+n-\alpha-1) = 0$, the following result can be directly stated:

Proposition 43. [14] *For $\alpha > 0$, and f defined in a suitable domain N_a, we have for $t \in N_{a+n} \subset N_a$*

$$\Delta^{\alpha}_{a+\alpha}\Delta^{-\alpha}f(t) = f(t), \tag{3.32}$$

and

$$\Delta^{-\alpha}_{a+n-\alpha-1}\Delta^{\alpha}f(t) = f(t), \quad \textit{when } \alpha \notin \mathbb{N}, \tag{3.33}$$

$$\Delta^{-\alpha}\Delta^{\alpha}f(t) = f(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{(k)}}{k!}\Delta^{k}f(a), \textit{when } \alpha = n \in \mathbb{N}. \tag{3.34}$$

Similarly, by the help of Lemma 38, Theorem 42, (3.9) and that $\nabla^{-(n-\alpha)}f(b-(n-\alpha)+1) = 0$, for the right sums and differences the following is obtained

Proposition 44. [14] *For $\alpha > 0$, and f defined in a suitable domain N_a, we have for $t \in {}_{b-n}N \subset {}_{b}N$*

$$\nabla^{\alpha}_{b-\alpha}\nabla^{-\alpha}f(t) = f(t), \tag{3.35}$$

and

$$\nabla^{-\alpha}_{b-(n-\alpha)+1}\nabla^{\alpha}f(t) = f(t), \quad \textit{when } \alpha \notin \mathbb{N}, \tag{3.36}$$

$$\nabla^{-\alpha}\nabla^{\alpha}_{b}f(t) = f(t) - \sum_{k=0}^{n-1}\frac{(b-t)^{(k)}}{k!}\nabla^{k}_{b}f(b), \textit{when } \alpha = n \in \mathbb{N} \tag{3.37}$$

3.2 Caputo Fractional Differences

In analogous to the usual fractional calculus we can define

Definition 45. *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then, the $\alpha-$order Caputo left fractional and right fractional differences of a function f defined on N_a and ${}_bN$, respectively, are defined by*

$$\Delta_C^\alpha f(t) \triangleq \Delta^{-(n-\alpha)}\Delta^n f(t) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} (t-\sigma(s))^{(n-\alpha-1)} \Delta_s^n f(s) \tag{3.38}$$

and

$$\nabla_C^\alpha f(t) \triangleq \nabla^{-(n-\alpha)}\nabla_b^n f(t) = \frac{1}{\Gamma(n-\alpha)} \sum_{s=t+(n-\alpha)}^{b} (\rho(s)-t)^{(n-\alpha-1)} \nabla_b^n f(s) \tag{3.39}$$

where $n = [\alpha] + 1$.

If $\alpha = n \in \mathbb{N}$, then

$$\Delta_C^\alpha f(t) \triangleq \Delta^n f(t) \text{ and } \nabla_C^\alpha f(t) \triangleq \nabla_b^n f(t). \tag{3.40}$$

Also, it is clear that Δ_C^α maps functions defined on N_a to functions defined on $N_{a+(n-\alpha)}$, and that ∇_C^α maps functions defined on ${}_bN$ to functions defined on ${}_{b-(n-\alpha)}N$

If, in Lemma 37 and Theorem 40 we replace α by $n-\alpha$ and p by n, where $n = [\alpha]+1$. Then, we can relate the Riemann and Caputo fractional left and right differences. Namely, we state

Theorem 46. [19] *For any $\alpha > 0$, we have*

$$\Delta_C^\alpha f(t) = \Delta^\alpha f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{(k-\alpha)}}{\Gamma(k-\alpha+1)} \Delta^k f(a) \tag{3.41}$$

and

$$\nabla_C^\alpha f(t) = \nabla^\alpha f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{(k-\alpha)}}{\Gamma(k-\alpha+1)} \nabla_b^k f(b). \tag{3.42}$$

In particular, when $0 < \alpha < 1$, we have

$$\Delta_C^\alpha f(t) = \Delta^\alpha f(t) - \frac{(t-a)^{(-\alpha)}}{\Gamma(1-\alpha)} f(a) \tag{3.43}$$

$$\nabla_C^\alpha f(t) = \nabla^\alpha f(t) - \frac{(b-t)^{(-\alpha)}}{\Gamma(1-\alpha)} f(b) \tag{3.44}$$

One can note that the Riemann and Caputo fractional differences, for $0 < \alpha < 1$, coincide when f vanishes at the end points.

The following identity is useful to transform Caputo fractional difference equations into fractional summations.

Proposition 47. [19] *Assume $\alpha > 0$ and f is defined on suitable domains $\mathbb{N}_a$ and ${}_b\mathbb{N}$. Then*

$$\Delta_{a+(n-\alpha)}^{-\alpha} \Delta_C^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{(k)}}{k!} \Delta^k f(a) \tag{3.45}$$

and

$$\nabla_{b-(n-\alpha)}^{-\alpha} \nabla_C^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{(k)}}{k!} \nabla_b^k f(b). \tag{3.46}$$

In particular, if $0 < \alpha \leq 1$ *then*

$$\Delta_{a+(n-\alpha)}^{-\alpha} \Delta_C^{\alpha} f(t) = f(t) - f(a) \text{ and } \nabla_{b-(n-\alpha)}^{-\alpha} \nabla_C^{\alpha} f(t) = f(t) - f(b) \tag{3.47}$$

Proof. The proof of (3.45) is followed by applying the definition and then using Lemma 35 and Proposition 43 (3.34). While the proof of (3.46) is followed by applying the definition and then using Theorem 42 and Proposition 44 (3.37) □

Using the identity (3.17) and the identity

$$\Delta^n \frac{(t-a)^{(n-\alpha-1)}}{\Gamma(n-\alpha)} = \frac{(t-a)^{(-\alpha-1)}}{\Gamma(-\alpha)} \tag{3.48}$$

we can compute the Riemann and Caputo differences of certain functions. For example, for $\beta > n = [\alpha] + 1$, we have

$$\Delta^{\alpha}(t-a)^{(\beta-1)} = \Delta_C^{\alpha}(t-a)^{(\beta-1)} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{(\beta-\alpha-1)} \tag{3.49}$$

and

$$\nabla^{\alpha}(b-t)^{(\beta-1)} = \nabla_C^{\alpha}(b-t)^{(\beta-1)} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{(\beta-\alpha-1)} \tag{3.50}$$

However,

$$\Delta_C^{\alpha} 1 = \nabla_C^{\alpha} 1 = 0 \tag{3.51}$$

while

$$\Delta^{\alpha}1 = \frac{(t-a)^{(-\alpha)}}{\Gamma(1-\alpha)}, \quad \nabla^{\alpha}1 = \frac{(b-t)^{(-\alpha)}}{\Gamma(1-\alpha)} \tag{3.52}$$

Remark 48. *Note that the results obtained in Theorem 46 and Proposition 47 agree with those in the usual continuous case (See [3] pages 91,96).*

Example 49. *[19] Let $0 < \alpha \leq 1$, $a = \alpha - 1$ and consider the left Caputo fractional difference equation*

$$\Delta_C^{\alpha} y(t) = \lambda y(t+\alpha-1), \quad y(a) = a_0, \ t \in N_0. \tag{3.53}$$

Note that the solution $y(t)$, if exists, is defined on N_a and hence $\Delta_C^{\alpha} y(t)$ becomes defined on $N_{a+(1-\alpha)} = N_0$. Thus, if we apply $\Delta_0^{-\alpha}$ on the equation (4.55) then we see that

$$y(t) = a_0 + \frac{\lambda}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} y(s+\alpha-1). \tag{3.54}$$

To obtain an explicit clear solution, we apply the method of successive approximation. Set $y_0(t) = a_0$ and

$$y_m(t) = a_0 + \lambda \Delta_0^{-\alpha} y_{m-1}(t+\alpha-1), m = 1, 2, 3, ... \tag{3.55}$$

For $m = 1$, we have by the power formula (3.30)

$$y_1(t) = a_0[1 + \frac{\lambda t^{(\alpha)}}{\Gamma(\alpha+1)}]. \tag{3.56}$$

For $m = 2$, we also see that

$$y_2(t) = a_0 + \lambda\Delta_0^{-\alpha}[a_0 + \frac{(t+\alpha-1)^{(\alpha)}}{\Gamma(\alpha+1)}] = a_0[1 + \frac{\lambda t^{(\alpha)}}{\Gamma(\alpha+1)} + \frac{\lambda^2(t+\alpha-1)^{(2\alpha)}}{\Gamma(2\alpha+1)}]$$

If we proceed inductively and let $m \to \infty$ we obtain the solution

$$y(t) = a_0[1 + \sum_{k=1}^{\infty} \frac{\lambda^k (t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}]. \tag{3.57}$$

If we set $\alpha = 1$ we come to the conclusion that $(\lambda+1)^t = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t^{(k)}$ on the time scale $\mathbb{Z}$.

Definition 50. [19]*(Discrete Mittag-Leffler) For $\lambda \in \mathbb{R}$ and $\alpha, \beta, z \in \mathbb{C}$ with $Re(\alpha) > 0$, the discrete (like) Mittag-Leffler functions are defined by*

$$E_{(\alpha,\beta)}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z+(k-1)(\alpha-1))^{(k\alpha)}(z+k(\alpha-1))^{(\beta-1)}}{\Gamma(\alpha k+\beta)} \tag{3.58}$$

For $\beta = 1$, it is written that

$$E_{(\alpha)}(\lambda, z) \triangleq E_{(\alpha,1)}(\lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{(z+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(\alpha k+1)} \tag{3.59}$$

The following specific semigroup property is useful to proceed in our main results:

Lemma 51. *[19] For $\alpha > 0$ and $k = 1, 2, 3, ...,$ we have*

$$\Delta_0^{-\alpha}(\Delta_0^{-k\alpha} f)(t + k(\alpha-1)) = (\Delta_0^{-(k+1)\alpha} f)(t + k(\alpha-1)) \tag{3.60}$$

Proof. We prove for $k = 1$ and the rest follows inductively. Hence we show that

$$\Delta_0^{-\alpha}(\Delta_0^{-\alpha} f)(t+\alpha-1) = (\Delta_0^{-2\alpha} f)(t+\alpha-1). \tag{3.61}$$

To this end,

$$\Delta_0^{-\alpha}(\Delta_0^{-\alpha}f)(t+\alpha-1) = \frac{1}{\Gamma(\alpha)^2}\sum_{r=0}^{t-\alpha}\sum_{s=0}^{r-1}(t-\sigma(r))^{(\alpha-1)}(r+\alpha-1-\sigma(s))^{(\alpha-1)}$$

If we change the order of summation we reach at

$$\Delta_0^{-\alpha}(\Delta_0^{-\alpha}f)(t+\alpha-1) = \tag{3.62}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha-1}f(s)\frac{1}{\Gamma(\alpha)}\sum_{r=s+1}^{t-\alpha}(t-\sigma(r))^{(\alpha-1)}(r+\alpha-1-\sigma(s))^{(\alpha-1)}. \tag{3.63}$$

That is

$$\Delta_0^{-\alpha}(\Delta_0^{-\alpha}f)(t+\alpha-1) = \tag{3.64}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha-1}f(s)\Delta_{s+1}^{-\alpha}(t-(s+2-\alpha))^{(\alpha-1)}. \tag{3.65}$$

Finally, by (3.30) we obtain

$$\Delta_0^{-\alpha}(\Delta_0^{-\alpha}f)(t+\alpha-1) = \frac{1}{\Gamma(2\alpha)}\sum_{s=0}^{t-\alpha-1}(t+\alpha-1-\sigma(s))^{(2\alpha-1)}f(s). \tag{3.66}$$

Which is exactly equal to $(\Delta_0^{-2\alpha}f)(t+\alpha-1)$. $\square$

Example 52. [19] *Let $0 < \alpha \leq 1$, $a = \alpha - 1$ and consider the left Caputo nonhomogeneous fractional difference equation*

$$\Delta_C^{\alpha}y(t) = \lambda y(t+\alpha-1) + f(t), \quad y(a) = a_0,\ t \in N_0. \tag{3.67}$$

Note that the solution $y(t)$, if exists, is defined on N_a and hence $\Delta_C^{\alpha} y(t)$ becomes defined on $N_{a+(1-\alpha)} = N_0$. Thus, if we apply $\Delta_0^{-\alpha}$ on the equation (4.55) then by (3.47) we see that

$$y(t) = a_0 + \frac{\lambda}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} y(s+\alpha-1) + \Delta_0^{-\alpha} f(t). \tag{3.68}$$

To obtain an explicit clear solution, we apply the method of successive approximation. Set $y_0(t) = a_0$ and

$$y_m(t) = a_0 + \Delta_0^{-\alpha}[\lambda y_{m-1}(t+\alpha-1) + f(t)], m = 1, 2, 3, ... \tag{3.69}$$

For $m = 1$, we have by the power formula (3.30)

$$y_1(t) = a_0[1 + \frac{\lambda t^{(\alpha)}}{\Gamma(\alpha+1)}] + \Delta_0^{-\alpha} f(t). \tag{3.70}$$

For $m = 2$, we also see by the help of Lemma 51 that

$$y_2(t) = a_0 + \lambda \Delta_0^{-\alpha}[a_0 + \frac{(t+\alpha-1)^{(\alpha)}}{\Gamma(\alpha+1)}] + \Delta_0^{-\alpha} f(t) + \lambda \Delta_0^{-2\alpha} f(t+\alpha-1) =$$

$$a_0[1 + \frac{\lambda t^{(\alpha)}}{\Gamma(\alpha+1)} + \frac{\lambda^2 (t+\alpha-1)^{(2\alpha)}}{\Gamma(2\alpha+1)}] + \Delta_0^{-\alpha} f(t) + \lambda \Delta_0^{-2\alpha} f(t+\alpha-1).$$

If we proceed inductively, make use of Lemma 51 and let $m \to \infty$ we obtain the solution

$$y(t) = a_0[1 + \sum_{k=1}^{\infty} \frac{\lambda^k (t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}] + \sum_{k=1}^{\infty} \lambda^{k-1} (\Delta_0^{-k\alpha} f)(t+(k-1)(\alpha-1)).$$

Then,

$$y(t) = a_0 E_{(\alpha)}(\lambda, t) + \sum_{k=0}^{\infty} \lambda^k \frac{1}{\Gamma(\alpha k + \alpha)} \sum_{s=0}^{t-k-\alpha} (t + k(\alpha - 1) - \sigma(s))^{(k\alpha+\alpha-1)} f(s) \tag{3.71}$$

Since division over ball yields zero then

$$y(t) = a_0 E_{(\alpha)}(\lambda, t) + \sum_{k=0}^{\infty} \lambda^k \frac{1}{\Gamma(\alpha k + \alpha)} \sum_{s=0}^{t-\alpha} (t + k(\alpha - 1) - \sigma(s))^{(k\alpha+\alpha-1)} f(s) \tag{3.72}$$

Interchanging the order of sum in (3.72) and making use of Lemma 34 (vi), then we conclude that

$$y(t) = a_0 E_{(\alpha)}(\lambda, t) + \tag{3.73}$$

$$\sum_{s=0}^{t-\alpha} \sum_{k=0}^{\infty} \lambda^k \frac{(t - \sigma(s) + (k-1)(\alpha - 1))^{(k\alpha)} (t - \sigma(s) + k(\alpha - 1))^{(\alpha-1)}}{\Gamma(\alpha k + \alpha)} f(s) \tag{3.74}$$

That is

$$y(t) = a_0 E_{(\alpha)}(\lambda, t) + \sum_{s=0}^{t-\alpha} E_{(\alpha,\alpha)}(\lambda, t - \sigma(s)) f(s) \tag{3.75}$$

3.3 Fractional Differences and Integration by Parts

After the left and right fractional sums are defined above, it becomes ready to obtain a by–part formula for them. Namely

Proposition 53. [14] *Let $\alpha > 0$, $a, b \in \mathbb{R}$ such that $a < b$ and $b \equiv a + \alpha$ (mod 1). If*

f is defined on N_a and g is defined on ${}_bN$, then we have

$$\sum_{s=a+\alpha}^{b} (\Delta^{-\alpha} f)(s)g(s) = \sum_{s=a}^{b-\alpha} f(s)\nabla^{-\alpha} g(s) \tag{3.76}$$

Proof. By direct substitution we have

$$\sum_{s=a+\alpha}^{b} (\Delta^{-\alpha} f)(s)g(s) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+\alpha}^{b} (\sum_{t=a}^{s-\alpha} (s-\sigma(t))^{(\alpha-1)} f(t))\ g(s) \tag{3.77}$$

Then, the result follows by interchanging the order of summation in (3.77) and noting that $(s - \sigma(t)) = (\rho(s) - t)$. □

Obeying the above by–parts formula for fractional sums, we can obtain a by–parts formula also for fractional differences.

Proposition 54. [14] *Let $\alpha > 0$ be non–integer and assume that $b \equiv a + (n - \alpha)\ (mod\ 1)$. If f is defined on ${}_bN$ and g is defined on N_a, then*

$$\sum_{s=a+(n-\alpha)-1}^{b-n+1} f(s)\Delta^{\alpha} g(s) = \sum_{s=a+n-1}^{b-(n-\alpha)+1} g(s)\nabla^{\alpha} f(s) \tag{3.78}$$

Proof. By the help of equation (3.36) of Proposition 44 we can write

$$\sum_{s=a+(n-\alpha)-1}^{b-n+1} f(s)\Delta^{\alpha} g(s) = \sum_{s=a+(n-\alpha)-1}^{b-n+1} \nabla^{-\alpha}_{b-(n-\alpha)+1}(\nabla^{\alpha}_{b} f(s))\Delta^{\alpha} g(s) \tag{3.79}$$

and by Proposition 53 we have

$$\sum_{s=a+(n-\alpha)-1}^{b-n+1} f(s)\Delta^{\alpha} g(s) = \sum_{s=a+n-1}^{b-(n-\alpha)+1} \nabla^{\alpha} f(s)\Delta^{-\alpha}_{a+(n-\alpha)-1}\Delta^{\alpha} g(s) \tag{3.80}$$

But then the result follows by equation (3.33) of Proposition 43. $\square$

When α is an integer it is possible to obtain, by the help of equation (3.37) of Proposition 44 and equation (3.34) of Proposition 43, the usual by–parts formula in difference calculus. For example, if $\alpha = 1$ we obtain

$$\sum_{s=a}^{b-1} f(s)\Delta g(s) = f(s)g(s)|_a^b - \sum_{s=a+1}^{b} g(s)\nabla f(s) = f(s)g(s)|_a^b - \sum_{s=a}^{b-1} g(s+1)\Delta f(s) \quad (3.81)$$

3.4 Application

In this section, we give an application to integration by parts for fractional differences as it was presented in [14]. Actually, we show how it is used to obtain Euler–Lagrange equations for a discrete variational problem in fractional calculus.

Consider the functional $J : S \to \mathbb{R}$,

$$J(y) = \sum_{s=a-\alpha}^{b} L(s, y(s), \Delta^\alpha y(s)) \quad (3.82)$$

where

$$a, b \in \mathbb{R},\ 0 < \alpha < 1 \quad (3.83)$$

$$L : (N_{a-\alpha} \cap\ {}_bN) \times (\mathbb{R}^n)^2 \to \mathbb{R},\ b \equiv a - \alpha\ (mod\ 1) \quad (3.84)$$

and

$$S = \{y : N_{a-\alpha} \cap\ {}_{b+\alpha}N \to \mathbb{R}^n : y(a) = y_0 \texttt{ and } y(b+\alpha) = y_1\} \quad (3.85)$$

Moreover, we assume that the function y fits the discrete time scales N_a and $N_{a-\alpha}$. That is, $y(s) = y(s+1-\alpha)$ for all $s \in N_a$.

We shortly write :

$$L(s) \equiv L(s, y(s), \Delta^\alpha y(s)) \tag{3.86}$$

We calculate the first variation of the functional J on the linear manifold S: Let $\eta \in H = \{h :\to \mathbb{R}^n : h(a) = h(b+\alpha) = 0\}$, then

$$\delta J(y(x), \eta(x)) = \frac{d}{d\epsilon} J(y(x) + \epsilon\eta(x))|_{\epsilon=0} \tag{3.87}$$

$$\sum_{s=a-\alpha}^{b} \left[\frac{\partial L(s)}{\partial y} \eta(s) + \frac{\partial L(s)}{\partial \Delta^\alpha y} \Delta^\alpha \eta(s) \right]. \tag{3.88}$$

Then we apply Proposition with $0 < \alpha < 1$ and $n = 1$ to get

$$\delta J(y(x), \eta(x)) = \sum_{s=a}^{b+\alpha} \left[\frac{\partial L(s)}{\partial y} + \nabla^\alpha \frac{\partial L(s)}{\partial \Delta^\alpha y} \right] \eta(s) = 0, \tag{3.89}$$

and by applying a suitable discrete fundamental lemma in calculus of variations we obtain the Euler–Lagrange equation:

$$\frac{\partial L(s)}{\partial y} + \nabla^\alpha \frac{\partial L(s)}{\partial \Delta^\alpha y} = 0. \tag{3.90}$$

CHAPTER 4

FRACTONAL Q–DIFFERENCE EQUATIONS AND Q–MITTAG–LEFFLER FUNCTION

This chapter mainly review the q–fractional calculus results obtained recently in [18], where the q–analogue for Mittag–Leffler functions was obtained by solving certain q–Caputo fractional difference equations by the method of successive approximation.

4.1 Fractional q–Integrals and q–Derivatives

For the theory of q-calculus we refer the reader to the survey [4] and for the basic definitions and results for the q-fractional calculus we refer to [13]. Here we shall summarize some of those basics.

For $0 < q < 1$, let T_q be the time scale

$$T_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}. \tag{4.1}$$

where Z is the set of integers. More generally, if α is a nonnegative real number then we define the time scale

$$T_q^{\alpha} = \{q^{n+\alpha} : n \in Z\} \cup \{0\}, \tag{4.2}$$

we write $T_q^0 = T_q$.

For a function $f : T_q \to \mathbb{R}$, the nabla q–derivative of f is given by

$$\nabla_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in T_q - \{0\} \tag{4.3}$$

The nabla q–integral of f is given by

$$\int_0^t f(s)\nabla_q s = (1-q)t\sum_{i=0}^{\infty} q^i f(tq^i) \tag{4.4}$$

and for $0 \leq a \in T_q$

$$\int_a^t f(s)\nabla_q s = \int_0^t f(s)\nabla_q s - \int_0^a f(s)\nabla_q s \tag{4.5}$$

On the other hand

$$\int_t^{\infty} f(s)\nabla_q s = (1-q)t\sum_{i=1}^{\infty} q^{-i} f(tq^{-i}) \tag{4.6}$$

and for $0 < b < \infty$ in T_q

$$\int_t^b f(s)\nabla_q s = \int_t^{\infty} f(s)\nabla_q s - \int_b^{\infty} f(s)\nabla_q s \tag{4.7}$$

By the fundamental theorem in q–calculus we have

$$\nabla_q \int_0^t f(s)\nabla_q s = f(t) \tag{4.8}$$

and if f is continuous at 0, then

$$\int_0^t \nabla_q f(s)\nabla_q s = f(t) - f(0) \tag{4.9}$$

Also the following identity will be helpful

$$\nabla_q \int_a^t f(t,s)\nabla_q s = \int_a^t \nabla_q f(t,s)\nabla_q s + f(qt,t) \tag{4.10}$$

Similarly the following identity will be useful as well

$$\nabla_q \int_t^b f(t,s)\nabla_q s = \int_{qt}^b \nabla_q f(t,s)\nabla_q s - f(t,t) \tag{4.11}$$

The q-derivative in (4.10) and (4.11) is applied with respect to t.

From the theory of q–calculus and the theory of time scale more generally, the following product rule is valid

$$\nabla_q (f(t)g(t)) = f(qt)\nabla_q g(t) + \nabla_q f(t) g(t). \tag{4.12}$$

The q-factorial function for $n \in \mathbb{N}$ is defined by

$$(t-s)_q^n = \prod_{i=0}^{n-1} (t - q^i s) \tag{4.13}$$

When α is a non positive integer, the q–factorial function is defined by

$$(t-s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \frac{1 - \frac{s}{t} q^i}{1 - \frac{s}{t} q^{i+\alpha}}. \tag{4.14}$$

We summarize some of the properties of q–factorial functions, which can be found mainly in [13], in the following lemma

Lemma 55. *(i)*$(t-s)_q^{\beta+\gamma} = (t-s)_q^{\beta}(t-q^{\beta}s)_q^{\gamma}$

(ii)$(at-as)_q^{\beta} = a^{\beta}(t-s)_q^{\beta}$

(iii) The nabla q–derivative of the q–factorial function with respect to t is

$$\nabla_q(t-s)_q^{\alpha} = \frac{1-q^{\alpha}}{1-q}(t-s)_q^{\alpha-1}$$

(iv)The nabla q–derivative of the q–factorial function with respect to s is

$$\nabla_q(t-s)_q^{\alpha} = -\frac{1-q^{\alpha}}{1-q}(t-qs)_q^{\alpha-1}$$

where $\alpha, \gamma, \beta \in \mathbb{R}$.

For the q–gamma function, $\Gamma_q(\alpha)$, we refer the reader to [13] and the references there in. We just mention here the identity

$$\Gamma_q(\alpha+1) = \frac{1-q^{\alpha}}{1-q}\Gamma_q(\alpha), \quad \Gamma_q(1) = 1, \ \alpha > 0. \tag{4.15}$$

The authors in [13] following [8] defines the left fractional q–integral of order $\alpha \neq 0, -1, -2, ...$ by

$$_qI^{\alpha}f(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)_q^{\alpha-1}f(s)\nabla_q s \tag{4.16}$$

In [8] it was proved that the left q–fractional integral obeys the identity

$$
{}_qI^{\beta}\ {}_qI^{\alpha}f(t) ={}_q I^{\alpha+\beta}f(t), \qquad \alpha\ ,\beta > 0 \tag{4.17}
$$

The left q–fractional integral ${}_qI_a^{\alpha}$ starting from $0 < a \in T_q$ is to be defined by

$$
{}_qI_a^{\alpha}f(t) = \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1}f(s)\nabla_q s \tag{4.18}
$$

It is clear, from the q–analogue of Cauchy's formula [5], that

$$
\nabla_q^n\ {}_qI_a^n f(t) = f(t) \tag{4.19}
$$

where n is a positive integer and $0 \leq a \in T_q$

Recently, in Theorem 5 of [9], the authors there have proved that

$$
{}_qI_a^{\beta}\ {}_qI_a^{\alpha}f(t) ={}_q I_a^{\alpha+\beta}f(t), \qquad \alpha\ ,\beta > 0 \tag{4.20}
$$

The right q–fractional integral of order α is defined by [8]

$$
I_q^{\alpha}f(t) = \frac{q^{-(1/2)\alpha(\alpha-1)}}{\Gamma_q(\alpha)}\int_t^{\infty} (s-t)_q^{\alpha-1}f(sq^{1-\alpha})\nabla_q s \tag{4.21}
$$

and the right q–fractional integral of order α ending at b for some $b \in T_q$ is defined by

$$
{}_bI_q^{\alpha}f(t) = \frac{q^{-(1/2)\alpha(\alpha-1)}}{\Gamma_q(\alpha)}\int_t^{b} (s-t)_q^{\alpha-1}f(sq^{1-\alpha})\nabla_q s \tag{4.22}
$$

Note that, while the left q–fractional integral ${}_qI_a^\alpha$ maps functions defined T_q to functions defined on T_q, the right q–fractional integral ${}_bI_q^\alpha$, $0 < b \leq \infty$, maps functions defined on $T_q^{1-\alpha}$ to functions defined on T_q.

It is clear, from the q–analogue of Cauchy's formula [5], that

$$\nabla_q^n I_q^n f(t) = (-1)^n f(t) \tag{4.23}$$

In [7] it was proved that the right q–fractional integral obeys the identity

$$I_q^\beta \ I_q^\alpha f(t) = I_q^{\alpha+\beta} f(t), \qquad \alpha \ , \beta > 0 \tag{4.24}$$

Taking into account the domain and the range of the right q-fractional integral, as mentioned above, we note that the formula (4.24) is valid under the condition that f must be at least defined on T_q, $T_q^{1-\beta}$, $T_q^{1-\alpha}$ and $T_q^{1-(\alpha+\beta)}$.

A particular case of the identity (4.24) is

$$I_q^{n-\alpha} \ I_q^\alpha f(t) = I_q^n f(t), \qquad \alpha \ > 0. \tag{4.25}$$

Lemma 56. *For $\alpha, \beta > 0$ and a function f fitting suitable domains, we have*

$$\int_b^\infty (t-x)_q^{\beta-1} \ {}_bI_q^\alpha f(tq^{1-\beta}) \nabla_q t = 0 \tag{4.26}$$

Proof. From (4.6) we can write

$$\textstyle\int_b^\infty (t-x)_q^{\beta-1} \ {}_bI_q^\alpha f(tq^{1-\beta}) \nabla_q t = \tag{4.27}$$

$$(1-q)b\sum_{i=0}^{\infty} q^{-i}(bq^{-i}-x)_q^{\beta-1}\ {}_bI_q^{\alpha}f(q^{1-\beta}bq^{-i}) \tag{4.28}$$

From the fact that $(t-r)_q^{\beta-1}=0$, when $t<r$ we conclude that ${}_bI_q^{\alpha}f(q^{1-\beta}bq^{-i})=0$ and hence the result follows. □

Problem 1: Can we use Lemma 56 and following similar ideas to that in [9] to prove that

$${}_bI_q^{\beta}\ {}_bI_q^{\alpha}f(t)=\ {}_bI_q^{\alpha+\beta}f(t),\qquad \alpha\ ,\beta>0,\ 0<b\in T_q\ ? \tag{4.29}$$

Alternatively, can we define the q–analogue of the Q–operator and prove that

$$Q\ {}_qI_a^{\alpha}f(t)=\ {}_bI_q^{\alpha}Qf(t)? \tag{4.30}$$

Then apply the Q–operator to the identity

$${}_qI_a^{\beta}\ {}_qI_a^{\alpha}g(t)={}_q\ I_a^{\alpha+\beta}g(t),\qquad \alpha\ ,\beta>0 \tag{4.31}$$

with $g(t)=Qf(t)$ to obtain (4.29). Recall that in the continuous case $Qf(t)=f(a+b-t)$.

In connection to Problem 1, the following open problem is also raised

Problem 2: Is it possible to obtain a by–part formula for q–fractional derivatives when the lower limit a and the upper limit b both exist. That is on the interval $[a,b]_q$. As for the $(0,\infty)$ case there is a formula was early obtained by Agarwal in [8].

As for the left and right (Riemann) q–fractional derivatives of order $\alpha > 0$, as traditionally done in fractional calculus, they are defined respectively by

$$_q\nabla_a^\alpha f(t) \triangleq \nabla_q^n \ _qI_a^{n-\alpha} f(t) \text{ and } _b\nabla_q^\alpha f(t) \triangleq (-1)^n \nabla_q^n \ _bI_q^{n-\alpha} f(t) \tag{4.32}$$

where $n = [\alpha] + 1$ and $a, b \in T_q \cup \{\infty\}$ with $0 \leq a < b \leq \infty$. We usually remove the endpoints in the notation when $a = 0$ or $b = \infty$. Here, we point that the operator $_q\nabla_a^\alpha$ maps functions defined on T_q to functions defined on T_q, while the operator $_b\nabla_q^\alpha$ maps functions defined on $T_q^{1-(n-\alpha)}$ to functions defined on T_q. Also, particularly, one has to note that

$$_q\nabla_a^n f(t) = \nabla_q^n f(t) \text{ and } _b\nabla_q^n f(t) = (-1)^n \nabla_q^n f(t) \tag{4.33}$$

where ∇_q^n always denotes the $n - th$ q–derivative (i.e. the q–derivative applied n times).

4.2 Caputo q–Fractional Derivative

In this section, before defining Caputo–type q–fractional derivatives and relating them to Riemann ones, we first state and prove some essential preparatory lemmas.

Lemma 57. *For any $\alpha > 0$, the following equality holds:*

$$_qI_a^\alpha \nabla_q f(t) = \nabla_q \ _qI_a^\alpha f(t) - \frac{(t-a)_q^{\alpha-1}}{\Gamma_q(\alpha)} f(a) \tag{4.34}$$

Proof. From (4.12) and (iv) of Lemma 55, we obtain the following q–integration by parts:

$$\nabla_q((t-s)_q^{\alpha-1} f(s)) = (t-qs)_q^{\alpha-1} \nabla_q f(s) - \frac{1-q^{\alpha-1}}{1-q}(t-qs)_q^{\alpha-2} f(s) \tag{4.35}$$

Applying (4.35) leads to

$$_qI_a^\alpha \nabla_q f(t) = \frac{(t-s)_q^{\alpha-1}}{\Gamma_q(\alpha)} f(s)|_a^t + \frac{1-q^{\alpha-1}}{1-q} \int_a^t (t-qs)_q^{\alpha-2} f(s) \nabla_q s \tag{4.36}$$

or

$$_qI_a^\alpha \nabla_q f(t) = -\frac{(t-a)_q^{\alpha-1}}{\Gamma_q(\alpha)} f(a) + \frac{1-q^{\alpha-1}}{1-q} \int_a^t (t-qs)_q^{\alpha-2} f(s) \nabla_q s \tag{4.37}$$

On the other hand, and by the help of (iii) of Lemma 55, (4.10) and the identity (4.15), we find that

$$\nabla_q\, {}_qI_a^\alpha f(t) = \frac{1-q^{\alpha-1}}{1-q} \int_a^t (t-qs)_q^{\alpha-2} f(s) \nabla_q s, \tag{4.38}$$

which completes the proof. □

Theorem 58. *For any real $\alpha > 0$ and any positive integer p such that $\alpha - p + 1$ is not negative integer or 0, in particular $\alpha > p - 1$, the following equality holds:*

$$_qI_a^\alpha \nabla_q^p f(t) = \nabla_q^p\, {}_qI_a^\alpha f(t) - \sum_{k=0}^{p-1} \frac{(t-a)_q^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} \nabla_q^k f(a) \tag{4.39}$$

Proof. The proof can be achieved by following inductively on p and making use of Lemma 57, (iii) of Lemma 55 and (4.15). □

Now we obtain an analogue to Lemma 57 for the right q-integrals.

Lemma 59. *For any $\alpha > 0$, the following equality holds:*

$$ {}_{q^{-1}b}I_q^{\alpha}\ {}_b\nabla_q f(t) = \ {}_b\nabla_q\ {}_bI_q^{\alpha} f(t) - \frac{r(\alpha)}{\Gamma_q(\alpha)}(b-qt)_q^{\alpha-1} f(q^{1-\alpha}q^{-1}b) \tag{4.40}$$

where

$$ r(\alpha) = q^{(-1/2)\alpha(\alpha-1)} \tag{4.41}$$

and

$$ {}_b\nabla_q f(t) = -\nabla_q f(t)$$

.

Proof. First, by the help of (iii) of Lemma 55 and (4.12), the following q–calculus by–parts version is valid:

$$ (s-t)_q^{\alpha-1}\nabla_q f(sq^{1-\alpha})q^{1-\alpha} = \tag{4.42}$$

$$ \nabla_q((s-t)_q^{\alpha-1} f(sq^{1-\alpha})) - \frac{1-q^{\alpha-1}}{1-q}(s-t)_q^{\alpha-2} f(sq^{2-\alpha}) \tag{4.43}$$

where the q–derivative is applied with respect to s. Using (4.43) we obtain

$$ {}_{q^{-1}b}I_q^{\alpha}\ {}_b\nabla_q f(t) = \tag{4.44}$$

$$ \frac{q^{\alpha-1}r(\alpha)}{\Gamma_q(\alpha)}\left(\frac{1-q^{\alpha-1}}{1-q}\int_t^{q^{-1}b}(s-t)_q^{\alpha-2} f(q^{2-\alpha}s)\nabla_q s - (s-t)_q^{\alpha-1} f(q^{1-\alpha}s)\Big|_t^{q^{-1}b}\right) \tag{4.45}$$

$$= \frac{q^{\alpha-1}r(\alpha)}{\Gamma_q(\alpha)}(\frac{1-q^{\alpha-1}}{1-q}\int_t^{q^{-1}b}(s-t)_q^{\alpha-2}f(q^{2-\alpha}s)\nabla_q s-(q^{-1}b-t)_q^{\alpha-1}f(q^{1-\alpha}q^{-1}b)) \quad (4.46)$$

On the other hand (4.11) and (iv) of Lemma 55 imply

$$_b\nabla_q\ _bI_q^{\alpha}f(t) = \frac{q^{\alpha-1}r(\alpha)}{\Gamma_q(\alpha)}\frac{1-q^{\alpha-1}}{1-q}\int_t^{q^{-1}b}(s-t)_q^{\alpha-2}f(q^{2-\alpha}s)\nabla_q s \quad (4.47)$$

Taking into account (4.46) and (4.47), identity (4.40) will follow and the proof is complete. □

One has to note that the above formula (4.40) holds under the request that f must be at least defined on T_q and $T_q^{1-\alpha}$.

Definition 60. *Let $\alpha > 0$. If $\alpha \notin \mathbb{N}$, then the $\alpha-$order Caputo left q-fractional and right q–fractional derivatives of a function f are, respectively, defined by*

$$_qC_a^{\alpha}f(t) \triangleq\ _qI_a^{(n-\alpha)}\nabla_q^n f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-qs)_q^{n-\alpha-1}\nabla_q^n f(s)\nabla_q s \quad (4.48)$$

and

$$_bC_q^{\alpha}f(t) \triangleq\ _bI_q^{(n-\alpha)}\nabla_b^n f(t) = \frac{q^{(-1/2)\alpha(\alpha-1)}}{\Gamma_q(n-\alpha)}\int_t^b (s-t)_q^{n-\alpha-1}\ _b\nabla_q^n f(sq^{1-\alpha})\nabla_q s \quad (4.49)$$

where $n = [\alpha]+1$.

If $\alpha \in \mathbb{N}$, then $_qC_a^{\alpha}f(t) \triangleq \nabla_q^n f(t)$ and $_bC_q^{\alpha}f(t) \triangleq\ _b\nabla_q^n = (-1)^n\nabla_q^n$

Also, it is clear that $_qC_a^{\alpha}$ maps functions defined on T_q to functions defined on T_q, and that $_bC_q^{\alpha}$ maps functions defined on $T_q^{1-\alpha}$ to functions defined on T_q

If, in Lemma 57 and Lemma 59 we replace α by $1-\alpha$. Then, we can relate the left and right Riemann and Caputo q–fractional derivatives. Namely, we state

Theorem 61. *For any* $0<\alpha<1$, *we have*

$$ {}_qC_a^\alpha f(t) = {}_q\nabla_a^\alpha f(t) - \frac{(t-a)_q^{-\alpha}}{\Gamma_q(1-\alpha)} f(a) \tag{4.50}$$

and

$$ {}_{q^{-1}b}C_q^\alpha f(t) = {}_b\nabla_q^\alpha f(t) - \frac{r(1-\alpha)}{\Gamma_q(1-\alpha)} (b-qt)_q^{-\alpha} f(q^\alpha q^{-1}b) \tag{4.51}$$

4.3 A Caputo q–fractional Initial Value Problem and q–Mittag–Leffler Function

The following identity which is useful to transform Caputo q–fractional difference equations into q–fractional integrals, will be our key in this section.

Proposition 62. *Assume* $\alpha>0$ *and* f *is defined in suitable domains. Then*

$$ {}_qI_a^\alpha \, {}_qC_a^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a) \tag{4.52}$$

and if $0<\alpha\le 1$ *then*

$$ {}_qI_a^\alpha \, {}_qC_a^\alpha f(t) = f(t) - f(a). \tag{4.53}$$

The proof followed by definition of Caputo q–fractional derivatives, (4.19), Lemma 57 and Theorem 58.

The following identity [9] is essential to solve linear q–fractional equations

$$_qI_a^\alpha(x-a)_q^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)}(x-a)_q^{\mu+\alpha} \quad (0<a<x<b) \tag{4.54}$$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1,\infty)$.

Example 63. *Let $0<\alpha\leq 1$ and consider the left Caputo q–fractional difference equation*

$$_qC_a^\alpha y(t) = \lambda y(t) + f(t), \quad y(a)=a_0, \; t\in T_q. \tag{4.55}$$

if we apply $_qI_a^\alpha$ on the equation (4.55) then by the help of (4.53) we see that

$$y(t) = a_0 + \lambda \; _qI_a^\alpha y(t) + \; _qI_a^\alpha f(t). \tag{4.56}$$

To obtain an explicit clear solution, we apply the method of successive approximation. Set $y_0(t)=a_0$ and

$$y_m(t) = a_0 + \lambda \; _qI_a^\alpha y_{m-1}(t) +_q I_a^\alpha f(t), m=1,2,3,.... \tag{4.57}$$

For $m=1$, we have by the power formula (4.54)

$$y_1(t) = a_0[1+\frac{\lambda(t-a)_q^{(\alpha)}}{\Gamma_q(\alpha+1)}] + \; _qI_a^\alpha f(t). \tag{4.58}$$

For $m=2$, we also see that

$$y_2(t) = a_0 + \lambda a_0 \; _qI_a^\alpha[1+\frac{(t-a)_q^\alpha}{\Gamma_q(\alpha+1)}] + \; _qI_a^\alpha f(t) + \lambda \; _qI_a^{2\alpha}f(t) \tag{4.59}$$

$$= a_0[1 + \frac{\lambda(t-a)_q^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{\lambda^2(t-a)_q^{2\alpha}}{\Gamma_q(2\alpha+1)}] + {}_qI_a^{\alpha}f(t) + \lambda\ {}_qI_a^{2\alpha}f(t) \tag{4.60}$$

If we proceed inductively and let $m \to \infty$ *we obtain the solution*

$$y(t) = a_0[1 + \sum_{k=1}^{\infty} \frac{\lambda^k (t-a)_q^{k\alpha}}{\Gamma_q(k\alpha+1)}] + \int_a^t [\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{\Gamma_q(\alpha k)} (t-qs)_q^{\alpha k-1}] f(s) \nabla_q s \tag{4.61}$$

$$= a_0[1 + \sum_{k=1}^{\infty} \frac{\lambda^k (t-a)_q^{k\alpha}}{\Gamma_q(k\alpha+1)}] + \int_a^t [\sum_{k=0}^{\infty} \frac{\lambda^{k}}{\Gamma_q(\alpha k+\alpha)} (t-qs)_q^{\alpha k+(\alpha-1)}] f(s) \nabla_q s \tag{4.62}$$

$$= a_0[1 + \sum_{k=1}^{\infty} \frac{\lambda^k (t-a)_q^{k\alpha}}{\Gamma_q(k\alpha+1)}] + \tag{4.63}$$

$$+ \int_a^t (t-qs)_q^{(\alpha-1)} [\sum_{k=0}^{\infty} \frac{\lambda^{k}}{\Gamma_q(\alpha k+\alpha)} (t-q^{\alpha}s)_q^{(\alpha k)}] f(s) \nabla_q s \tag{4.64}$$

If we set $\alpha = 1$, $\lambda = 1$, $a = 0$ *and* $f(t) = 0$ *we come to a q-exponential formula* $e_q(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma_q(k+1)}$ *on the time scale* T_q, *where* $\Gamma_q(k+1) = [k]_q! = [1]_q[2]_q...[k]_q$ *with* $[r]_q = \frac{1-q^r}{1-q}$. *It is known that* $e_q(t) = E_q((1-q)t)$, *where* $E_q(t)$ *is a special case of the basic hypergeometric series, given by*

$$E_q(t) = {}_1\phi_0(0; q, t) = \Pi_{n=0}^{\infty}(1-q^n t)^{-1} = \sum_{n=0}^{\infty} \frac{t^n}{(q)_n}, \tag{4.65}$$

where $(q)_n = (1-q)(1-q^2)...(1-q^n)$ *is the q–Pochhammer symbol.*

If we compare with the classical case, then the above example suggests the following q–analogue of Mittag–Leffler function.

Definition 64. *For $z, z_0 \in \mathbf{C}$ and $\Re(\alpha) > 0$, the q-Mittag-Leffler function is defined by*

$$
{}_qE_{\alpha,\beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \lambda^k \frac{(z - z_0)_q^{\alpha k}}{\Gamma_q(\alpha k + \beta)}. \tag{4.66}
$$

When $\beta = 1$ we simply use ${}_qE_{\alpha}(\lambda, z - z_0) := {}_qE_{\alpha,1}(\lambda, z - z_0)$.

According to Definition 64 above, the solution of the q-Caputo-fractional equation in Example 63 is expressed by

$$
y(t) = a_0 \, {}_qE_{\alpha}(\lambda, t - a) + \int_a^t (t - qs)_q^{\alpha-1} \, {}_qE_{\alpha,\alpha}(\lambda, t - q^{\alpha}s) f(s) \nabla_q s. \tag{4.67}
$$

Remark 65. *1) Note that the above proposed definition of the q-analogue of Mittag–Leffler function agrees with time scale definition of exponential functions. As it depends on the three parameters other than α and β.*

2) The power term of the q–Mittag–Leffler function contains α (the term $(z - z_0)_q^{\alpha k}$). We include this α in order to express the solution of q–Caputo initial value problem explicitly by means of the q–Mittag–Leffler function. This is due to that in general it is not true for the q–factorial function to satisfy the power formula $(z - z_0)_q^{\alpha k} = [(z - z_0)_q^{\alpha}]^k$. But for example the latter power formula is true when $z_0 = 0$. Therefore, for the case $z_0 = 0$, we may drop α from the power so that the q–Mittag–Leffler function will tend to the classical one when $q \to 1$.

3) Once Problem 1 raised in section 2 is solved an analogue result to Proposition 62 can be obtained for right Caputo q–fractional derivatives.

CHAPTER 5

GENERALIZED Q–MITTAG–LEFFLER FUNCTION

This chapter contains the main results introduced in [34]. A new Caputo linear type q–fractional differential equation is solved. The solution is expressed by a q–analogue of the generalized Mittag–Leffler function introduced by Kilbas and Saigo [17] (see also [3] page 48 or 2.56).

Definition 66. [34] *For $\alpha, l, \lambda \in \mathbb{C}$ are complex numbers and $m \in \mathbb{R}$ such that $\Re(\alpha) > \mathfrak{o},\ \mathfrak{m} > \mathfrak{o}, \mathfrak{a} \geq \mathfrak{o}$ and $\alpha(jm+l) \neq -1, -2, -3, ...,$, the generalized q–Mittag–Leffler function (of order 0) is defined by*

$$
{}_qE_{\alpha,m,l}(\lambda, x-a) = 1 + \sum_{k=1}^{\infty} \lambda^k q^{-\frac{k(k-1)}{2}\alpha(m-1)(\alpha l+\alpha)} c_k (x-a)_q^{\alpha k m} \tag{5.1}
$$

where

$$
c_k = \prod_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]},\ k = 1, 2, 3, \cdots. \tag{5.2}
$$

While the the generalized q–Mittag–Leffler function (of order r), $r = 0, 1, 2, 3, ...,$ is defined by

$$
{}_qE^r_{\alpha,m,l}(\lambda, x-a) = 1 + \sum_{k=1}^{\infty} \lambda^k q^{-k\alpha(m-1)r} q^{-\frac{k(k-1)}{2}\alpha(m-1)(\alpha l+\alpha)} c_k (x-q^r a)_q^{\alpha k m}. \tag{5.3}
$$

Note that ${}_qE^0_{\alpha,m,l}(\lambda, x-a) = {}_qE_{\alpha,m,l}(\lambda, x-a)$.

Remark 67. *[34] In particular, if $m = 1$, then the generalized q–Mittag–Leffler function is reduced to the q–Mittag–Leffler function, apart from a constant factor $\Gamma_q(\alpha l + 1)$. Namely,*

$$ {}_qE_{\alpha,1,l}(\lambda, x-a) = \Gamma_q(\alpha l + 1)\ {}_qE_{\alpha,\alpha l+1}(\lambda, x-a). \tag{5.4} $$

This turns to be the q–analogue of the identity $E_{\alpha,1,l}(z) = \Gamma(\alpha l + 1)E_{\alpha,\alpha l+1}(z)$ (see [3]) page 48).

Example 68. [34] *Consider the q–Caputo difference equation*

$$ ({}_qC^\alpha_a y)(x) = \lambda(x-a)^\beta_q y(q^{-\beta}x),\ y(a) = b, \tag{5.5} $$

where

$$ 0 < \alpha < 1,\ \beta > -\alpha,\ \lambda \in \mathbb{R},\ \ \in \mathbb{R}. \tag{5.6} $$

Applying proposition (62) we have

$$ y(x) = y(a) + \lambda\ {}_qI^\alpha_a[(x-a)^\beta_q y(q^{-\beta}x)]. \tag{5.7} $$

The method of successive applications implies that

$$ y_m(x) = y(a) + \lambda_q I^\alpha_a[(x-a)^\beta_q y_{m-1}(q^{-\beta}x)],\quad m = 1, 2, 3, ..., \tag{5.8} $$

where $y_0(x) = b$. *Then by the help of (4.54) we have*

$$y_1(x) = b + b\lambda \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(x-a)_q^{\beta+\alpha}, \tag{5.9}$$

and

$$y_2(x) = b + b\lambda \; {}_qI_a^{\alpha}[(x-a)_q^{\beta}\{1 + \lambda \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(q^{-\beta}x-a)_q^{\beta+\alpha}\}]. \tag{5.10}$$

Then by (i) and (ii) of Lemma 55

$$y_2(x) = b + b\lambda \; {}_qI_a^{\alpha}[(x-a)_q^{\beta} + \lambda \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)} q^{-\beta(\alpha+\beta)}(x-a)_q^{2\beta+\alpha}]. \tag{5.11}$$

Again by (4.54) we conclude

$$y_2(x) = b + b\lambda \; {}_qI_a^{\alpha}[(x-a)_q^{\beta} + \lambda \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)} q^{-\beta(\alpha+\beta)}(x-a)_q^{2\beta+\alpha}]. \tag{5.12}$$

Then (4.54) leads to

$$y_2(x) = \tag{5.13}$$

$$b[1+\lambda \; \frac{\Gamma_q(\beta+1)}{\Gamma_q^{(}\beta+\alpha+1)}(x-a)_q^{\beta+\alpha} + \lambda^2 \frac{\Gamma_q(2\beta+\alpha+1)}{\Gamma_q(2\beta+2\alpha+1)} q^{-\beta(\alpha+\beta)}(x-a)_q^{2\beta+2\alpha}]. \tag{5.14}$$

Proceeding inductively, for each $m = 1, 2, ..$ *we obtain*

$$y_m(x) = b[1 + \sum_{k=1}^{m} \lambda^k q^{-\beta \frac{k(k-1)}{2}(\alpha+\beta)} c_k (x-a)_q^{k(\alpha+\beta)}], \tag{5.15}$$

where

$$c_k = \prod_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]}, \quad m = 1 + \frac{\beta}{\alpha},\ l = \frac{\beta}{\alpha},\ k = 1, 2, 3, \cdots. \tag{5.16}$$

If we let $m \to \infty$, then we obtain the solution

$$y(x) = b\ [1 + \sum_{k=1}^{\infty} \lambda^k q^{-\beta \frac{k(k-1)}{2}(\alpha+\beta)} c_k (x-a)_q^{k(\alpha+\beta)}] \tag{5.17}$$

which is exactly

$$y(x) = b\ {}_qE_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(\lambda, x-a). \tag{5.18}$$

Remark 69. [34] *1)If in (5.5) $\beta = 0$, then in accordance with (5.4) and Example 10 in [18] we have*

$${}_qE_{\alpha,1,0}(\lambda, x-a) = {}_qE_{\alpha,1}(\lambda, x-a) = {}_qE_{\alpha}(\lambda, x-a). \tag{5.19}$$

2) The solution of the q–Cauchy problem

$$({}_qC_a^{\frac{1}{2}} y)(x) = \lambda(x-a)_q^{\beta} y(q^{-\beta}x),\ y(a) = b, \tag{5.20}$$

where

$$0 < \alpha < 1,\ \beta > -\tfrac{1}{2},\ \lambda \in \mathbb{R},\ \in \mathbb{R} \tag{5.21}$$

is given by

$$y(x) = b\ {}_qE_{\frac{1}{2}, 1+2\beta, 2\beta}(\lambda, x-a). \tag{5.22}$$

3) By the help of (4.52) and Lemma 55 and by applying the successive approximation with $y_0(x) = \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a)$*, Example 68 can be generalized for arbitrary* $\alpha > 0$*. Namely, the solution of the q–initial value problem,*

$$({}_qC_a^\alpha y)(x) = \lambda(x-a)_q^\beta y(q^{-\beta}x),\ y^{(k)}(a) = b_k \quad (b_k \in \mathbb{R},\ k = 0, 1, ..., n-1), \tag{5.23}$$

where

$$n - 1 < \alpha < n,\ \beta > -\alpha,\ \lambda \in \mathbb{R},\ b \in \mathbb{R} \tag{5.24}$$

is given by

$$y(x) = \sum_{r=0}^{n-1} \frac{b_r}{\Gamma_q(r+1)} (x-a)_q^r \, {}_qE^r_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta+r}{\alpha}}(\lambda, x-a). \tag{5.25}$$

Note that when $0 < \alpha < 1$*, i.e,* $n = 1$*, the solution of Example 68 is recovered.*

CHAPTER 6

CONCLUSION

Mittag–Leffler functions generalize exponential functions through fractional calculus. As the theory of difference equations and q–difference equations act as the discrete face for the theory of differential equation, we find it worthwile in this thesis to review the way how the discrete analogue and q–analogue of Mittag–Leffler functions have been obtained through discrete and q–fractional calculus. Also, as an original contribution, we were able to obtain a q–analogue of a certain generalized q–Mittag–Leffler function. This function was obtained by solving a certain linear Caputo type q–fractional initial value problem. The choice of Caputo fractional derivative in our approach was due to many advantages that these derivatives have over Riemann ones.

We believe that that such discrete analogue concepts will play an important role in studying discrete and q–fractional calculus in the near future. In particular, when the Mittag–Leffler type stability for such systems is studied.

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APPENDIX A

CURRICULUM VITAE

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1. Abdeljawad T., Betul Benli, A Quantum Generalized Mittag-Leffler Function Via Caputo q-Fractional Equations, arXiv:1102.1585v1, submitted.
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