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*Research Article*

# Fractional Odd-Dimensional Mechanics

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The classical Nambu mechanics is generalized to involve fractional derivatives using two different methods. The first method is based on the definition of fractional exterior derivative and the second one is based on extending the standard velocities to the fractional ones. Fractional Nambu mechanics may be used for nonintegrable systems with memory. Further, Lagrangian which is generate fractional Nambu equations is defined.

## 1. Introduction

Derivatives and integrals of fractional-order have found many applications in recent studies in mechanics and physics, for example, in chaotic dynamics, quantum mechanics, plasma physics, anomalous diffusion, and so many fields of physics [1–12]. Fractional mechanics describes both conservative and nonconservative systems [13, 14]. In mechanics, Riewe has shown that Lagrangian involving fractional time derivatives leads to equation of motion with nonconservative classical derivatives such as friction [13, 14]. Motivated by this approach many researchers have explored this area giving new insight into this problem [15–37]. Agrawal has presented fractional Euler-Lagrangian equation involving Riemann-Liouville derivatives [16, 17]. Further fractional single and multi-time Hamiltonian formulation has been developed by Baleanu and coworkers [38].

In 1973, Nambu generalized Hamiltonian mechanics which is called now Nambu mechanics. This formalism is shown that provide a suitable framework for the odd dimensional phase space and nonintegrable systems [39–43]. By this motivation the authors have fractionalized this formalism [21].

In this work two methods are introduced for fractionalizing of Nambu mechanics. The first method is based on the definition of fractional exterior derivative and fractional forms. The second methods is based on fractionalizing of classical velocity. The resulted equations using these methods may use for complex memorial systems.

This paper is organized as follows.

Section 2 is devoted to a brief review of the fractional derivative definitions and fractional forms. Section 3 contains the classical Nambu mechanics. Section 4 deals with fractionalizing Hamiltonian mechanics using fractional differential forms. Using two different methods in Section 5 the Nambu mechanics has been fractionalized. In Section 6 is defined a Lagrangian which its variation gives the fractional Hamiltonian equations. In Section 7 we present our conclusions.

## 2. Basic Tools

The following subsections contain all mathematical tools used in this manuscript.

### *2.1. Fractional Derivatives*

In this section it is briefly presented the definition of the left and the right fractional derivatives of Riemann-Liouville as well as Caputo [6, 7]. The left Riemann-Liouville fractional derivative is defined as

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \tag{2.1} $$

and the right Riemann-Liouville fractional derivative,

$$ {}_tD_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha+1-n}} d\tau, \tag{2.2} $$

where the order $\alpha$ fulfills $n-1 \leq \alpha < n$ and $\Gamma$ represent the gamma function. An alternative definition of Riemann-Liouville fractional derivative called Caputo derivative that introduced by Caputo in 1967. The left Caputo derivative defined as

$$ {}_a^CD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau) d\tau, \tag{2.3} $$

and the right Caputo fractional derivative

$$ {}_t^CD_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} \left(-\frac{d}{d\tau}\right)^n f(\tau) d\tau, \tag{2.4} $$

where the order $\alpha$ satisfies $n-1 \leq \alpha < n$. The Riemann-Liouville derivative of constant isn't zero, but the Caputo derivative of a constant is zero.

### *2.2. Fractional Forms*

The calculus of classical differential forms is a powerful tool in applied mathematics. There are so many books that give a clear introduction to this field [44]. In calculus when a new function appears in the scene, it is natural to ask what its derivative is. Similarly with form, it is reasonable to ask what its exterior derivative is. For example a 1-form, integer-order one, can be shown as follows:

$$\omega = \sum_{i=1}^{n} a_i dx_i. \tag{2.5}$$

The classical exterior derivative is defined as

$$d = dx_i \frac{\partial}{\partial x_i}. \tag{2.6}$$

In [45, 46] the author generalizes the definition of integer-order vector spaces form to fractional-order one, and denotes it by $F(\nu, m, n)$. In this notation $\nu$ is the order of differential form, $m$ the number of coordinate differential appearing in the basis elements, $n$ the number of coordinates. For instance (2.5) is an element of $F(1, 1, n)$.

*The Definition of Fractional Exterior Derivative*

If the partial derivative in the definition of the classical exterior derivative, is replaced by the fractional-order, definition of fractional exterior derivative is obtained,

$$d^{\nu} = \sum_{i=1}^{n} dx_i^{\nu} \, {}_0D_{x_i}^{\nu}, \tag{2.7}$$

where ${}_0D_x^{\nu}$ is left fractional derivative. Let $\sigma \in F(\nu, 1, n)$,

$$\sigma = \sum_{i=1}^{n} \sigma_i dx_i^{\nu}, \tag{2.8}$$

and consider its fractional exterior derivative

$$d^{\nu}\sigma = \sum_{i=1}^{n} d^{\nu}\left(\sigma_i dx_i^{\nu}\right). \tag{2.9}$$

Using the product rule of exterior fractional derivative,

$$d^{\nu}\sigma = \sum_{i=1}^{n}\sum_{j=1}^{n} {}_0D_{x^j}^{\nu}\sigma_i dx_j^{\nu} \wedge dx_i^{\nu}. \tag{2.10}$$

If $d^{\nu}\sigma = 0$, then

$${}_0D_{x^j}^{\nu}\sigma_i - {}_0D_{x^i}^{\nu}\sigma_j = 0. \tag{2.11}$$

Note that in the following equations, the $\wedge$ sign is omitted between the differential forms.

## 3. Nambu Mechanics

Consider the Hamilton equations

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}, \\ \dot{p} &= -\frac{\partial H}{\partial p}. \end{aligned} \tag{3.1}$$

In another notation

$$\begin{aligned} \dot{q} &= \frac{\partial q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial q}{\partial p} = \frac{\partial(q,H)}{\partial(q,p)}, \\ \dot{p} &= \frac{\partial p}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial p}{\partial p} = \frac{\partial(p,H)}{\partial(q,p)}. \end{aligned} \tag{3.2}$$

Nambu generalized these equations to triplet $p, q, r$ dynamical variables with two Hamiltonians $H_1, H_2$ as follows:

$$\dot{q} = \begin{pmatrix} \frac{\partial q}{\partial q} & 0 & 0 \\ \frac{\partial H_1}{\partial q} & \frac{\partial H_1}{\partial p} & \frac{\partial H_1}{\partial r} \\ \frac{\partial H_2}{\partial q} & \frac{\partial H_2}{\partial p} & \frac{\partial H_2}{\partial r} \end{pmatrix} = \frac{\partial(q, H_1, H_2)}{\partial(q, p, r)},$$

$$\dot{p} = \begin{pmatrix} 0 & \dfrac{\partial p}{\partial p} & 0 \\ \dfrac{\partial H_1}{\partial q} & \dfrac{\partial H_1}{\partial p} & \dfrac{\partial H_1}{\partial r} \\ \dfrac{\partial H_2}{\partial q} & \dfrac{\partial H_2}{\partial p} & \dfrac{\partial H_2}{\partial r} \end{pmatrix} = \frac{\partial(q, H_1, H_2)}{\partial(q, p, r)},$$

$$\dot{r} = \begin{pmatrix} 0 & 0 & \dfrac{\partial r}{\partial r} \\ \dfrac{\partial H_1}{\partial q} & \dfrac{\partial H_1}{\partial p} & \dfrac{\partial H_1}{\partial r} \\ \dfrac{\partial H_2}{\partial q} & \dfrac{\partial H_2}{\partial p} & \dfrac{\partial H_2}{\partial r} \end{pmatrix} = \frac{\partial(q, H_1, H_2)}{\partial(q, p, r)}. \tag{3.3}$$

Considering these equations in terms of differential from, take the following 1-form:

$$\Omega^{(1)} = pdq - H(p, q)dt. \tag{3.4}$$

The exterior derivative of $\Omega^{(1)}$ is

$$d\Omega^{(1)} = \left(dp + \frac{\partial H}{\partial q}dt\right) \wedge \left(dq - \frac{\partial H}{\partial p}dt\right). \tag{3.5}$$

The Pfaffian equations is obtained, and then the Hamiltonian equations are resulted.

Now in terms of differential forms the Nambu mechanics is obtained using the following form:

$$\Omega^{(2)} = qdp \wedge dr - H_1 dH_2 \wedge dt. \tag{3.6}$$

The exterior derivative of this form is as follows:

$$d\Omega^{(2)} = \left(dq - \frac{\partial(H_1, H_2)}{\partial(p, r)}dt\right) \wedge \left(dp - \frac{\partial(H_1, H_2)}{\partial(r, q)}dt\right) \wedge \left(dr - \frac{\partial(H_1, H_2)}{\partial(q, p)}dt\right) = \theta \wedge \rho \wedge \sigma, \tag{3.7}$$

where

$$\theta = dq - \frac{\partial(H_1, H_2)}{\partial(p, r)} dt,$$
$$\rho = dp - \frac{\partial(H_1, H_2)}{\partial(r, q)} dt, \tag{3.8}$$
$$\theta = dr - \frac{\partial(H_1, H_2)}{\partial(q, p)} dt.$$

Now the Paffian equations. Equating them with zero we lead to Nambu mechanics equations.

## 4. Fractional Hamilton's Equations

The fractional generalization of (3.4) can be defined by

$$\Omega_\alpha^{(1)} = p(dq)^\alpha - H(p, q)(dt)^\alpha. \tag{4.1}$$

The fractional exterior derivative of this form is as follows:

$$d^\alpha\left(\Omega_\alpha^{(1)}\right) = d^\alpha(p) \wedge (dq)^\alpha - d^\alpha H(p, q) \wedge (dt)^\alpha. \tag{4.2}$$

Taking

$$d^\alpha p = \frac{p^{1-\alpha}}{\Gamma(2-\alpha)} (dp)^\alpha,$$
$$d^\alpha H = \frac{\partial^\alpha H}{\partial p^\alpha} (dp)^\alpha + \frac{\partial^\alpha H}{\partial q^\alpha} (dq)^\alpha, \tag{4.3}$$

we have

$$\begin{aligned} d^\alpha\left(\Omega_\alpha^{(1)}\right) &= \frac{p^{\alpha-1}}{\Gamma(2-\alpha)} (dp)^\alpha \wedge (dq)^\alpha - \frac{\partial^\alpha H}{\partial p^\alpha} (dp)^\alpha \wedge (dt)^\alpha + \frac{\partial^\alpha H}{\partial q^\alpha} (dq)^\alpha \wedge (dt)^\alpha, \\ &= \left(\frac{p^{1-\alpha}}{\Gamma(2-\alpha)} (dp)^\alpha - \frac{\partial^\alpha H}{\partial q^\alpha} (dt)^\alpha\right) \wedge \left((dq)^\alpha - \frac{\Gamma(2-\alpha)}{p^{1-\alpha}} \frac{\partial^\alpha H}{\partial p^\alpha} (dt)^\alpha\right). \end{aligned} \tag{4.4}$$

The fractional Pfaffian equations are as follows:

$$\theta_\alpha = \frac{p^{1-\alpha}}{\Gamma(2-\alpha)} (dp)^\alpha - \frac{\partial^\alpha H}{\partial q^\alpha} (dt)^\alpha,$$
$$\rho_\alpha = (dp)^\alpha - \frac{\Gamma(2-\alpha)}{p^{1-\alpha}} \frac{\partial^\alpha H}{\partial p^\alpha} (dt)^\alpha. \tag{4.5}$$

The fractional Hamiltonian equations is resulted

$$\frac{p^{1-\alpha}}{\Gamma(2-\alpha)}(dp)^{\alpha}=\frac{\partial^{\alpha}H}{\partial q^{\alpha}}(dt)^{\alpha},$$
$$(dq)^{\alpha}=\frac{\Gamma(2-\alpha)}{p^{1-\alpha}}\frac{\partial^{\alpha}H}{\partial p^{\alpha}}(dt)^{\alpha}. \tag{4.6}$$

## 5. Fractional Nambu Mechanics

*Method 1.* In this part, we present the fractional Nambu mechanic based on fractional generalization of the form (3.6). As a result, we obtain

$$\Omega_{\alpha}^{(2)}=q(dp)^{\alpha}\wedge(dr)^{\alpha}-H_1\left(d^{\alpha}H_2-(dt)^{\alpha}\right). \tag{5.1}$$

Taking

$$d^{\alpha}=\frac{\partial^{\alpha}}{\partial p^{\alpha}}(dp)^{\alpha}+\frac{\partial^{\alpha}}{\partial q^{\alpha}}(dq)^{\alpha},$$
$$\frac{\partial^{\alpha}q}{\partial q^{\alpha}}=\frac{q^{1-\alpha}}{\Gamma(2-\alpha)}, \tag{5.2}$$

the exterior of this form is

$$d^{\alpha}\Omega_{\alpha}^{(2)}=\left(\frac{q^{1-\alpha}}{\Gamma(2-\alpha)}(dq)^{\alpha}-\frac{\partial^{\alpha}(H_1,H_2)}{\partial^{\alpha}(p,r)}(dt)^{\alpha}\right)\wedge\left((dp)^{\alpha}-\frac{\Gamma(2-\alpha)}{q^{1-\alpha}}\frac{\partial^{\alpha}(H_1,H_2)}{\partial^{\alpha}(q,r)}(dt)^{\alpha}\right)$$
$$\wedge\left((dr)^{\alpha}-\frac{\Gamma(2-\alpha)}{q^{1-\alpha}}\frac{\partial^{\alpha}(H_1,H_2)}{\partial^{\alpha}(q,p)}(dt)^{\alpha}\right). \tag{5.3}$$

We lead to Pfaffian equations

$$\frac{q^{1-\alpha}}{\Gamma(2-\alpha)}(dq)^{\alpha}-\frac{\partial^{\alpha}(H_1,H_2)}{\partial^{\alpha}(p,r)}(dt)^{\alpha}=0,$$
$$(dp)^{\alpha}-\frac{\Gamma(2-\alpha)}{q^{1-\alpha}}\frac{\partial^{\alpha}(H_1,H_2)}{\partial^{\alpha}(q,r)}(dt)^{\alpha}=0, \tag{5.4}$$
$$(dr)^{\alpha}-\frac{\Gamma(2-\alpha)}{q^{1-\alpha}}\frac{\partial^{\alpha}(H_1,H_2)}{\partial^{\alpha}(q,p)}(dt)^{\alpha}=0.$$

It can be generalized as follows:

$$\Omega_{\alpha}^{(n-1)}=x_1(dx_2)^{\alpha}\wedge\cdots\wedge(dx_n)^{\alpha}-H_1d^{\alpha}H_2\wedge\cdots\wedge d^{\alpha}H_{n-1}\wedge(dt)^{\alpha}. \tag{5.5}$$

*Method 2.* In this part we present the fractional Nambu mechanics based on two Lagrangian formulation. In simple fractional Hamiltonian equations we have

$$
\begin{aligned}
{}_aD_t^\alpha q &= \frac{\partial H}{\partial p} = \frac{\partial(q,H)}{\partial(q,p)},\\
-{}_tD_b^\alpha p &= \frac{\partial H}{\partial q} = \frac{\partial(p,H)}{\partial(q,p)}.
\end{aligned}
\tag{5.6}
$$

If we generalize these equations to Nambu fractional:

$$
\begin{aligned}
{}_aD_t^\alpha q = \frac{\partial(q,H_1,H_2)}{\partial(q,p,r)} &= \begin{pmatrix} \dfrac{\partial q}{\partial q} & 0 & 0 \\ \dfrac{\partial H_1}{\partial q} & \dfrac{\partial H_1}{\partial p} & \dfrac{\partial H_1}{\partial r} \\ \dfrac{\partial H_2}{\partial q} & \dfrac{\partial H_2}{\partial p} & \dfrac{\partial H_2}{\partial r} \end{pmatrix} = \frac{\partial H_1}{\partial p}\frac{\partial H_2}{\partial r} - \frac{\partial H_2}{\partial r}\frac{\partial H_1}{\partial p},\\
-{}_tD_b^\alpha p &= \frac{\partial(p,H_1,H_2)}{\partial(q,p,r)},\\
-{}_tD_b^\alpha r &= \frac{\partial(r,H_1,H_2)}{\partial(q,p,r)}.
\end{aligned}
\tag{5.7}
$$

In classical mechanics only variable, $q$, is considered as a configuration variables and variables, $p, r$ are first and second canonical variables, respectively. Suppose that it could be existed as many Lagrangian as the number of Hamiltonian. In the simplest case suppose that $L_1(q, {}_aD_t^\alpha q)$ and $L_2(q, {}_aD_t^\alpha q)$

$$
\begin{aligned}
&\delta \int L_1(q, {}_aD_t^\alpha q)\,dt,\\
&\delta \int L_2(q, {}_aD_t^\alpha q)\,dt.
\end{aligned}
\tag{5.8}
$$

Then the Fractional Euler-Lagrangian equations are

$$
\begin{aligned}
\frac{\partial L_1}{\partial q} + {}_tD_b^\alpha \frac{\partial L_1}{\partial_a D_t^\alpha q} &= 0,\\
\frac{\partial L_2}{\partial q} + {}_tD_b^\alpha \frac{\partial L_2}{\partial_a D_t^\alpha q} &= 0.
\end{aligned}
\tag{5.9}
$$

Next are defined the first and second canonical momentum $p, r$

$$p = \frac{\partial L_1}{\partial_a D_t^{\alpha} q},\qquad r = \frac{\partial L_2}{\partial_a D_t^{\alpha} q}, \tag{5.10}$$

and then the fractional Euler-Lagrange equations are

$${}_t D_b^{\alpha} p = \frac{\partial L_1}{\partial q},\qquad {}_t D_b^{\alpha} r = \frac{\partial L_2}{\partial q}. \tag{5.11}$$

Now we define two Hamiltonians $H_1$ and $H_2$ as follows:

$$\frac{1}{{}_a D_t^{\alpha} q} d\left[p_a D_t^{\alpha} q - L_1\right] \wedge d\left[r_a D_t^{\alpha} q - L_2\right] = dH_1 \wedge dH_2. \tag{5.12}$$

Therefore we obtain

$${}_a D_t^{\alpha} q dp \wedge dr + {}_t D_b^{\alpha} p dr \wedge dq + {}_t D_b^{\alpha} r dq \wedge dp = dH_1 \wedge dH_2. \tag{5.13}$$

By expanding the right hand and by comparing the coefficients of form we lead to fractional Nambu mechanic equations.

This result can be generalized to the $n$-dimensional case, namely $L_1(q, {}_a D_t^{\alpha} q), \ldots, L_{n-1}(q, {}_a D_t^{\alpha} q)$. As before we calculate the momenta $p_1 = \partial L_1/\partial_a D_t^{\alpha} q, \ldots, p_n = \partial L_(n - 1)/\partial_a D_t^{\alpha} q$, and so on

$$dH_1 \wedge \cdots dH_{n-1} = \frac{1}{\left({}_a D_t^{\alpha} q\right)^{n-2}} d\left(p_{1a} D_t^{\alpha} q - L_1\right) \wedge \cdots d\left(p_{n-1a} D_t^{\alpha} q - L_{n-1}\right). \tag{5.14}$$

## 6. A Fractional Lagrangian Formulation

Let us assumed that the functions $q(t', t), p(t', t), r(t', t)$ are of two variable $t, t'$. Then the corresponding Lagrangian is given by the following form:

$$L = \int \left[q\left( {}_t D_b^{\alpha} p \frac{\partial r}{\partial t'} - \frac{\partial p}{\partial t'} {}_t D_b^{\alpha} r\right) - H_1\left(\frac{\partial H_2}{\partial q}\frac{\partial q}{\partial t'} + \frac{\partial H_2}{\partial p}\frac{\partial p}{\partial t'} + \frac{\partial H_2}{\partial r}\frac{\partial r}{\partial t'}\right)\right] dt'. \tag{6.1}$$

Imposing

$$\delta S = 0, \tag{6.2}$$

we obtain

$$\delta S=\iint\Big[\delta q\Big({}_tD_b^{\alpha}p\frac{\partial r}{\partial t'}-\frac{\partial p}{\partial t'}\,{}_tD_b^{\alpha}r\Big)-\delta H_1\Big(\frac{\partial H_2}{\partial q}\frac{\partial q}{\partial t'}+\frac{\partial H_2}{\partial p}\frac{\partial p}{\partial t'}+\frac{\partial H_2}{\partial r}\frac{\partial r}{\partial t'}\Big)$$
$$-H_1\delta\Big(\frac{\partial H_2}{\partial q}\frac{\partial q}{\partial t'}+\frac{\partial H_2}{\partial p}\frac{\partial p}{\partial t'}+\frac{\partial H_2}{\partial r}\frac{\partial r}{\partial t'}\Big)\Big]dt'dt, \tag{6.3}$$

or

$$\delta S=\iint\Big[\delta q\Big(-\frac{\partial p}{\partial t'}\,{}_tD_b^{\alpha}r+{}_tD_b^{\alpha}p\frac{\partial r}{\partial t'}\Big)-\frac{\partial H_1}{\partial q}\delta q\Big(\frac{\partial H_2}{\partial q}\frac{\partial q}{\partial t'}+\frac{\partial H_2}{\partial p}\frac{\partial p}{\partial t'}+\frac{\partial H_2}{\partial r}\frac{\partial r}{\partial t'}\Big)$$
$$-H_1\Big(\frac{\partial^2 H_2}{\partial q^2}\delta q\frac{\partial q}{\partial t'}+\frac{\partial H_2}{\partial q}\frac{\partial}{\partial t'}\delta q+\frac{\partial^2 H_2}{\partial p\partial q}\frac{\partial p}{\partial t'}\delta q+\frac{\partial^2 H_2}{\partial r\partial q}\frac{\partial r}{\partial t'}\delta q\Big)\Big]dt'dt. \tag{6.4}$$

After some calculations, we obtain

$$\delta S=\iint\Big\{\Big[\Big({}_tD_b^{\alpha}r-\frac{\partial(H_1,H_2)}{\partial(q,p)}\Big)\frac{\partial p}{\partial t'}-\Big({}_tD_b^{\alpha}p-\frac{\partial(H_1,H_2)}{\partial(r,q)}\Big)\frac{\partial r}{\partial t'}\Big]\delta q$$
$$+\Big[\Big({}_aD_t^{\alpha}q-\frac{\partial(H_1,H_2)}{\partial(p,r)}\Big)\frac{\partial r}{\partial t'}-\Big({}_tD_b^{\alpha}r-\frac{\partial(H_1,H_2)}{\partial(q,p)}\Big)\frac{\partial q}{\partial t'}\Big]\delta p \tag{6.5}$$
$$+\Big[\Big({}_tD_b^{\alpha}p-\frac{\partial(H_1,H_2)}{\partial(r,q)}\Big)\frac{\partial q}{\partial t'}-\Big({}_aD_t^{\alpha}q-\frac{\partial(H_1,H_2)}{\partial(p,r)}\Big)\frac{\partial p}{\partial t'}\Big]\delta r\Big\}dt'dt.$$

We conclude that if (5.7) is satisfied, then $\delta S = 0$ for arbitrary variations of $\delta q, \delta p$ and $\delta r$.

## 7. Conclusions

In this paper, we defined new equations corresponding to the complex systems described by the Nambu mechanics within the languages of the fractional differential forms. It is shown that variation of the corresponding new action using fractional Lagrangian gives fractional Nambu equations. The equivalent methods presented in this manuscript can be applied to investigate the dynamics of the complex nonintegrable systems with memory. The classical results are obtained in the limiting case $\alpha \to 1$.

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