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Research Article

A Generalized q-Mittag-Leffler Function by q-Captuo Fractional Linear Equations

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Some Caputo q-fractional difference equations are solved. The solutions are expressed by means of a new introduced generalized type of q-Mittag-Leffler functions. The method of successive approximation is used to obtain the solutions. The obtained q-version of Mittag-Leffler function is thought as the q-analogue of the one introduced previously by Kilbas and Saigo (1995).

1. Introduction and Preliminaries

The concept of fractional calculus is not new. However, it has gained its popularity and importance during the last three decades or so. This is due to its distinguished applications in numerous diverse fields of science and engineering (see, e.g., [1–6] and the references therein). The q-calculus is also not of recent appearance. It was initiated in the twenties of the last century. For the basic concepts in q-calculus we refer the reader to [7]. Discrete and q-fractional difference equations are discrete versions of fractional differential equations. An extensive work has been done in discrete fractional dynamic equations and discrete fractional variational calculus (see [8–12]). Some of the authors applied the delta analysis and some applied nabla analysis. Since the domain of nabla operatos is more stable, the nabla approach could be preferable. In this paper we apply the nabla approach in the quantum case with $0 < q < 1$, but also the delta approach is possible [13]. During the last decade many authors applied diverse methods, such as homotopy perturbation method, to derive approximate analytical solutions of systems of fractional differential equations into Caputo and Riemann (see [14–18]). In this paper, we apply a direct method to express the solution of a certain linear Caputo q-fractional differential equation by means of a new introduced generalized q-Mittag-Leffler function.

Starting from the q-analogue of Cauchy formula [19], Al-Salam started the fitting of the concept of q-fractional calculus. After that he [20, 21] and Agarwal [22] continued on by studying certain q-fractional integrals and derivatives, where they proved the semigroup properties for left and right (Riemann) type fractional integrals but without variable lower limit and variable upper limit, respectively. Recently, the authors in [23] generalized the notion of the (left) fractional q-integral and q-derivative by introducing variable lower limit and proved the semigroup properties.

Very recently and after the appearance of time-scale calculus (see, e.g., [24]), some authors started to pay attention and apply the techniques of time scale to discrete fractional calculus (see [25–28]) benefitting from the results announced before in [29]. All of these results are mainly about fractional calculus on the time scales $\mathbb{T}_q = \{q^n : n \in \mathbb{Z}\} \cup \{0\}$ and $h\mathbb{Z}$ [30]. As a contribution in this direction and being motivated by what is mentioned before, in this paper we introduce the q-analogue of a generalized type Mittag-Leffler function used before by Kilbas and Saigo in [31]. Such functions are obtained by solving linear q-Caputo initial value problems. The results obtained in this paper generalize also the results of [32]. Indeed, the authors in [32] solved a linear Caputo q-fractional difference equation of the form

$$\left({}_qC_a^{\alpha}y\right)(x) = \lambda y(x) + f(x), \quad y(a) = b, \quad 0 < \alpha < 1, \tag{1.1}$$

where the solution was expressed by means of discrete q-Mittag-Leffler functions. In this paper, we solve an equation of the form

$$\begin{gathered}\left({}_qC_a^{\alpha}y\right)(x) = \lambda (x-a)_q^{\beta}\, y\left(q^{-\beta}x\right), \qquad y(a) = b, \\ 0 < \alpha < 1,\ \beta > -\alpha,\ \lambda \in \mathbb{R},\ b \in \mathbb{R},\end{gathered} \tag{1.2}$$

where the solution is expressed by means of a more general discrete q-Mittag-Leffler functions generalizing the ones obtained by (1.1), as (1.1) is obtained from (1.2) by setting $\beta = 0$. Finally, we generalize to the higher-order case for any $\alpha > 0$, where higher-order q-Mittag-Leffler functions are obtained.

For the theory of q-calculus we refer the reader to the survey of [7], and for the basic definitions and results for the q-fractional calculus we refer to [28]. Here we will summarize some of those basics.

For $0 < q < 1$, let $\mathbb{T}_q$ be the time scale:

$$\mathbb{T}_q = \left\{q^n : n \in \mathbb{Z}\right\} \cup \{0\}, \tag{1.3}$$

where $\mathbb{Z}$ is the set of integers. More generally, if α is a nonnegative real number, then we define the time scale

$$\mathbb{T}_q^{\alpha} = \left\{q^{n+\alpha} : n \in \mathbb{Z}\right\} \cup \{0\}, \tag{1.4}$$

and we write $\mathbb{T}_q^0 = \mathbb{T}_q$.

For a function $f : T_q \to \mathbb{R}$, the nabla q-derivative of f is given by

$$\nabla_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in \mathbb{T}_q - \{0\}. \tag{1.5}$$

The nabla q-integral of f is given by

$$\int_0^t f(s)\nabla_q s = (1-q)t\sum_{i=0}^{\infty} q^i f\left(tq^i\right), \tag{1.6}$$

and for $0 \le a \in T_q$,

$$\int_a^t f(s)\nabla_q s = \int_0^t f(s)\ \nabla_q s - \int_0^a f(s)\ \nabla_q s. \tag{1.7}$$

On the other hand

$$\int_t^{\infty} f(s)\nabla_q s = (1-q)t\sum_{i=1}^{\infty} q^{-i} f\left(tq^{-i}\right), \tag{1.8}$$

and for $0 < b < \infty$ in $\mathbb{T}_q$,

$$\int_t^b f(s)\nabla_q s = \int_t^{\infty} f(s)\nabla_q s - \int_b^{\infty} f(s)\nabla_q s. \tag{1.9}$$

By the fundamental theorem in q-calculus we have

$$\nabla_q \int_0^t f(s)\nabla_q s = f(t), \tag{1.10}$$

and if f is continuous at 0, then

$$\int_0^t \nabla_q f(s)\nabla_q s = f(t) - f(0). \tag{1.11}$$

Also the following identity will be helpful:

$$\nabla_q \int_a^t f(t,s)\nabla_q s = \int_a^t \nabla_q f(t,s)\ \nabla_q s + f(qt,t). \tag{1.12}$$

Similarly the following identity will be useful as well:

$$\nabla_q \int_t^b f(t,s)\nabla_q s = \int_{qt}^b \nabla_q f(t,s)\ \nabla_q s - f(t,t). \tag{1.13}$$

The q-derivative in (1.12) and (1.13) is applied with respect to t.

From the theory of q-calculus and the theory of time scale more generally, the following product rule is valid:

$$\nabla_q\left(f(t)g(t)\right) = f\left(qt\right)\nabla_q g(t) + \nabla_q f(t)g(t). \tag{1.14}$$

The q-factorial function for $n \in \mathbb{N}$ is defined by

$$(t-s)_q^n = \prod_{i=0}^{n-1}\left(t - q^i s\right). \tag{1.15}$$

When α is a nonpositive integer, the q-factorial function is defined by

$$(t-s)_q^\alpha = t^\alpha \prod_{i=0}^{\infty} \frac{(1-(s/t))q^i}{(1-(s/t))q^{i+\alpha}}. \tag{1.16}$$

We summarize some of the properties of q-factorial functions, which can be found mainly in [28], in the following lemma.

Lemma 1.1. *One has the following.*

(i) $(t-s)_q^{\beta+\gamma} = (t-s)_q^\beta (t-q^\beta s)_q^\gamma$.

(ii) $(at-as)_q^\beta = a^\beta (t-s)_q^\beta$.

(iii) *The nabla q-derivative of the q-factorial function with respect to t is*

$$\nabla_q (t-s)_q^\alpha = \frac{1-q^\alpha}{1-q}(t-s)_q^{\alpha-1}. \tag{1.17}$$

(iv) *The nabla q-derivative of the q-factorial function with respect to s is*

$$\nabla_q (t-s)_q^\alpha = -\frac{1-q^\alpha}{1-q}\left(t-qs\right)_q^{\alpha-1}, \tag{1.18}$$

where $\alpha, \gamma, \beta \in \mathbb{R}$.

Definition 1.2 (see [32]). Let $\alpha > 0$. *If* $\alpha \notin \mathbb{N}$, then the α-order Caputo (left) q-fractional derivative of a function f is defined by

$$ {}_qC_a^\alpha f(t) \triangleq {}_qI_a^{(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-qs)_q^{n-\alpha-1} \nabla_q^n f(s) \nabla_q s, \tag{1.19}$$

where $n = [\alpha] + 1$.

If $\alpha \in \mathbb{N}$, then ${}_qC_a^\alpha f(t) \triangleq \nabla_q^n f(t)$.

It is clear that ${}_qC_a^\alpha$ maps functions defined on T_q to functions defined on T_q, and that ${}_bC_q^\alpha$ maps functions defined on $T_q^{1-\alpha}$ to functions defined on T_q.

The following identity which is useful to transform Caputo q-fractional difference equations into q-fractional integrals, will be our key in solving the q-fractional linear type equation by using successive approximation.

Proposition 1.3 ([32]). *Assume that* $\alpha > 0$ *and* f *is defined in suitable domains. Then*

$$ {}_qI_a^\alpha \, {}_qC_a^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a), \tag{1.20}$$

and if $0 < \alpha \leq 1$, *then*

$$ {}_qI_a^\alpha {}_qC_a^\alpha f(t) = f(t) - f(a). \tag{1.21}$$

The following identity [23] is essential to solve linear q-fractional equations:

$$ {}_qI_a^\alpha (x-a)_q^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)} (x-a)_q^{\mu+\alpha} \quad (0 < a < x < b), \tag{1.22}$$

where $\alpha \in \mathbb{R}^+$ and $\mu \in (-1, \infty)$. The q-analogue of Mittag-Leffler function with double index (α, β) is introduced in [32]. It was defined as follows.

Definition 1.4 ([32]). For $z, z_0 \in \mathbf{C}$ and $\Re(\alpha) > 0$, the q-Mittag-Leffler function is defined by

$$ {}_qE_{\alpha,\beta}(\lambda, z - z_0) = \sum_{k=0}^{\infty} \lambda^k \frac{(z-z_0)_q^{\alpha k}}{\Gamma_q(\alpha k + \beta)}. \tag{1.23}$$

When $\beta = 1$, we simply use ${}_qE_\alpha(\lambda, z - z_0) := {}_qE_{\alpha,1}(\lambda, z - z_0)$.

2. Main Results

The following is to be the q-analogue of the generalized Mittag-Leffler function introduced by Kilbas and Saigo [31] (see also [3] page 48).

Definition 2.1. For $\alpha, l, \lambda \in \mathbb{C}$ are complex numbers and $m \in \mathbb{R}$ such that $\Re(\alpha) > 0, m > 0, a \geq 0$, and $\alpha(jm+l) \neq -1, -2, -3, \ldots$, the generalized q-Mittag-Leffler function (of order 0) is defined by

$$ {}_qE_{\alpha,m,l}(\lambda, x-a) = 1 + \sum_{k=1}^{\infty} \lambda^k q^{-(k(k-1)/2)\alpha(m-1)(\alpha l+\alpha)} c_k (x-a)_q^{\alpha k m}, \tag{2.1}$$

where

$$ c_k = \prod_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]}, \qquad k = 1, 2, 3, \ldots, \tag{2.2}$$

while the generalized q-Mittag-Leffler function (of order r), $r = 0, 1, 2, 3, \ldots$, is defined by

$$ {}_qE^r_{\alpha,m,l}(\lambda, x-a) = 1 + \sum_{k=1}^{\infty} \lambda^k q^{-k\alpha(m-1)r} q^{-(k(k-1)/2)\alpha(m-1)(\alpha l+\alpha)} c_k \left(x - q^r a\right)_q^{\alpha k m}. \tag{2.3}$$

Note that ${}_qE^0_{\alpha,m,l}(\lambda, x-a) = {}_qE_{\alpha,m,l}(\lambda, x-a)$.

Remark 2.2. In particular, if $m = 1$, then the generalized q-Mittag-Leffler function is reduced to the q-Mittag-Leffler function, apart from a constant factor $\Gamma_q(\alpha l + 1)$. Namely,

$$ {}_qE_{\alpha,1,l}(\lambda, x-a) = \Gamma_q(\alpha l + 1)\, {}_qE_{\alpha,\alpha l+1}(\lambda, x-a). \tag{2.4}$$

This turns to be the q-analogue of the identity $E_{\alpha,1,l}(z) = \Gamma(\alpha l + 1)E_{\alpha,\alpha l+1}(z)$ (see [3] page 48).

Example 2.3. Consider the q-Caputo difference equation:

$$ \begin{gathered} \left({}_qC_a^{\alpha} y\right)(x) = \lambda (x-a)_q^{\beta} y\left(q^{-\beta}x\right), \qquad y(a) = b, \\ 0 < \alpha < 1,\ \beta > -\alpha,\ \lambda \in \mathbb{R},\ b \in \mathbb{R}. \end{gathered} \tag{2.5}$$

Applying Proposition 1.3 we have

$$ y(x) = y(a) + \lambda\, {}_qI_a^{\alpha}\left[(x-a)_q^{\beta} y\left(q^{-\beta}x\right)\right]. \tag{2.6}$$

The method of successive applications implies that

$$ y_m(x) = y(a) + \lambda\, {}_qI_a^{\alpha}\left[(x-a)_q^{\beta} y_{m-1}\left(q^{-\beta}x\right)\right], \qquad m = 1, 2, 3, \ldots, \tag{2.7}$$

where $y_0(x) = b$. Then by the help of (1.22) we have

$$y_1(x) = b + b\lambda \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(x-a)_q^{\beta+\alpha},$$
$$y_2(x) = b + b\lambda\, {}_qI_a^{\alpha}\left[(x-a)_q^{\beta}\left\{1+\lambda\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}\left(q^{-\beta}x-a\right)_q^{\beta+\alpha}\right\}\right]. \tag{2.8}$$

Then by (i) and (ii) of Lemma 1.1,

$$y_2(x) = b + b\lambda\, {}_qI_a^{\alpha}\left[(x-a)_q^{\beta} + \lambda\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}q^{-\beta(\alpha+\beta)}(x-a)_q^{2\beta+\alpha}\right]. \tag{2.9}$$

Again by (1.22) we conclude that

$$y_2(x) = b + b\lambda\, {}_qI_a^{\alpha}\left[(x-a)_q^{\beta} + \lambda\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}q^{-\beta(\alpha+\beta)}(x-a)_q^{2\beta+\alpha}\right]. \tag{2.10}$$

Then (1.22) leads to

$$y_2(x) = b\left[1+\lambda\frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(x-a)_q^{\beta+\alpha} + \lambda^2\frac{\Gamma_q(2\beta+\alpha+1)}{\Gamma_q(2\beta+2\alpha+1)}q^{-\beta(\alpha+\beta)}(x-a)_q^{2\beta+2\alpha}\right]. \tag{2.11}$$

Proceeding inductively, for each $m = 1, 2, \dots$ we obtain

$$y_m(x) = b\left[1+\sum_{k=1}^{m}\lambda^k q^{-\beta(k(k-1)/2)(\alpha+\beta)}c_k(x-a)_q^{k(\alpha+\beta)}\right], \tag{2.12}$$

where

$$c_k = \prod_{j=0}^{k-1}\frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]}, \qquad m = 1+\frac{\beta}{\alpha},\ l = \frac{\beta}{\alpha},\ k = 1,2,3,\dots. \tag{2.13}$$

If we let $m \to \infty$, then we obtain the solution

$$y(x) = b\left[1+\sum_{k=1}^{\infty}\lambda^k q^{-\beta(k(k-1)/2)(\alpha+\beta)}c_k(x-a)_q^{k(\alpha+\beta)}\right]. \tag{2.14}$$

Now, by means of Definition 2.1, we can state the following.

Theorem 2.4. *The solution of the q-Caputo difference equation* (2.5) *is given by*

$$y(x) = b\, {}_qE_{\alpha,(1+(\beta/\alpha)),\beta/\alpha}(\lambda, x-a). \tag{2.15}$$

Remark 2.5. (1) If in (2.5) $\beta = 0$, then in accordance with (2.4) and Example 9 in [32] we have

$$ {}_qE_{\alpha,1,0}(\lambda, x-a) = {}_qE_{\alpha,1}(\lambda, x-a) = {}_qE_{\alpha}(\lambda, x-a). \tag{2.16}$$

(2) The solution of the q-Cauchy problem

$$\begin{gathered} \left({}_qC_a^{1/2}y\right)(x) = \lambda\ (x-a)_q^{\beta} y\left(q^{-\beta}x\right), \qquad y(a) = b, \\ 0<\alpha<1,\ \beta > -\frac{1}{2},\ \lambda \in \mathbb{R},\ b \in \mathbb{R}, \end{gathered} \tag{2.17}$$

is given by

$$y(x) = b\,{}_qE_{1/2,1+2\beta,2\beta}(\lambda, x-a). \tag{2.18}$$

For the sake of generalization to the higher-order case, we consider the fractional q-initial value problem:

$$\left({}_qC_a^{\alpha}y\right)(x) = \lambda(x-a)_q^{\beta} y\left(q^{-\beta}x\right), \quad y^{(k)}(a) = b_k \quad (b_k \in \mathbb{R}, k = 0,1,\dots,n-1), \tag{2.19}$$

where

$$n-1<\alpha<n, \quad \beta > -\alpha, \quad \lambda \in \mathbb{R}, \quad b \in \mathbb{R}. \tag{2.20}$$

Theorem 2.6. *The solution of the fractional q-initial value problem* (2.19) *is of the following form:*

$$y(x) = \sum_{r=0}^{n-1} \frac{b_r}{\Gamma_q(r+1)}(x-a)_q^r{}_qE^r_{\alpha,((1+\beta)/\alpha),((\beta+r)/\alpha)}(\lambda, x-a). \tag{2.21}$$

Proof. The proof follows by the help of (1.20) and let Lemma 1.1 and by applying the successive approximation with

$$y_0(x) = \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a), \tag{2.22}$$

□

Note that when $0<\alpha<1$, that is, $n=1$, the solution of Example 2.3 is recovered.
Next, we solve a nonhomogenous versions of (2.5).

Lemma 2.7. *Let $r \in \mathbb{N}$, $\alpha > 0$, and let f be defined on $\mathbb{T}_q$. Then*

$$ {}_qI_a f(q^{-r}t) = q^{r\alpha}\left({}_qI_{q^{-r}a}f\right)(q^{-r}t) \quad \forall t \in \mathbb{T}_q. \tag{2.23}$$

In particular, if $a = 0$, then

$$ {}_qI_0 f(q^{-r}t) = q^{r\alpha}\left({}_qI_0 f\right)(q^{-r}t) \quad \forall t \in \mathbb{T}_q. \tag{2.24}$$

Proof. The proof can be achieved by making use of Theorem 1 in [28] for integration by substitution (for details see [24]). Indeed,

$$\begin{aligned} {}_qI_a f(q^{-r}t) &= \frac{1}{\Gamma_q(\alpha)}\int_a^t (t-qs)_q^{\alpha-1} f(q^{-r}s)\nabla_q s \\ &= \frac{q^r}{\Gamma_q(\alpha)}\int_{q^{-r}a}^{q^{-r}t} (t-qq^r s)_q^{\alpha-1} f(s)\nabla_q s \\ &= \frac{q^{r\alpha}}{\Gamma_q(\alpha)}\int_{q^{-r}a}^{q^{-r}t} (q^{-r}t-qs)_q^{\alpha-1} f(s)\nabla_q s \\ &= q^{r\alpha}\left({}_qI_{q^{-r}a} f\right)(q^{-r}t). \end{aligned} \tag{2.25}$$

□

Consider the q-fractional initial value problem:

$$\left({}_qC_0^\alpha y\right)(x) = \lambda x^\beta y\left(q^{-\beta}x\right) + f(x), \qquad y(0) = b, \tag{2.26}$$

where

$$0 < \alpha < 1, \quad \beta > -\alpha, \quad \beta \in \mathbb{N}_0, \quad \lambda \in \mathbb{R}, \quad b \in \mathbb{R}. \tag{2.27}$$

If we apply the successive approximation as in Example 2.3 and use Lemma 2.7, then we can state the following

Theorem 2.8. *The solution of the initial value problem* (2.26) *is expressed by*

$$y(x) = b\, {}_qE_{\alpha,(1+(\beta/\alpha)),\beta/\alpha}(\lambda, x) + \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma_q(\alpha k+\alpha)} q^{-\alpha\beta(k(k+1)/2)}\int_0^x (x-qt)_q^{\alpha k+\alpha} f\left(q^{-k\beta}t\right)\nabla_q t. \tag{2.28}$$

Remark 2.9. If in (2.26) we set $\beta = 0$, then Example 9 in [32] is recovered for a = 0.

Definition 2.10. A function $f : \mathbb{T}_q \to \mathbb{R}$ is called periodic with period $\beta \in \mathbb{N}_1$ *if* β is the smallest natural number such that $f(q^\beta t) = f(t)$, for all $t \in \mathbb{T}_q$.

Consider the nonhomogeneous initial value problem:

$$\left({}_qC_0^\alpha y\right)(x) = \lambda (x-a)_q^\beta y\left(q^{-\beta}x\right) + f(x), \qquad y(a) = b, \tag{2.29}$$

where

$$0 < \alpha < 1, \quad \beta > -\alpha, \quad \beta \in \mathbb{N}_0, \quad \lambda \in \mathbb{R}, \quad b \in \mathbb{R}. \tag{2.30}$$

If we apply the successive approximation as in Example 2.3, then we state the following.

Theorem 2.11. *If in* (2.29) *either* $\beta = 0$ *or* f *is periodic with period dividing* β*, then the solution is given by*

$$y(x) = b_q E_{\alpha,(1+(\beta/\alpha)),\beta/\alpha}(\lambda, x-a) + \int_a^x (x-qt)_q^{\alpha-1} {}_qE_{\alpha,\alpha}(\lambda, x-q^\alpha t) f(t) \nabla_q t. \tag{2.31}$$

Clearly, if $\beta = 0$, then the result in Example 9 in [32] is recovered as well.

For the sake of completeness, it would be interesting if the h-discrete fractional analogue, or more generally the (q,h)-analogue of the general q-Mittag-Leffler functions are obtained, possibly better, by applying nabla calculus (see [33–35]). However, this needs preparations in the Caputo case and it might be very complicated.

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