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Nonlocal nonlinear integrodifferential equations of fractional orders

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Abstract

In this paper, Schauder fixed point theorem, Gelfand-Shilov principles combined with semigroup theory are used to prove the existence of mild and strong solutions for nonlinear fractional integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces. To illustrate our abstract results, an example is given.

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1 Introduction

We are concerned with the nonlocal nonlinear fractional problem

$$\frac{d^\alpha(Bu(t))}{dt^\alpha} + Au(t) = f(t, W(t)) + \int_0^t g(t, s, W(s)) ds, \quad (1.1)$$

$$u(0) + \sum_{k=1}^p c_k u(t_k) = u_0, \quad (1.2)$$

where $\frac{d^\alpha}{dt^\alpha}$, $0 < \alpha \leq 1$ is the Riemann-Liouville fractional derivative, $0 \leq t_1 < \dots < t_p \leq a$, c_1, \dots, c_p are real numbers, B and A are linear closed operators with domains contained in a Banach space X and ranges contained in a Banach space Y , $W(t) = (B_1(t)u(t), \dots, B_r(t)u(t))$, $\{B_i(t) : i = 1, \dots, r, t \in I = [0, a]\}$ is a family of linear closed operators defined on dense sets $S_1, \dots, S_r \supset D(A) \supset D(B)$ respectively in X into X , $f : I \times X^r \rightarrow Y$ and $g : \Delta \times X^r \rightarrow Y$ are given abstract functions. Here $\Delta = \{(s, t) : 0 \leq s \leq t \leq a\}$.

Fractional differential equations have attracted many authors [1, 6–8, 21, 24, 25, 30]. This is mostly because it efficiently describes many phenomena arising in engineering, physics, economics and science. In fact, we can find several applications in viscoelasticity, electrochemistry, electromagnetic, *etc.* For example, Machado [22] gave a novel method for the design of fractional order digital controllers.

Following Gelfand and Shilov [20], we define the fractional integral of order $\alpha > 0$ as

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

also, the (Riemann-Liouville) fractional derivative of the function f of order $0 < \alpha < 1$ as

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) ds,$$

where f is an abstract continuous function on the interval $[a, b]$ and $\Gamma(\alpha)$ is the Gamma function, see also [14, 24].

The existence results to evolution equations with nonlocal conditions in a Banach space was studied first by Byszewski [9, 10]; subsequently, many authors were pointed to the same field, see for instance [2–4, 11–13, 19, 28].

Deng [15] indicated that using the nonlocal condition $u(0) + h(u) = u_0$ to describe, for instance, the diffusion phenomenon of a small amount of gas in a transparent tube can give a better result than using the usual local Cauchy problem $u(0) = u_0$. Let us observe also that since Deng's papers, the function h is considered

$$h(u) = \sum_{k=1}^p c_k u(t_k), \tag{1.3}$$

where $c_k, k = 1, 2, \dots, p$ are given constants and $0 \leq t_1 < \dots < t_p \leq a$.

However, among the previous research on nonlocal Cauchy problems, few authors have been concerned with mild solutions of fractional semilinear differential equations [23].

Recently, many authors have extended this work to various kinds of nonlinear evolution equations [2, 3, 5, 11, 12, 18]. Balachandran and Uchiyama [3] proved the existence of mild and strong solutions of a nonlinear integrodifferential equation of Sobolev type with nonlocal condition.

In this paper, motivated by [3, 13, 17, 19], we use Schauder fixed point theorem and the semigroup theory to investigate the existence and uniqueness of mild and strong solutions for nonlinear fractional integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces, the solutions were obtained by using Gelfand-Shilov approach in fractional calculus and are given in terms of some probability density functions such that their Laplace transforms are indicated [17].

Our paper is organized as follows. Section 2 is devoted to the review of some essential results. In Section 3, we state and prove our main results; the last section deals with giving an example to illustrate the abstract results.

2 Preliminary results

In this section, we mention some results obtained by Balachandran [3], El-Borai [19] and Pazy [26], which will be used to get our new results. Let X and Y be Banach spaces with norm $|\cdot|$ and $\|\cdot\|$ respectively. The operator $B : D(B) \subset X \rightarrow Y$ satisfies the following hypotheses:

- (H₁) B is bijective,
- (H₂) $B^{-1} : Y \rightarrow D(B)$ is compact.

The above fact and the closed graph theorem imply the boundedness of the linear operator $AB^{-1} : Y \rightarrow Y$. Further $E = -AB^{-1}$ generates a uniformly continuous semigroup $Q(t), t \geq 0$ such that $\max_{t \in I} \|Q(t)\| \leq K, Q(t)h \in D(A), \|EQ(t)h\| \leq \frac{K}{t} \|h\|$ for every $h \in X$ and all $t \in (0, a]$, see [29].

Let $\lambda = \|B^{-1}\|$, $c = \sum_{k=1}^p |c_k|$ and $\Lambda_\tau = \{(u_1, \dots, u_r) : u_i \in X, \sum_{i=1}^r |u_i| \leq \tau\}$.

It is supposed that

(H₃) f and g are continuous in t on I , Δ respectively, and there exist constants $M_1, M_2 > 0$ such that $\|f(t, W)\| \leq M_1$, $\|g(t, s, W)\| \leq M_2$ for all $t \in I$, $(s, t) \in \Delta$ and $W \in \Lambda_\tau$.

Definition 2.1 By a strong solution of the nonlocal Cauchy problem (1.1), (1.2), we mean a function u with values in X such that

- (i) u is a continuous function in $t \in I$ and $u(t) \in D(A)$,
- (ii) $\frac{d^\alpha u}{dt^\alpha}$ exists and is continuous on $(0, a]$, $0 < \alpha < 1$, and u satisfies (1.1) on $(0, a]$ and (1.2).

Remark 2.1 Let us take in the considered problem B is the identity, the inhomogeneous part is equal to an abstract continuous function $f(t)$, and the nonlocal condition is reduced to the initial condition $u(0) = u_0$, i.e.,

$$D_t^\alpha u(t) + Au(t) = f(t), \tag{2.1}$$

$$u(0) = u_0. \tag{2.2}$$

According to El-Borai [17–19], we first apply the fractional integral on both sides of (2.1) and then using (2.2), we apply the Laplace transform on the new integral equations by considering a suitable one-sided stable probability density whose Laplace transform is given. Hence we can conclude that a solution of the problem (2.1)-(2.2) can be formally represented by

$$u(t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) u_0 d\theta + \alpha \int_0^t \int_0^\infty \theta(t-s)^{\alpha-1} \zeta_\alpha(\theta) Q((t-s)^\alpha \theta) f(s) d\theta ds, \tag{2.3}$$

where ζ_α is a probability density function defined on $(0, \infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\theta x} \zeta_\alpha(\theta) d\theta = \sum_{j=0}^\infty \frac{(-x)^j}{\Gamma(1 + \alpha j)}, \quad 0 < \alpha \leq 1, x > 0.$$

For more details, we refer to Zhou *et al.* [27, 31], see also [14, 16].

Using Gelfand-Shilov principle [20], it is suitable to rewrite (1.1), (1.2) in the form

$$Bu(t) = Bu(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} \times \left[-Au(\eta) + f(\eta, W(\eta)) + \int_0^\eta g(\eta, s, W(s)) ds \right] d\eta, \tag{2.4}$$

where $\Gamma(\alpha)$ is the Gamma function.

According to [17–19], the equation (2.4) is equivalent to the integral equation

$$Bu(t) = \Psi(t)Bu(0) + \int_0^t \Phi(t-\eta) \left[f(\eta, W(\eta)) + \int_0^\eta g(\eta, s, W(s)) ds \right] d\eta, \quad t > 0, \tag{2.5}$$

where

$$\Psi(t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) d\theta,$$

$$\Phi(t) = \alpha \int_0^\infty \theta t^{\alpha-1} \zeta_\alpha(\theta) Q(t^\alpha \theta) d\theta.$$

It is assumed that there exists an operator ψ on $D(\psi) = X$ given by the formula

$$\psi = \left[I + \sum_{k=1}^p c_k B^{-1} \Psi(t_k) B \right]^{-1},$$

satisfying $\psi u_0 \in D(B)$ and for $k = 1, \dots, p$

$$\psi \int_0^{t_k} B^{-1} \Phi(t_k - \eta) \left[f(\eta, W(\eta)) + \int_0^\eta g(\eta, s, W(s)) ds \right] d\eta \in D(B),$$

also

$$(H_4) \quad K\lambda \|B\psi u_0\| [\lambda^2 K^2 c\alpha^\alpha \|B\psi\| + \lambda K\alpha^\alpha] (M_1 + aM_2) \leq \tau.$$

Further we assume

(H₅) There is a number $\gamma \in (0, 1)$ such that

$$\|B_i(t_2) Q(t_1) h\| \leq \frac{K_1}{t_1^\gamma} \|h\|,$$

where $t_1 \in (0, a]$, $t_2 \in I$, $h \in X$ and K_1 is a positive constant, $i = 1, \dots, r$.

(H₆) The functions $B_1(t)h, \dots, B_r(t)h$ are uniformly Hölder continuous in $t \in I$ for every element h in $\bigcap_i S_i$.

Suppose that $\{Q(t)\}$ is a C_0 -semigroup of operators on X such that $\|B^{-1}Q(t_k)B\| \leq Ce^{-\delta t_k}$, where δ is a positive constant and $C \geq 1$. Noting that $\int_0^\infty \zeta_\alpha(\theta) d\theta = 1$ (see [14, p.4]).

If $\sum_{k=1}^p |c_k| e^{-\delta t_k} < \frac{1}{C}$, then $\|\sum_{k=1}^p c_k B^{-1} \psi(t_k) B\| < 1$, which achieves that ψ exists on X .

3 Main results

The following is different from [3, 19, 26] and represents the new result.

Lemma 3.1 *If u is a continuous solution of (2.5), then u satisfies the integral equation*

$$\begin{aligned} u(t) &= B^{-1} \Psi(t) B \psi u_0 \\ &\quad - \sum_{k=1}^p c_k B^{-1} \Psi(t) B \psi \\ &\quad \times \int_0^{t_k} B^{-1} \Phi(t_k - s) \left[f(s, W(s)) + \int_0^s g(s, \eta, W(\eta)) d\eta \right] ds \\ &\quad + \int_0^t B^{-1} \Phi(t - s) \left[f(s, W(s)) + \int_0^s g(s, \eta, W(\eta)) d\eta \right] ds. \end{aligned} \tag{3.1}$$

Proof Using (2.5) and (1.2), we get

$$\begin{aligned} \sum_{k=1}^p c_k Bu(t_k) &= \sum_{k=1}^p c_k \Psi(t_k) Bu_0 - \sum_{k=1}^p c_k \Psi(t_k) B \sum_{k=1}^p c_k u(t_k) \\ &\quad + \sum_{k=1}^p c_k \int_0^{t_k} \Phi(t_k - s) \left[f(s, W(s)) + \int_0^s g(s, \eta, W(\eta)) d\eta \right] ds. \end{aligned}$$

Then

$$\begin{aligned} \sum_{k=1}^p c_k u(t_k) \left[I + \sum_{k=1}^p c_k B^{-1} \Psi(t_k) B \right] &= \sum_{k=1}^p c_k B^{-1} \Psi(t_k) Bu_0 + \sum_{k=1}^p c_k \int_0^{t_k} B^{-1} \Phi(t_k - s) \\ &\quad \times \left[f(s, W(s)) + \int_0^s g(s, \eta, W(\eta)) d\eta \right] ds. \end{aligned}$$

Thus

$$\begin{aligned} \Psi(t) Bu(0) &= \Psi(t) \left[Bu_0 - \sum_{k=1}^p c_k Bu(t_k) \right] \\ &= \Psi(t) Bu_0 - \Psi(t) B \psi \sum_{k=1}^p c_k B^{-1} \Psi(t_k) Bu_0 \\ &\quad - \Psi(t) B \psi \sum_{k=1}^p c_k \int_0^{t_k} B^{-1} \Phi(t_k - s) \left[f(s, W(s)) + \int_0^s g(s, \eta, W(\eta)) d\eta \right] ds \\ &= \Psi(t) B \psi u_0 \left[\psi^{-1} - \sum_{k=1}^p c_k B^{-1} \Psi(t_k) B \right] \\ &\quad - \Psi(t) B \psi \sum_{k=1}^p c_k \int_0^{t_k} B^{-1} \Phi(t_k - s) \left[f(s, W(s)) + \int_0^s g(s, \eta, W(\eta)) d\eta \right] ds. \end{aligned}$$

Hence the required result. □

Definition 3.1 A continuous solution of the integral equation (3.1) is called a mild solution of the nonlocal problem (1.1), (1.2) on I .

Theorem 3.2 *If the assumptions $(H_1) \sim (H_4)$ hold and $W(t) = u(t)$, then the problem (1.1), (1.2) has a mild solution on I .*

Proof Let $Z = C(I, X)$ and $Z_0 = \{u \in Z : u(t) \in \Lambda_\tau, t \in I\}$. It is easy to see that Z_0 is a bounded closed convex subset of Z . We define a mapping $\varphi : Z_0 \rightarrow Z_0$ by

$$\begin{aligned} (\varphi u)(t) &= B^{-1} \Psi(t) B \psi u_0 \\ &\quad - \sum_{k=1}^p c_k B^{-1} \Psi(t) B \psi \int_0^{t_k} B^{-1} \Phi(t_k - s) \left[f(s, W(s)) + \int_0^s g(s, \eta, W(\eta)) d\eta \right] ds \\ &\quad + \int_0^t B^{-1} \Phi(t - s) \left[f(s, W(s)) + \int_0^s g(s, \eta, W(\eta)) d\eta \right] ds, \quad t \in I. \end{aligned}$$

Noting also that $\int_0^\infty \theta \zeta_\alpha(\theta) d\theta = 1$ (see [14, p.4]), we have

$$\begin{aligned} \|(\varphi u)(t)\| &\leq K\lambda \|B\psi u_0\| + (M_1 + aM_2)\lambda^2 K^2 c a^\alpha \|B\psi\| \\ &\quad + (M_1 + aM_2)\lambda K a^\alpha \leq \tau. \end{aligned}$$

We deduce that φ is continuous and maps Z_0 into itself. Moreover, φ maps Z_0 into a precompact subset of Z_0 . Note that the set $Z_0(t) = \{(\varphi u)(t) : u \in Z_0\}$ is precompact in X , for every fixed $t \in I$. We shall show that $\varphi(Z_0) = S = \{\varphi u : u \in Z_0\}$ is an equicontinuous family of functions. For $0 < s < t$, we have

$$\begin{aligned} &\|(\varphi u)(t) - (\varphi u)(s)\| \\ &\leq [\lambda \|B\psi u_0\| + c\lambda^2 K a^\alpha (M_1 + aM_2) \|B\psi\|] \|\Psi(t) - \Psi(s)\| \\ &\quad + \lambda(M_1 + aM_2) \int_s^t \|\Phi(t - \eta)\| d\eta \\ &\quad + \lambda(M_1 + aM_2) \int_0^s \|\Phi(t - \eta) - \Phi(s - \eta)\| d\eta. \end{aligned}$$

The right-hand side of the above inequality is independent of $u \in Z_0$ and tends to zero as $s \rightarrow t$ as a consequence of the continuity of $\Psi(t)$ and $\Phi(t)$ in the uniform operator topology for $t > 0$. It is clear that S is bounded in Z . Thus by Arzela-Ascoli's theorem, S is precompact. Hence by the Schauder fixed point theorem, φ has a fixed point in Z_0 and any fixed point of φ is a mild solution of (1.1), (1.2) on I such that $u(t) \in X$ for all $t \in I$. \square

Theorem 3.3 *Assume that*

- (i) *Conditions $(H_1) \sim (H_6)$ hold,*
- (ii) *Y is a reflexive Banach space with norm $\|\cdot\|$,*
- (iii) *there are numbers $L_1, L_2 > 0$ and $p, q \in (0, 1]$ such that*

$$\begin{aligned} \|f(t_1, W) - f(t_2, W^*)\| &\leq L_1 \left(|t_1 - t_2|^p + \sum_{i=1}^r |w_i - w_i^*| \right), \\ \|g(s_1, \eta, W) - g(s_2, \eta, W)\| &\leq L_2 |s_1 - s_2|^q \end{aligned}$$

for all $t_1, t_2 \in I, (s_1, \eta), (s_2, \eta) \in \Delta$ and all $W, W^ \in \Lambda_\tau$, where $w_i = B_i u$ and $w_i^* = B_i u^*$. Then the problem (1.1), (1.2) has a unique strong solution on I .*

Proof Applying Theorem 3.2, the problem (1.1), (1.2) has a mild solution $u \in C(I, \Lambda_\tau)$. Now, we shall show that u is a unique strong solution of the considered problem on I .

According to (H_6) , $\sum_{i=1}^r |w_i - w_i^*|$ is uniformly Hölder continuous in $t \in I$ for every element u in $\bigcap_i S_i$ combined with (iii), which implies that $t \rightarrow f(t, W(t))$ and $t \rightarrow \int_0^t g(t, s, W(s)) ds$ are uniformly Hölder continuous on I .

Set

$$V(t) = f(t, W(t)) + \int_0^t g(t, s, W(s)) ds.$$

From (3.1), the solution u of the considered problem can be written in the form

$$u(t) = B^{-1}\Psi(t)B\psi u_0 - B^{-1}\Psi(t)B\psi \sum_{k=1}^p c_k \int_0^{t_k} B^{-1}\Phi(t_k - s)V(s) ds + \int_0^t B^{-1}\Phi(t - s)V(s) ds.$$

Noting that Ψ and ψ are bounded, using our assumptions, we observe that there exists a unique function $V \in C(I, X)$ which satisfies the equation

$$\frac{d^\alpha(Bu(t))}{dt^\alpha} + Au(t) = V(t).$$

Also as in [19, p.409], we deduce that

$$\int_0^t B^{-1}\Phi(t - s)V(s) ds \in D(E)$$

for all $t \in I$ and $\psi u_0 \in D(E)$. It follows that $u(t) \in D(E)$ for all $t \in I$. □

4 Example

Consider the nonlinear integro-partial differential equation of fractional order

$$\frac{\partial^\alpha [\sum_{|q| \leq 2m} b_q(x) D_x^q u(x, t)]}{\partial t^\alpha} + \sum_{|q| \leq 2m} a_q(x) D_x^q u(x, t) = F(x, t, W) + \int_0^t G(x, t, s, W(s)) ds, \tag{4.1}$$

with nonlocal condition

$$u(x, 0) + \sum_{k=1}^p c_k u(x, t_k) = g(x), \tag{4.2}$$

where $0 < \alpha \leq 1$, $0 \leq t_1 < \dots < t_p \leq a$, $x \in R^n$, $D_x^q = D_{x_1}^{q_1} \dots D_{x_n}^{q_n}$, $D_{x_i} = \frac{\partial}{\partial x_i}$, $q = (q_1, \dots, q_n)$ is an n -dimensional multi-index, $|q| = q_1 + \dots + q_n$, $W = (w_1, \dots, w_r)$,

$$w_i(x, t) = \sum_{|q| \leq 2m-1} b_{q_i}(x, t) D_x^q u(x, t) + \int_\Omega \sum_{|q| \leq 2m-1} c_{q_i}(x, t) D_y^q u(y, t) dy,$$

and Ω is an open subset of R^n . Let $L_2(R^n)$ be the set of all square integrable functions on R^n . We denote by $C^m(R^n)$ the set of all continuous real-valued functions defined on R^n which have continuous partial derivatives of order less than or equal to m . By $C_0^m(R^n)$ we denote the set of all functions $f \in C^m(R^n)$ with compact supports. Let $H^m(R^n)$ be the completion of $C_0^m(R^n)$ with respect to the norm

$$\|f\|_m^2 = \sum_{|q| \leq m} \int_{R^n} |D_x^q f(x)|^2 dx.$$

It is supposed that

(i) The operator $E = -\sum_{|q|=2m} e_q(x) D_x^q$ is uniformly elliptic on R^n . In other words, all the coefficients e_q , $|q| = 2m$ are continuous and bounded on R^n , and there is a positive number c such that

$$(-1)^{m+1} \sum_{|q|=2m} e_q(x) \xi^q \geq c |\xi|^{2m},$$

for all $x \in R^n$ and all $\xi \neq 0$, $\xi \in R^n$, where $e_q = a_q b_q^{-1}$, $\xi^q = \xi_1^{q_1} \dots \xi_n^{q_n}$ and $|\xi|^2 = \xi_1^2 + \dots + \xi_n^2$.

(ii) All the coefficients e_q , $|q| = 2m$, satisfy a uniform Hölder condition on R^n . Under these conditions, the operator E with the domain of definition $D(E) = H^{2m}(R^n)$ generates an analytic semigroup $Q(t)$ defined on $L_2(R^n)$, and it is well known that $H^{2m}(R^n)$ is dense in $Y = L_2(R^n)$, see [17, p.438].

Lemma 4.1 *The solution representation of (4.1), (4.2) can be written explicitly.*

Proof Let $\{E_q(x) : |q| \leq 2m\}$ be a family of deterministic square matrices of order k and let $L(x, D) = \{E_q(x) : |q| = 2m\}$. We assume that

$$\det\{(-1)^m L(x, \sigma) - \lambda I\} = 0$$

has roots which satisfy the inequality $\text{Re } \lambda < -\delta$, $\delta > 0$ for all $x \in R^n$ and for any real vector σ , $\sigma_1^2 + \dots + \sigma_n^2 = 1$. If Θ is a matrix of order $m \times n$, then we introduce $|\Theta| = \sum_{i,j} |b_{ij}|$.

It is well known that there exists a fundamental matrix solution $Z(x, y, t)$ which satisfies the system

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= L(x, D)u(x, t), \quad t > 0, x \in R^n, \\ u(x, t_0) + \sum_{i=1}^N c_i u(x, t_i) &= g(x). \end{aligned}$$

This fundamental matrix also satisfies the inequality

$$|D_x^q Z(x, y, t)| \leq K_1 t^{-\rho_1} \exp(-K_2 \rho_2),$$

where $|q| \leq 2m$, $\rho_1 = -\frac{n+|q|}{2m}$, $\rho_2 = \sum_{i=1}^n |x_i - y_i|^\lambda t^{-\frac{1}{2m-1}}$, $\lambda = \frac{2m}{2m-1}$ and K_1, K_2 are positive constants. From [13, p.58], if the nonlocal function $g(x)$ is an element in Hilbert space $H^{2m}(R^n)$, then we can write

$$Q(t)g(x) = \int_{R^n} Z(x - y, t)g(y) dy.$$

It can be proved that

$$\|D_x^q Q(t)g\| \leq \frac{M}{t^\beta} \|g\|,$$

where $0 < \beta < 1$, M is a positive constant, $|q| \leq 2m - 1$, $t > 0$ and $\|g\|^2 = \int_{R^n} g^2(x) dx$.

([17]) The nonlocal Cauchy problems (4.1), (4.2) are equivalent to the integral equation

$$\begin{aligned}
 u(x, t) = & \int_0^\infty \int_{R^n} \zeta_\alpha(\theta) Q(x - \xi, t^\alpha \theta) u(\xi, 0) d\xi d\theta \\
 & + \alpha \int_0^t \int_0^\infty \int_{R^n} \theta (t - \eta)^{\alpha-1} \zeta_\alpha(\theta) Q(x - \xi, (t - \eta)^\alpha \theta) \\
 & \times \left[F(\xi, \eta, W) + \int_0^\eta G(\xi, \eta, s, W(s)) ds \right] d\xi d\theta d\eta,
 \end{aligned}$$

where the explicit form of Q is given by

$$Q(x, t) = \frac{e^{-|x|^2/4t}}{(\sqrt{4\pi t})^n},$$

$|x|^2 = x_1^2 + x_2^2 + \dots + x_n^2$. Applying Theorem 3.2, we achieve the proof of the existence of mild solutions of the problems (4.1), (4.2). In addition, if the operators F and G satisfy the following:

(iii) There are numbers $L_1, L_2 \geq 0$ and $0 < p, q \leq 1$ such that

$$\sum_{|q| \leq 2m-1} \int_{R^n} |F(x, t, D_x^q W) - F(x, s, D_x^q W^*)|^2 dx \leq L_1 \left(|t - s|^p + \sum_{i=1}^r |w_i - w_i^*|^2 \right)$$

and

$$\sum_{|q| \leq 2m-1} \int_{R^n} |G(x, t, \eta, D_x^q W) - G(x, s, \eta, D_x^q W)|^2 dx \leq L_2 |t - s|^q$$

for all $t, s \in I$, $(t, \eta), (s, \eta) \in \Delta$, $W, W^* \in \Lambda_\tau$ and all $x \in R^n$. Then applying Theorem 3.3, we deduce that (4.1), (4.2) has a unique strong solution. \square

5 Conclusion

In this article, a new solution representation for Sobolev type fractional evolution equation has been proved using Deng's nonlocal condition, a suitable explicit form of the semigroup has been discussed. Moreover, the existence result of mild solutions for nonlinear fractional integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces has been established by using Arzela-Ascoli's theorem and Schauder fixed point theorem. Further, the uniformly Hölder continuous condition has been applied for the existence of strong solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

AD wrote the first draft, DB corrected and improved it and RPA prepared the final version. All authors read and approved the final draft.

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