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OSCILLATION OF EVEN ORDER NONLINEAR DELAY DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. One of the important methods for studying the oscillation of higher order differential equations is to make a comparison with second order differential equations. The method involves using Taylor's Formula. In this paper we show how such a method can be used for a class of even order delay dynamic equations on time scales via comparison with second order dynamic inequalities. In particular, it is shown that nonexistence of an eventually positive solution of a certain second order delay dynamic inequality is sufficient for oscillation of even order dynamic equations on time scales. The arguments are based on Taylor monomials on time scales.

Keywords: time scale, even order, delay, oscillation, Taylor monomial

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1. Introduction

In this paper we consider the oscillation of solutions of higher order nonlinear delay dynamic equations with forcing terms of the form

$$x^{\Delta^n}(t)+q(t)|x(\varphi(t))|^{\alpha-1}x(\varphi(t))=g(t),\quad t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

where n is even, $t_0\in\mathbb{T}$, and $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$ denotes a time scale interval with $\sup\mathbb{T}=\infty$. The equation is called sublinear if $0<\alpha<1$, and superlinear if $\alpha>1$.

These problems for second order and higher order nonlinear differential and difference equations with/without delay have been considered by many authors and we mention just a few [3], [4], [5], [7], [11], [13], [14], [15], [16].

A *time scale* $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. The most well-known examples are $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{Z}$, and $\mathbb{T}=\overline{q^{\mathbb{Z}}}:=\{q^n\colon n\in\mathbb{Z}\}\cup\{0\}$, where

$q > 1$. The forward and backward jump operators $\sigma, \varrho\colon\ \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T}\colon\ s > t\} \quad \text{and} \quad \varrho(t) := \sup\{s \in \mathbb{T}\colon\ s < t\},$$

respectively, where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is said to be left-dense if $t > \inf \mathbb{T}$ and $\varrho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\varrho(t) < t$, and right-scattered if $\sigma(t) > t$. A function $f\colon\ \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions is denoted by C_{rd}. For more details, we refer the reader to [2], [8].

Throughout the paper we assume that

(i) $q \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $q(t) \geqslant 0$, $q(t) \not\equiv 0$ on $[T, \infty)_{\mathbb{T}}$ for any $T \geqslant t_0$;
(ii) $g \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$;
(iii) $\varphi\colon\ \mathbb{T} \to \mathbb{T}$ is rd-continuous, $\varphi(t) \leqslant t$, $\lim\limits_{t\to\infty} \varphi(t) = \infty$.

By a solution of Equation (1.1) we mean a function $x \in C^n_{\mathrm{rd}}([t_x, \infty)_{\mathbb{T}}, \mathbb{R})$ that satisfies Equation (1.1) for all $t \geqslant t_x \geqslant t_0$. Here we are concerned with proper solutions of Equation (1.1), i.e., those solutions x which satisfy $\sup\{|x(t)|\colon\ t \geqslant T\} > 0$ for every $T \geqslant t_x$. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative; i.e., if for any given $t_1 \in [t_x, \infty)_{\mathbb{T}}$ there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $x(t_2)x(\sigma(t_2)) \leqslant 0$. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Definition 1.1 ([2]). The Taylor monomials are the functions $g_k, h_k\colon\ \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$, which are defined recursively as follows:

$$g_0(t,s) = h_0(t,s) \equiv 1 \quad \text{for all } t, s \in \mathbb{T},$$

and for $k \in \mathbb{N}_0$,

$$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau), s)\,\Delta\tau \quad \text{for all } t, s \in \mathbb{T},$$
$$h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\,\Delta\tau \quad \text{for all } t, s \in \mathbb{T}.$$

We recall the definition of $\mathbb{T}^\kappa$ and $\mathbb{T}^{\kappa^k}$, $k \geqslant 2$.

Definition 1.2 ([2]). If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa := \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa := \mathbb{T}$. And $\mathbb{T}^{\kappa^k} := (\mathbb{T}^\kappa)^{\kappa^{k-1}}$, $k \geqslant 2$.

Theorem 1.1 ([2, Theorem 1.112]). *The functions h_k and g_k satisfy*

$$h_k(t,s) = (-1)^k g_k(s,t) \quad \textit{for all } t \in \mathbb{T} \textit{ and all } s \in \mathbb{T}^{\kappa^k}.$$

It is clear from Definition 1.1 that $h_1(t,s) = g_1(t,s) = t - s$ for all $t, s \in \mathbb{T}$. However, finding g_k, h_k for $k > 1$ is not easy in general. But for a particular given time scale, for example for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, one can easily find the functions g_k and h_k. We have for $k \in \mathbb{N}_0$,

$$h_k(t,s) = g_k(t,s) = \frac{(t-s)^k}{k!} \quad \text{for all } t, s \in \mathbb{R} \tag{1.2}$$

and

$$h_k(t,s) = \frac{(t-s)^{\underline{k}}}{k!} \quad \text{and} \quad g_k(t,s) = \frac{(t-s+k-1)^{\underline{k}}}{k!} \quad \text{for all } t, s \in \mathbb{Z}, \tag{1.3}$$

where $t^{\underline{m}}$, $m \in \mathbb{N}_0$, is the usual factorial function; $t^{\underline{m}} = (t-m+1)t^{\underline{m-1}}$, $t^{\underline{0}} = 1$.

We will also use the following result:

Theorem 1.2 ([2, Theorem 1.117]). *Let $a \in \mathbb{T}^\kappa$, $b \in \mathbb{T}$ and assume $f\colon \mathbb{T}\times\mathbb{T}^\kappa \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^\kappa$ with $t > a$. Also assume that $f^\Delta(t,.)$ is* rd-*continuous on $[a, \sigma(t)]_{\mathbb{T}}$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [a,\sigma(t)]_{\mathbb{T}}$, such that*

$$|f(\sigma(t),\tau) - f(s,\tau) - f^\Delta(t,\tau)\,(\sigma(t)-s)| \leqslant \varepsilon|\sigma(t)-s| \quad \textit{for all} \quad s \in U,$$

where f^Δ denotes the derivative of f with respect to the first variable. Then

(1) $g(t) := \int_a^t f(t,\tau)\,\Delta\tau$ *implies* $g^\Delta(t) = \int_a^t f^\Delta(t,\tau)\,\Delta\tau + f(\sigma(t),t)$.

(2) $h(t) := \int_t^b f(t,\tau)\,\Delta\tau$ *implies* $h^\Delta(t) = \int_t^b f^\Delta(t,\tau)\,\Delta\tau - f(\sigma(t),t)$.

2. Preparatory lemmas

The following lemmas will be essential in order to obtain the main results. The first one is the time scale analog of the well-known lemmas due to Kiguradze and Kneser in the case $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively, see [1].

Lemma 2.1. *Let $n \in \mathbb{N}$ and f be n-times differentiable on $\mathbb{T}$. Assume* $\sup \mathbb{T} = \infty$ *and for any $\varepsilon > 0$, the set $L_\varepsilon(\infty) := \{t \in \mathbb{T}\colon\ t > 1/\varepsilon\}$. Suppose there exists $\varepsilon > 0$ such that*

$$f(t) > 0,\ \operatorname{sgn}(f^{\Delta^n}(t)) \equiv s \in \{-1, 1\} \quad \text{for all } t \in L_\varepsilon(\infty)$$

and $f^{\Delta^n}(t) \not\equiv 0$ on $L_\delta(\infty)$ for any $\delta > 0$. Then there exists $l \in [0, n] \cap \mathbb{N}_0$ such that $n + l$ is even for $s = 1$ and odd for $s = -1$ with

$$\begin{aligned} f^{\Delta^i}(t) > 0 \quad &\text{for all } t \in L_{\delta_i}(\infty) \ (\text{with } \delta_i \in (0, \varepsilon)),\ i \in [1, l-1] \cap \mathbb{N}_0, \\ (-1)^{l+i} f^{\Delta^i}(t) > 0 \quad &\text{for all } t \in L_\varepsilon(\infty),\ i \in [l, n-1] \cap \mathbb{N}_0. \end{aligned}$$

The following result provides an explicit formula for the Taylor monomials $h_k(t, s)$ on time scales $\mathbb{T}$ unbounded from above, for which the forward jump operator has a certain explicit form given by $\sigma(t) = at + b$, where $a \geqslant 1$, $b \geqslant 0$ are constants. In addition to the fact that it unifies the formulas (1.2) and (1.3), it can also be applied to time scales $\mathbb{T}$ that are different from $\mathbb{R}$ and $\mathbb{Z}$; for example, $\mathbb{T} = h\mathbb{Z} := \{hn\colon\ n \in \mathbb{Z}\}$ with $h > 0$, or

$$\mathbb{T} = \overline{q^{\mathbb{Z}}} := \{q^n\colon\ n \in \mathbb{Z}\} \cup \{0\} \quad \text{with } q > 1$$

(see [2, Example 1.104]).

Lemma 2.2. *Let $\mathbb{T}$ be a time scale which is unbounded above with $\sigma(t) = at + b$, where $a \geqslant 1$, $b \geqslant 0$ are constants. Then the Taylor monomials $h_k(t, s)$ on $\mathbb{T}$ are given by the formula*

$$h_k(t, s) = \prod_{i=0}^{k-1} \frac{(t - \sigma^i(s))}{\beta_i}, \quad t, s \in \mathbb{T},\ k \in \mathbb{N}_0, \tag{2.1}$$

where $\beta_i := \sum\limits_{j=0}^{i} a^j$ and $\sigma^0(s) := s$.

P r o o f. We will establish (2.1) by induction. Let us denote the right hand side of (2.1) by $\tilde{h}_k(t, s)$. It is clear that $\tilde{h}_0(t, s) = 1 = h_0(t, s)$ (observe that the empty product is considered to be 1, as usual) and $\tilde{h}_1(t, s) = t - s = h_1(t, s)$. If we assume $\tilde{h}_n = h_n$ for all $n \leqslant k$ for some $k \in \mathbb{N}$, then we have

$$\begin{aligned} \tilde{h}_{k+1}^{\Delta}(t, s) &= \Big[\tilde{h}_k(t, s) \frac{(t - \sigma^k(s))}{\beta_k}\Big]^{\Delta} \\ &= \frac{1}{\beta_k} [h_k(t, s)(t - a^k s - b\beta_{k-1})]^{\Delta} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\beta_k}[h_k^{\Delta}(t,s)(\sigma(t) - a^k s - b\beta_{k-1}) + h_k(t,s)] \\
&= \frac{1}{\beta_k}[h_{k-1}(t,s)(at - a^k s - ab\beta_{k-2}) + h_k(t,s)] \\
&= \frac{1}{\beta_k}[\tilde{h}_{k-1}(t,s)(t - a^{k-1} s - b\beta_{k-2})a + \tilde{h}_k(t,s)] \\
&= \frac{1}{\beta_k}[\tilde{h}_{k-1}(t,s)(t - \sigma^{k-1}(s))a + \tilde{h}_k(t,s)] \\
&= \frac{1}{\beta_k}[a\beta_{k-1}\tilde{h}_k(t,s) + \tilde{h}_k(t,s)] \\
&= \frac{1}{\beta_k}\{1 + a\beta_{k-1}\}\tilde{h}_k(t,s) = \tilde{h}_k(t,s).
\end{aligned}
$$

Since $\tilde{h}_{k+1}(s,s) = 0$, we have $\tilde{h}_{k+1} = h_{k+1}$, and hence the formula (2.1) holds for all $k \in \mathbb{N}_0$. $\square$

The analog of the Kiguradze lemma is the following.

Lemma 2.3. *Let $\mathbb{T}$ be a time scale which is unbounded above with $\sigma(t) = at + b$, where $a \geqslant 1$, $b \geqslant 0$ are constants. If x is an $(n+1)$-times differentiable function on $[t_0, \infty)_{\mathbb{T}}$ with $x^{\Delta^i}(t) \geqslant 0$, $i = 1, 2, \ldots, n$ and $x^{\Delta^{n+1}}(t) \leqslant 0$, then*

$$
x(t) \geqslant \frac{h_{n-1}(t, \sigma(t_0))}{\beta_{n-1}} x^{\Delta^{n-1}}(t), \quad t \geqslant \sigma^{n-1}(t_0), \tag{2.2}
$$

where, as earlier, $\beta_i := \sum\limits_{j=0}^{i} a^j$.

P r o o f. For the case $n = 1$, (2.2) is obvious. So let us assume $n > 1$. We have

$$
x^{\Delta^{n-1}}(t) = x^{\Delta^{n-1}}(t_0) + \int_{t_0}^{t} x^{\Delta^n}(s)\,\Delta s
$$

and so

$$
x^{\Delta^{n-1}}(t) \geqslant x^{\Delta^n}(t)(t - t_0), \quad t \geqslant t_0. \tag{2.3}
$$

From the product rule and $\sigma(t) = at + b$, we have

$$
x^{\Delta^n}(t)(t - t_0) = \Big(x^{\Delta^{n-1}}(t)\frac{t - \sigma(t_0)}{a}\Big)^{\Delta} - \frac{1}{a} x^{\Delta^{n-1}}(t).
$$

Integrating (2.3) from $\sigma(t_0)$ to t, we obtain

$$
x^{\Delta^{n-2}}(t) \geqslant \frac{t - \sigma(t_0)}{\beta_1} x^{\Delta^{n-1}}(t) = \frac{h_1(t, \sigma(t_0))}{\beta_1} x^{\Delta^{n-1}}(t), \quad t \geqslant \sigma(t_0). \tag{2.4}
$$

Integrating (2.4) from $\sigma^2(t_0)$ to t, and using

$$x^{\Delta^{n-1}}(t)\frac{t-\sigma(t_0)}{\beta_1} = \Big(x^{\Delta^{n-2}}(t)\frac{t-\sigma^2(t_0)}{a\beta_1}\Big)^{\Delta} - \frac{1}{a\beta_1}x^{\Delta^{n-2}}(t),$$

we see that

$$x^{\Delta^{n-3}}(t) \geqslant \frac{t-\sigma^2(t_0)}{\beta_2}\,x^{\Delta^{n-2}}(t), \quad t \geqslant \sigma^2(t_0). \tag{2.5}$$

Now from (2.4) and (2.5), it follows that

$$x^{\Delta^{n-3}}(t) \geqslant \frac{(t-\sigma(t_0))(t-\sigma^2(t_0))}{\beta_1\beta_2}x^{\Delta^{n-1}}(t) = \frac{h_2(t,\sigma(t_0))}{\beta_2}x^{\Delta^{n-1}}(t), \quad t \geqslant \sigma^2(t_0).$$

Continuing in this manner, we obtain

$$x^{\Delta^{n-i}}(t) \geqslant \frac{h_{i-1}(t,\sigma(t_0))}{\beta_{i-1}}x^{\Delta^{n-1}}(t), \quad 2 \leqslant i \leqslant n,\ t \geqslant \sigma^{i-1}(t_0).$$

Setting $i = n$ in the above inequality, we find that

$$x(t) \geqslant \frac{h_{n-1}(t,\sigma(t_0))}{\beta_{n-1}}x^{\Delta^{n-1}}(t), \quad t \geqslant \sigma^{n-1}(t_0),$$

as desired. □

We next establish the time scale version of a lemma due to Onose [12] in $\mathbb{T} = \mathbb{R}$. The result here is for an arbitrary time scale which we state as:

Lemma 2.4. *Let n be even and consider the equation*

$$x^{\Delta^n}(t) + f(t, x(\varphi(t))) = 0, \quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{2.6}$$

and the inequality

$$x^{\Delta^n}(t) + f(t, x(\varphi(t))) \leqslant 0, \quad t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.7}$$

Here we assume $f\colon [t_0,\infty)_{\mathbb{T}} \times (0,\infty) \to (0,\infty)$ is a function with the property $f(\cdot, w(\cdot))\colon [t_0,\infty)_{\mathbb{T}} \to (0,\infty)$ is rd-*continuous for any* rd-*continuous function $w\colon [t_0,\infty)_{\mathbb{T}} \to (0,\infty)$, and $f(.,u)$ is continuous and nondecreasing.*

If Equation (2.6) is oscillatory, then Inequality (2.7) has no eventually positive solution.

P r o o f. Assume to the contrary that there exists a positive solution x of Inequality (2.7) on $[t_1,\infty)_{\mathbb{T}}$, for some $t_1 \geqslant t_0$. Choose $t_2 \geqslant t_1$ so that $\varphi(t) \geqslant t_1$ for $t \geqslant t_2$. From (2.7), we have

$$x^{\Delta^n}(t) \leqslant -f(t, x(\varphi(t))) < 0, \quad t \geqslant t_2. \tag{2.8}$$

By repeated integration of (2.8), we get

$$\begin{aligned} x(t) &\geqslant x(t_2) + \int_{t_2}^{t}\int_{t_2}^{u_{n-1}}\dots\int_{t_2}^{u_{n-l+1}}\int_{u_{n-l}}^{\infty}\dots\int_{u_1}^{\infty} f(s, x(\varphi(s)))\Delta s\Delta u_1\dots\Delta u_{n-1} \\ &:= x(t_2) + \Phi(t, x(\varphi(t))), \quad t \geqslant t_2, \end{aligned}$$

where the integer $1 \leqslant l \leqslant n-1$ is from Lemma 2.1.

Now consider the equation

$$z(t) = x(t_2) + \Phi(t, z(\varphi(t))). \tag{2.9}$$

We note that if (2.9) has a solution $z(t) > 0$ for $t \geqslant t_2$, then z is a positive solution of Equation (2.6). To establish the existence of z, we define the sequence $\{z_n\}$ as follows:

$$z_1(t) := x(t), \quad t \geqslant t_1,$$

and for $n \geqslant 1$,

$$z_{n+1}(t) := \begin{cases} x(t) & \text{for } t_1 \leqslant t \leqslant t_2, \\ x(t_2) + \Phi(t, z_n(\varphi(t))) & \text{for } t \geqslant t_2. \end{cases}$$

Then we see that the z_n's are well defined and satisfy

$$0 < x(t_2) \leqslant z_{n+1}(t) \leqslant z_n(t), \quad t \geqslant t_2.$$

If we put

$$z(t) := \lim_{n\to\infty} z_n(t), \quad t \geqslant t_2,$$

then it follows from Lebesgue's dominated convergence theorem that

$$z(t) = x(t_2) + \Phi(t, z(\varphi(t))), \quad t \geqslant t_3$$

for some $t_3 > t_2$. Hence z satisfies (2.9) and the proof is complete. $\square$

Remark 2.1. Clearly, Lemma 2.4 holds for any finite number of delays.

3. The main results

We shall need to make some assumptions on the form of the forcing term g in the following.

Theorem 3.1. *Suppose the following additional conditions hold:*

(iv) $g(t) = h^{\Delta^n}(t)$ *for some* $h \in C^n_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$;

(v) h *is oscillatory and there exist two sequences* $\{s_n\}$ *and* $\{\bar{s}_n\}$ *tending to infinity such that for all* n,

$$\begin{aligned} h(s_n) &= \inf\{h(t)\colon\ t \geqslant s_n\},\\ h(\bar{s}_n) &= \sup\{h(t)\colon\ t \geqslant \bar{s}_n\}. \end{aligned} \tag{3.1}$$

If the second order delay dynamic inequality

$$u^{\Delta\Delta}(t) + c\lambda^{\alpha} g^{\alpha}_{l-1}(\sigma(T),\varphi(t)) g_{n-l-1}(\varphi(t),T) q(t) u^{\alpha}(\varphi(t)) \leqslant 0, \tag{3.2}$$

where $c := 1/\beta_{n-l-1}(\beta_{l-1})^{\alpha}$, *has no eventually positive solution for every constant* $0 < \lambda < 1$, *every* $T \geqslant t_0$, *and every integer* $1 \leqslant l \leqslant n-1$, *then Equation* (1.1) *is oscillatory.*

P r o o f. Assume to the contrary that there exists a nonoscillatory solution x of Equation (1.1). Without loss of generality, we may assume that

$$x(t) > 0 \quad \text{and} \quad x(\varphi(t)) > 0, \quad t \geqslant t_1$$

for some sufficiently large $t_1 \geqslant t_0$.

Define y by the equation

$$y(t) := x(t) - h(t), \quad t \geqslant t_1. \tag{3.3}$$

From Equation (1.1), we have

$$y^{\Delta^n}(t) + q(t)x^{\alpha}(\varphi(t)) = 0, \quad t \geqslant t_1.$$

Then there exists $t_2 \geqslant t_1$ such that

$$y(t) > 0, \quad y^{\Delta^n}(t) \leqslant 0$$

for all $t \geqslant t_2$. By Lemma 2.1, there exists $T \geqslant t_2$ and an odd integer $1 \leqslant l \leqslant n-1$ such that

$$\begin{aligned} y^{\Delta^i}(t) &> 0, \quad i = 0,1,\ldots,l-1,\ t \geqslant T,\\ (-1)^{i-l} y^{\Delta^i}(t) &> 0, \quad i = l, l+1,\ldots,n-1,\ t \geqslant T. \end{aligned} \tag{3.4}$$

By Taylor's formula, we may write

$$\begin{aligned} y^{\Delta^l}(t) &= \sum_{k=0}^{n-l-1} y^{\Delta^{l+k}}(\tau)h_k(t,\tau) + \int_t^\tau h_{n-l-1}(t,\sigma(s))(-y^{\Delta^n}(s))\,\Delta s \\ &= \sum_{k=0}^{n-l-1} (-1)^k y^{\Delta^{l+k}}(\tau)g_k(\tau,t) + \int_t^\tau g_{n-l-1}(\sigma(s),t)(-y^{\Delta^n}(s))\,\Delta s. \end{aligned}$$

Using (3.4), we get

$$y^{\Delta^l}(t) \geqslant \int_t^\tau g_{n-l-1}(\sigma(s),t)q(s)x^\alpha(\varphi(s))\,\Delta s, \quad T \leqslant t \leqslant \tau.$$

Now letting $\tau \to \infty$ and integrating from T to t, we have

$$\begin{aligned} y^{\Delta^{l-1}}(t) &\geqslant y^{\Delta^{l-1}}(T) + \int_T^t \int_r^\infty g_{n-l-1}(\sigma(s),r)q(s)x^\alpha(\varphi(s))\,\Delta s\Delta r \\ &= y^{\Delta^{l-1}}(T) + \int_T^t \left[\int_T^{\sigma(s)} g_{n-l-1}(\sigma(s),r)\,\Delta r\right] q(s)x^\alpha(\varphi(s))\,\Delta s \\ &\quad + \int_t^\infty \left[\int_T^t g_{n-l-1}(\sigma(s),r)\,\Delta r\right] q(s)x^\alpha(\varphi(s))\,\Delta s, \quad t \geqslant T. \end{aligned}$$

Notice that in changing the order of integration, we have used the following equalities:

$$\begin{aligned} &\left[\int_t^\infty \int_T^t g_{n-l-1}(\sigma(s),r)q(s)x^\alpha(\varphi(s))\,\Delta r\Delta s\right]^\Delta \\ &\qquad = \int_t^\infty g_{n-l-1}(\sigma(s),t)q(s)x^\alpha(\varphi(s))\,\Delta s \\ &\qquad\quad - \int_T^{\sigma(t)} g_{n-l-1}(\sigma(t),r)q(t)x^\alpha(\varphi(t))\,\Delta r, \\ &\int_t^\infty g_{n-l-1}(\sigma(s),t)q(s)x^\alpha(\varphi(s))\,\Delta s \\ &\qquad = \left[\int_T^t \int_r^\infty g_{n-l-1}(\sigma(s),r)q(s)x^\alpha(\varphi(s))\,\Delta s\Delta r\right]^\Delta, \\ &\int_T^{\sigma(t)} g_{n-l-1}(\sigma(t),r)q(t)x^\alpha(\varphi(t))\,\Delta r \\ &\qquad = \left[\int_T^t \int_T^{\sigma(s)} g_{n-l-1}(\sigma(s),r)q(s)x^\alpha(\varphi(s))\,\Delta r\Delta s\right]^\Delta, \end{aligned}$$

all of which follow from Theorem 1.2. By direct integration, it follows that

$$\begin{aligned} y^{\Delta^{l-1}}(t) &\geqslant y^{\Delta^{l-1}}(T) + \int_T^t g_{n-l}(\sigma(s),T)q(s)x^\alpha(\varphi(s))\,\Delta s \\ &\quad + \int_t^\infty [g_{n-l}(\sigma(s),T) - g_{n-l}(\sigma(s),t)]q(s)x^\alpha(\varphi(s))\,\Delta s, \quad t \geqslant T. \end{aligned}$$

It can easily be verified that we have for $s \geqslant t \geqslant T$,

$$g_{n-l}(\sigma(s),T) - g_{n-l}(\sigma(s),t) \geqslant (t-T)\frac{\prod_{i=2}^{n-l}(\sigma^i(s)-T)}{\prod_{i=1}^{n-l-1}\beta_i}.$$

In view of this inequality,

$$\begin{aligned}(3.5)\qquad y^{\Delta^{l-1}}(t) \geqslant y^{\Delta^{l-1}}(T) &+ \frac{1}{\prod_{i=1}^{n-l-1}\beta_i}\int_T^t \prod_{i=1}^{n-l}(\sigma^i(s)-T)q(s)x^\alpha(\varphi(s))\,\Delta s \\ &+ \frac{(t-T)}{\prod_{i=1}^{n-l-1}\beta_i}\int_t^\infty \prod_{i=2}^{n-l}(\sigma^i(s)-T)q(s)x^\alpha(\varphi(s))\,\Delta s, \quad t \geqslant T.\end{aligned}$$

Let us denote the right-hand side of (3.5) by $u(t)$. It is easy to see that $u(t) > 0$ and satisfies the second order dynamic equation

$$(3.6)\qquad u^{\Delta\Delta}(t) + \frac{g_{n-l-1}(\sigma^2(t),T)}{\beta_{n-l-1}}q(t)x^\alpha(\varphi(t)) = 0, \quad t \geqslant T.$$

On the other hand, from (3.1), (3.3), and the fact that $x(t) > 0$, $y(t) > 0$, $y^\Delta(t) > 0$, it follows that there exists a constant $0 < \lambda < 1$ such that for T sufficiently large,

$$(3.7)\qquad x(t) \geqslant \lambda y(t), \quad t \geqslant T.$$

Combining (2.2) with x replaced by y and n replaced by l, (3.7), and the fact that $y^{\Delta^{l-1}}(t) \geqslant u(t)$, we obtain

$$x(\varphi(t)) \geqslant \frac{h_{l-1}(\varphi(t),\sigma(T))}{\beta_{l-1}}\lambda u(\varphi(t))$$

for $t \geqslant T_1 > T$ sufficiently large. In view of this inequality, it follows from (3.6) that

$$u^{\Delta\Delta}(t) + \lambda^\alpha \frac{g_{n-l-1}(\varphi(t),T)g_{l-1}^\alpha(\sigma(T),\varphi(t))}{\beta_{n-l-1}(\beta_{l-1})^\alpha}q(t)u^\alpha(\varphi(t)) \leqslant 0, \quad t \geqslant T_1.$$

We have shown that (3.2) has an eventually positive solution. This, however, contradicts the assumption of the theorem. The proof is similar if x is an eventually negative solution of Equation (1.1). □

Remark 3.1. In the proof of Theorem 3.1, the fact that $l \geqslant 1$ is crucial. It should also be noted that the case $l = 0$ is possible only when n is odd. It follows that, for unbounded solutions of Equation (1.1), the integer l associated with y is $\geqslant 2$. Hence unbounded solutions of Equation (1.1) must be oscillatory if n is odd.

Theorem 3.2. *Assume that $q(t) > 0$. If the second order delay dynamic equation*

$$u^{\Delta\Delta}(t) + c\lambda^{\alpha} g_{l-1}^{\alpha}(\sigma(T), \varphi(t)) g_{n-l-1}(\varphi(t), T) q(t) u^{\alpha}(\varphi(t)) = 0,$$

where $c := 1/\beta_{n-l-1}(\beta_{l-1})^{\alpha}$, is oscillatory for every constant $0 < \lambda < 1$, every $T \geqslant t_0$, and every integer $1 \leqslant l \leqslant n-1$, then Equation (1.1) is oscillatory.

P r o o f. The proof follows from Theorem 3.1 and Lemma 2.4. □

Corollary 3.1. *Assume that $\alpha = 1$ and $q(t) > 0$. If the second order delay dynamic equation*

$$u^{\Delta\Delta}(t) + \frac{\lambda}{\left(\beta_{n-2} \prod_{i=1}^{n-3} \beta_i\right)^2} (\varphi(t) - \sigma(T))^{n-2} q(t) u(\varphi(t)) = 0$$

is oscillatory for every constant $0 < \lambda < 1$ and every $T \geqslant t_0$, then Equation (1.1) is oscillatory.

Several oscillation criteria for Equation (1.1) can now be obtained from known oscillation criteria that already exist for (3.2) by means of Theorem 3.1. At this stage, we will give some examples to illustrate the extent of the use of Theorem 3.1. The following results deal with the second order delay dynamic equation

$$x^{\Delta\Delta}(t) + p(t)|x(\varphi(t))|^{\alpha-1} x(\varphi(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{3.8}$$

where $p \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $p(t) \geqslant 0$, $p(t) \not\equiv 0$ on $[T, \infty)_{\mathbb{T}}$ for any $T \geqslant t_0$. The first several results consider the superlinear and linear cases.

Theorem 3.3 ([6]). *Suppose that $\alpha \geqslant 1$ and*

$$\int^{\infty} \sigma(s) p(s) \, \Delta s = \infty. \tag{3.9}$$

If

$$\limsup_{t \to \infty} \left\{ t \int_t^{\infty} p(s) \left(\frac{\varphi(s)}{s} \right)^{\alpha} \Delta s \right\} = \infty, \tag{3.10}$$

then Equation (3.8) is oscillatory.

Theorem 3.4 ([6]). *Suppose that $\alpha > 1$ and (3.9) holds. If*

$$\limsup_{t \to \infty} \int_{t_0}^{t} p(s) \sigma(s) \left(\frac{\varphi(s)}{\sigma(s)} \right)^{\alpha} \Delta s = \infty, \tag{3.11}$$

then Equation (3.8) is oscillatory.

Theorem 3.5 ([9]). *Suppose that* $\alpha > 1$ *and* $\sigma(t) = O(\varphi(t))$ *as* $t \to \infty$. *If* (3.9) *holds, then Equation* (3.8) *is oscillatory.*

Corollary 3.2 ([6]). *Suppose that* $\alpha \geqslant 1$ *and* (3.9) *holds. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} p(s)\Big(\frac{\varphi(s)}{\sigma(s)}\Big)^{\alpha} \Delta s = \infty, \tag{3.12}$$

then Equation (3.8) *is oscillatory.*

Similarly, for the sublinear case we have:

Theorem 3.6 ([10]). *Suppose that* $0 < \alpha < 1$. *If*

$$\int^{\infty} (\varphi(s))^{\alpha} p(s)\, \Delta s = \infty, \tag{3.13}$$

then Equation (3.8) *is oscillatory.*

Corollary 3.3 ([6]). *Suppose that* $0 < \alpha \leqslant 1$ *and* (3.9) *holds. If* (3.12) *holds, then Equation* (3.8) *is oscillatory.*

4. Applications

Corollary 4.1. *Suppose that* $\alpha \geqslant 1$ *and*

$$\int^{\infty} \sigma(s)(\varphi(s))^{n-2} q(s)\, \Delta s = \infty. \tag{4.1}$$

If

$$\limsup_{t\to\infty} \Big\{ t \int_{t}^{\infty} s^{-\alpha}(\varphi(s))^{n+\alpha-2} q(s) \Delta s \Big\} = \infty, \tag{4.2}$$

then Equation (1.1) *is oscillatory.*

Proof. Conditions (4.1) and (4.2) are sufficient for (3.9) and (3.10) to hold with

$$p(t) = \lambda^{\alpha} \frac{g_{n-l-1}(\varphi(t), T) g_{l-1}^{\alpha}(\sigma(T), \varphi(t))}{\beta_{n-l-1}(\beta_{l-1})^{\alpha}}\, q(t),$$

respectively. Note that if the conditions (3.9) and (3.10) are satisfied for $l = 1$, then they hold for all $1 \leqslant l \leqslant n-1$. Hence (3.2) can not have an eventually positive solution. $\square$

Corollary 4.2. *Suppose that* $\alpha > 1$ *and* (4.1) *holds. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} (\varphi(s))^{n+\alpha-2}(\sigma(s))^{1-\alpha} q(s)\,\Delta s = \infty, \tag{4.3}$$

then Equation (1.1) *is oscillatory.*

P r o o f. Conditions (4.1) and (4.3) are sufficient for (3.9) and (3.11) to hold with

$$p(t) = \lambda^{\alpha} \frac{g_{n-l-1}(\varphi(t), T) g_{l-1}^{\alpha}(\sigma(T), \varphi(t))}{\beta_{n-l-1}(\beta_{l-1})^{\alpha}} q(t),$$

respectively. Note that if the conditions (3.9) and (3.11) are satisfied for $l = 1$, then they hold for all $1 \leqslant l \leqslant n-1$. Hence (3.2) can not have an eventually positive solution. $\square$

Corollary 4.3. *Suppose that* $\alpha > 1$ *and* $\sigma(t) = O(\varphi(t))$ *as* $t \to \infty$. *If* (4.1) *holds, then Equation* (1.1) *is oscillatory.*

P r o o f. Condition (4.1) is sufficient for (3.9) to hold with

$$p(t) = \lambda^{\alpha} \frac{g_{n-l-1}(\varphi(t), T) g_{l-1}^{\alpha}(\sigma(T), \varphi(t))}{\beta_{n-l-1}(\beta_{l-1})^{\alpha}} q(t).$$

Note that if (3.9) is satisfied for $l = 1$, then it holds for all $1 \leqslant l \leqslant n-1$. Hence (3.2) can not have an eventually positive solution. $\square$

Corollary 4.4. *Suppose that* $\alpha \geqslant 1$ *and* (4.1) *holds. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} (\varphi(s))^{n+\alpha-2}(\sigma(s))^{-\alpha} q(s)\,\Delta s = \infty, \tag{4.4}$$

then Equation (1.1) *is oscillatory.*

P r o o f. Conditions (4.1) and (4.4) are sufficient for (3.9) and (3.12) to hold with

$$p(t) = \lambda^{\alpha} \frac{g_{n-l-1}(\varphi(t), T) g_{l-1}^{\alpha}(\sigma(T), \varphi(t))}{\beta_{n-l-1}(\beta_{l-1})^{\alpha}} q(t),$$

respectively. Note that if the conditions (3.9) and (3.12) are satisfied for $l = 1$, then they hold for all $1 \leqslant l \leqslant n-1$. Hence (3.2) can not have an eventually positive solution. $\square$

Corollary 4.5. *Suppose that* $0 < \alpha < 1$. *If*

$$\int^{\infty} (\varphi(s))^{\alpha(n-1)} q(s)\,\Delta s = \infty, \tag{4.5}$$

then Equation (1.1) *is oscillatory.*

Proof. Condition (4.5) is sufficient for (3.13) to hold with

$$p(t) = \lambda^{\alpha} \frac{g_{n-l-1}(\varphi(t), T) g_{l-1}^{\alpha}(\sigma(T), \varphi(t))}{\beta_{n-l-1}(\beta_{l-1})^{\alpha}}\, q(t).$$

Note that if (3.13) is satisfied for $l = n-1$, then it holds for all $1 \leqslant l \leqslant n-1$. Hence (3.2) can not have an eventually positive solution. □

Corollary 4.6. *Suppose that* $0 < \alpha \leqslant 1$ *and*

$$\int^{\infty} \sigma(s)(\varphi(s))^{\alpha(n-2)} q(s)\,\Delta s = \infty. \tag{4.6}$$

If

$$\limsup_{t\to\infty} \int_{t_0}^{t} (\varphi(s))^{\alpha(n-1)} (\sigma(s))^{-\alpha} q(s)\,\Delta s = \infty, \tag{4.7}$$

then Equation (1.1) *is oscillatory.*

Proof. Conditions (4.6) and (4.7) are sufficient for (3.9) and (3.12) to hold with

$$p(t) = \lambda^{\alpha} \frac{g_{n-l-1}(\varphi(t), T) g_{l-1}^{\alpha}(\sigma(T), \varphi(t))}{\beta_{n-l-1}(\beta_{l-1})^{\alpha}}\, q(t),$$

respectively. Note that if the conditions (3.9) and (3.12) are satisfied for $l = n-1$, then they hold for all $1 \leqslant l \leqslant n-1$. Hence (3.2) can not have an eventually positive solution. □

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