Higher Order Fractional Variational Optimal Control Problems with Delayed Arguments

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Abstract

This article deals with higher order Caputo fractional variational problems with the presence of delay in the state variables and their integer higher order derivatives.

1 Introduction

Lagrangian theories involving higher-order derivatives appear of great interest as an imbedding for field theories with fields of subcanonical dimension,e.g. the Heisenberg nonlinear spinor theory, for which local interactions are less singular than in the canonical case.

The calculus of variation has a long history of communications with other fields of mathematics such as geometry and differential equations, and with physics. Recently the calculus of variations has found applications in economics

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and some branches of engineering such as electrical engineering. Optimal control which is a rapidly expanded field can be regarded as a part of the calculus of variations.

Recently, the fractional calculus which is as old as the classical calculus has become a candidate in solving problems of complex systems which appear in various branches of science [1, 2, 6, 4, 26, 7, 12, 13].

Several authors found interesting results when they used the fractional calculus in control theory [23, 24].

Experimentally, the use of delay together with the fractional calculus may give better results.

Optimal control problems with time delay in calculus of variations were discussed in [25]. Variational optimal control problems within fractional derivatives were considered in [27]. Fractional variational problems in the presence of delay were studied in [9, 10].

The aim of this paper is to deal with optimal control fractional variational problems in the presence of delay in the state variable and its higher order derivatives

The structure of the paper is as follows:

In section 2 the necessary definitions in the fractional calculus used in this manuscript are reviewed. In section 3 the unconstrained fractional Euler-Lagrange equations with delay are discussed . The fractional control problem is presented in Section 5.

2 Basic Definitions

We present in this section some basic definitions related to fractional derivatives.

The Left Riemann-Liouville Fractional Integral and *The Right Riemann-Liouville Fractional Integral* are defined respectively by

$$ {}_aI^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{1}$$

$$ I_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau) d\tau, \tag{2}$$

where $\alpha > 0,\ n-1 < \alpha < n$. Here and in the following $\Gamma(\alpha)$ represents the Gamma function.

The Left Riemann-Liouville Fractional Derivative is defined by

$$ {}_aD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \tag{3}$$

The Right Riemann-Liouville Fractional Derivative is defined by

$$ D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau, \tag{4}$$

The fractional derivative of a constant takes the form

$$ {}_aD^\alpha C = C\ \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{5} $$

and the fractional derivative of a power of t has the following form

$$ {}_aD^\alpha (t-a)^\beta = \frac{\Gamma(\alpha+1)(t-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, \tag{6} $$

for $\beta > -1, \alpha \geq 0$.

The Caputo's fractional derivatives are defined as follows:

The Left Caputo Fractional Derivative

$$ {}_a^CD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau)d\tau, \tag{7} $$

and

The Right Caputo Fractional Derivative

$$ {}^CD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} \left(-\frac{d}{d\tau}\right)^n f(\tau)d\tau, \tag{8} $$

where α represents the order of the derivative such that $n-1 < \alpha < n$. By definition the Caputo fractional derivative of a constant is zero.

The Riemann-Liouville fractional derivatives and Caputo fractional derivatives are connected with each other by the following relations:

$$ {}_a^CD^\alpha f(t) = {}_aD^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}, \tag{9} $$

$$ {}^CD_b^\alpha f(t) = D_b^\alpha f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha}. \tag{10} $$

In [1], a formula for the fractional integration by parts on the whole interval $[a, b]$ was given by the following lemma

Lemma 2.1. *Let $\alpha > 0$, $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$)*

(a) If $\varphi \in L_p(a, b)$ and $\psi \in L_q(a, b)$, then

$$ \int_a^b \varphi(t)({}_aI^\alpha\psi)(t)dt = \int_a^b \psi(t)(I_b^\alpha\varphi)(t)dt \tag{11} $$

(b) If $g \in I_b^\alpha(L_p)$ and $f \in {}_aI^\alpha(L_q)$, then

$$\int_a^b g(t)({}_aD^\alpha f)(t)dt = \int_a^b f(t)(D_b^\alpha g)(t)dt, \tag{12}$$

where ${}_aI^\alpha(L_p) := \{f : f = {}_aI^\alpha g, g \in L_p(a,b)\}$ and
$I_b^\alpha(L_p) := \{f : f = I_b^\alpha g, g \in L_p(a,b)\}$.

In [9] and [10] , other formulas for the fractional integration by parts on the subintervals $[a,r]$ and $[r,b]$ were given by the following lemmas

Lemma 2.2. *Let $\alpha > 0$, $p, q \geq 1$, $r \in (a,b)$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$).*

(a)If $\varphi \in L_p(a,b)$ and $\psi \in L_q(a,b)$, then

$$\int_a^r \varphi(t)({}_aI^\alpha \psi)(t)dt = \int_a^r \psi(t)(I_r^\alpha \varphi)(t)dt \tag{13}$$

and thus if $g \in I_r^\alpha(L_p)$ and $f \in {}_aI^\alpha(L_q)$, then

$$\int_a^r g(t)(\ {}_aD^\alpha f)(t)dt = \int_a^r f(t)(D_r^\alpha g)(t)dt \tag{14}$$

(b)If $\varphi \in L_p(a,b)$ and $\psi \in L_q(a,b)$, then

$$\int_r^b \varphi(t)({}_aI^\alpha \psi)(t)dt = \int_r^b \psi(t)(\ I_b^\alpha \varphi)(t)dt$$
$$+ \frac{1}{\Gamma(\alpha)} \int_a^r \psi(t)(\int_r^b \varphi(s)(s-t)^{\alpha-1}ds)dt \tag{15}$$

and hence if $g \in I_b^\alpha(L_p)$ and $f \in {}_aI^\alpha(L_q)$, then

$$\int_r^b g(t)(\ {}_aD^\alpha f)(t)dt = \int_r^b f(t)(D_b^\alpha g)(t)dt$$
$$- \frac{1}{\Gamma(\alpha)} \int_a^r (\ {}_aD^\alpha f)(t)(\int_r^b (D_b^\alpha g)(s)(s-t)^{\alpha-1}ds)dt. \tag{16}$$

That is

$$\int_r^b g(t)(\ {}_aD^\alpha f)(t)dt = \int_r^b f(t)(D_b^\alpha g)(t)dt$$
$$- \frac{1}{\Gamma(\alpha)} \int_a^r f(t)D_r^\alpha(\int_r^b (D_b^\alpha g)(s)(s-t)^{\alpha-1}ds)dt. \tag{17}$$

Lemma 2.3. *Let $\alpha > 0$, $p, q \geq 1$, $r \in (a, b)$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$).*

(a)If $\varphi \in L_p(a, b)$ and $\psi \in L_q(a, b)$, then

$$\int_r^b \varphi(t)(I_b^\alpha \psi)(t)dt = \int_r^b \psi(t)({}_rI^\alpha \varphi)(t)dt \tag{18}$$

and thus if $g \in {}_rI^\alpha(L_p)$ and $f \in I_b^\alpha(L_q)$, then

$$\int_r^b g(t)(D_b^\alpha f)(t)dt = \int_r^b f(t)({}_rD^\alpha g)(t)dt \tag{19}$$

(b)If $\varphi \in L_p(a, b)$ and $\psi \in L_q(a, b)$, then

$$\int_a^r \varphi(t)({}_bI^\alpha \psi)(t)dt = \int_a^r \psi(t)(I_a^\alpha \varphi)(t)dt$$

$$+ \frac{1}{\Gamma(\alpha)} \int_r^b \psi(t)(\int_a^r \varphi(s)(t - s)^{\alpha-1}ds)dt \tag{20}$$

and hence if $g \in {}_aI^\alpha(L_p)$ and $f \in I_b^\alpha(L_q)$, then

$$\int_a^r g(t)(D_b^\alpha f)(t)dt = \int_a^r f(t)({}_aD^\alpha g)(t)dt$$

$$- \frac{1}{\Gamma(\alpha)} \int_r^b (D_b^\alpha f)(t)(\int_a^r ({}_aD^\alpha g)(s)(t - s)^{\alpha-1}ds)dt. \tag{21}$$

That is

$$\int_a^r g(t)(D_b^\alpha f)(t)dt = \int_a^r f(t)({}_aD^\alpha g)(t)dt$$

$$- \frac{1}{\Gamma(\alpha)} \int_r^b f(t){}_rD^\alpha(\int_a^r ({}_aD^\alpha g)(s)(t - s)^{\alpha-1}ds)dt. \tag{22}$$

3 The Unconstrained Caputo Fractional Variation with delay

Before we consider the fractional control problem, let us consider the following fractional variational problem with delay

Minimize

$$J(y) = \int_a^b L[t,\ {}_a^CD^{\alpha_1}y(t),\ {}_a^CD^{\alpha_2}y(t), ...,\ {}_a^CD^{\alpha_n}y(t),$$

$${}^CD_b^{\beta_1}y(t),\ {}^CD_b^{\beta_2}y(t), ...,\ {}^CD_b^{\beta_m}y(t), y(t), y'(t), ..., y^{(k)}(t),$$

$$y(t-\tau), y'(t-\tau), ..., y^{(k)}(t-\tau)]dt, \tag{23}$$

such that

$$k-1 \leq \alpha\text{max} < k, \quad \alpha\text{max} = \max\{\alpha_i, \beta_j\}_{1\leq i\leq n, 1\leq j\leq m},$$
$$y^{(l)}(b) = c_l, \ l = 0,1,2,...,k-1, \quad y(t) = \phi(t) \ t \in [a-\tau, a],$$
$$\tau > 0, \ \tau < b-a, \quad \alpha_i, \beta_j \in \mathbb{R} \ \ \forall \ i = 1,2,...,n, \quad \forall j = 1,2,...,m, \tag{24}$$

where c_l are constant, $\phi(t)$ is a smooth function and L is a function with continuous first and second partial derivatives with respect to all of its arguments.

If the above variational problem (23) has a minimum at $y_0(t)$ and $\eta(t) \in \mathbb{R}$ is an admissible function such that $\eta(t) \equiv 0$ in the interval $[a-\tau, a]$ then the function

$$\xi(t) = J(y_0 + \epsilon\eta) \tag{25}$$

where $\epsilon \in \mathbb{R}$ admits a minimum at $\epsilon = 0$. Hence

$$\int_a^b [\sum_{i=1}^n \partial_{i+1} L(t) \ {}_a^C D^{\alpha_i} \eta(t) + \sum_{j=1}^m \partial_{n+j+1} L(t) {}^C D_b^{\beta_j} \eta(t) +$$
$$\sum_{p=0}^k \partial_{m+n+p+2} L(t) \eta^{(p)}(t) + \sum_{p=0}^k \partial_{m+n+k+3} L(t) \eta^{(p)}(t-\tau)]dt = 0. \tag{26}$$

where $\partial_i L$ is the partial derivative of L with respect to its i^{th} argument.

On using the connection formulas (9) and (10), (26) reads

$$\int_a^b [\sum_{i=1}^n \partial_{i+1} L(t) \ {}_a D^{\alpha_i} \eta(t) + \sum_{j=1}^m \partial_{n+j+1} L(t) D_b^{\beta_j} \eta(t) +$$
$$\sum_{p=0}^k \partial_{m+n+p+2} L(t) \eta^{(p)}(t) + \sum_{p=0}^k \partial_{m+n+k+3} L(t) \eta^{(p)}(t-\tau)]dt = 0. \tag{27}$$

Now if one splits the integral, makes the change of variables for $t-\tau$ and uses the fact that $\eta \equiv 0$ in $[a-\tau, a]$, (27) becomes

$$\int_a^{b-\tau} [\sum_{i=1}^n \partial_{i+1} L(t) \ {}_a D^{\alpha_i} \eta(t) + \sum_{j=1}^m \partial_{n+j+1} L(t) D_b^{\beta_j} \eta(t) +$$
$$\sum_{p=0}^k \partial_{m+n+p+2} L(t) \eta^{(p)}(t) + \sum_{p=0}^k \partial_{m+n+k+3} L(t+\tau) \eta^{(p)}(t)]dt +$$

$$\int_{b-\tau}^{b}[\sum_{i=1}^{n}\partial_{i+1}L(t)\ _aD^{\alpha_i}\eta(t)+\sum_{j=1}^{m}\partial_{n+j+1}L(t)D_b^{\beta_j}\eta(t)+$$

$$\sum_{p=0}^{k}\partial_{m+n+p+2}L(t)\eta^{(p)}(t)]dt=0 \tag{28}$$

By using the integration by parts formulas in the mentioned above Lemma 2.1, Lemma 2.2 and Lemma 2.3 and the usual integration by parts formula , one obtains the following

$$\int_{a}^{b-\tau}[\sum_{i=1}^{n}D_{b-\tau}^{\alpha_i}(\partial_{i+1}L)(t)+\sum_{j=1}^{m}\ _aD^{\beta_j}(\partial_{n+j+1}L)(t)+$$

$$\sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}\ (\partial_{m+n+p+2}L)(t)+\sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}\ (\partial_{m+n+k+p+3}L)(t+\tau)-$$

$$\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}D_{b-\tau}^{\alpha_i}(\int_{b-\tau}^{b}(D_b^{\alpha_i}\partial_{i+1}L)(s)(s-t)^{\alpha_i-1}ds)]\eta(t)dt+$$

$$\int_{b-\tau}^{b}[\sum_{i=1}^{n}D_b^{\alpha_i}(\partial_{i+1}L)(t)+\sum_{j=1}^{m}\ _{b-\tau}D^{\beta_j}(\partial_{n+j+1}L)(t)-$$

$$\sum_{j=1}^{m}\frac{1}{\Gamma(\beta_j)}\ _{b-\tau}D^{\beta_j}(\int_{a}^{b-\tau}(\ _aD^{\beta_j}\partial_{n+j+1}L)(s)(t-s)^{\beta_j-1}ds)]\eta(t)dt+$$

$$\sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}\ (\partial_{m+n+p+2}L)(t)]\eta(t)dt+$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\partial_{m+n+p+2}L)(t)\eta^{p-q-1}(t)|_a^{b-\tau}+$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\partial_{m+n+k+3}L)(t+\tau)\eta^{p-q-1}(t)|_a^{b-\tau}+$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\partial_{m+n+p+2}L)(t)\eta^{p-q-1}(t)|_{b-\tau}^{b}=0 \tag{29}$$

In equation (29) if one chooses η such that $\eta(a)=0$ and $\eta\equiv 0$ on $[b-\tau,b]$, one gets

$$\sum_{i=1}^{n}D_{b-\tau}^{\alpha_i}(\partial_{i+1}L)(t)+\sum_{j=1}^{m}\ _aD^{\beta_j}(\partial_{n+j+1}L)(t)+$$

$$\sum_{p=0}^{k}(-1)^p \frac{d^p}{dt^p}\ (\partial_{m+n+p+2}L)(t)+\sum_{p=0}^{k}(-1)^p \frac{d^p}{dt^p}\ (\partial_{m+n+k+p+3}L)(t+\tau)-$$

$$\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}D^{\alpha_i}_{b-\tau}(\int_{b-\tau}^{b}(D^{\alpha_i}_{b}\partial_{i+1}L)(s)(s-t)^{\alpha_i-1}ds)=0. \tag{30}$$

In equation (29) if one chooses η such that $\eta^{(l)}(b)=0$ and $\eta\equiv 0$ on $[a,b-\tau]$, one gets

$$\sum_{i=1}^{n}D^{\alpha_i}_{b}(\partial_{i+1}L)(t)+\sum_{j=1}^{m}\ {}_{b-\tau}D^{\beta_j}(\partial_{n+j+1}L)(t)-$$

$$\sum_{j=1}^{m}\frac{1}{\Gamma(\beta_j)}\ {}_{b-\tau}D^{\beta_j}(\int_{a}^{b-\tau}(\ {}_aD^{\beta_j}\partial_{n+j+1}L)(s)(t-s)^{\beta_j-1}ds)+$$

$$\sum_{p=0}^{k}(-1)^p \frac{d^p}{dt^p}\ (\partial_{m+n+p+2}L)(t)=0. \tag{31}$$

Now since both integrals in (29) are now zero, one gets

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q \frac{d^q}{dt^q}\ (\partial_{m+n+p+2}L)(t)\eta^{p-q-1}(t)|_{a}^{b-\tau}+$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q \frac{d^q}{dt^q}\ (\partial_{m+n+k+3}L)(t)\eta^{p-q-1}(t+\tau)|_{a}^{b-\tau}+$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q \frac{d^q}{dt^q}\ (\partial_{m+n+p+2}L)(t)\eta^{p-q-1}(t)|_{b-\tau}^{b}=0. \tag{32}$$

Thus one can state the following theorem

Theorem 3.1. *Let $J(y)$ be a functional of the form (23) defined on a set of continuous functions $y(t)$ which have continuous Caputo fractional left order derivatives of orders α_i and right derivative of order β_j in $[a,b]$ and satisfy the conditions in (24. Let $L:[a-\tau,b]\times\mathbb{R}^{m+n+2k+2}\to\mathbb{R}$ have continuous first and second partial derivatives with respect to all of its arguments. Then the necessary conditions that $J(y)$ possesses a minimum at $y(x)$ are the Euler-Lagrange equations*

$$\sum_{i=1}^{n}D^{\alpha_i}_{b-\tau}(\frac{\partial L}{\partial\ {}_a^CD^{\alpha_i}y(t)})(t)+\sum_{j=1}^{m}\ {}_aD^{\beta_j}(\frac{\partial L}{\partial\ {}^CD^{\beta_j}_{b}y(t)})(t)+$$

$$\sum_{p=0}^{k}(-1)^p \frac{d^p}{dt^p}(\frac{\partial L}{\partial y^{(p)}(t)})(t)+\sum_{p=0}^{k}(-1)^p \frac{d^p}{dt^p}(\frac{\partial L}{\partial y^{(p)}(t-\tau)})(t+\tau)-$$

$$\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}D_{b-\tau}^{\alpha_i}(\int_{b-\tau}^{b}(D_b^{\alpha_i}(\frac{\partial L}{\partial\ {}_a^C D^{\alpha_i}y(t)})(s)(s-t)^{\alpha_i-1}ds)=0 \tag{33}$$

for $a\leq t\leq b-\tau$,

$$\sum_{i=1}^{n}D_b^{\alpha_i}(\frac{\partial L}{\partial\ a^C D^{\alpha_i}y(t)})(t)+\sum_{j=1}^{m}D_{b-\tau}^{\beta_j}(\frac{\partial L}{\partial\ {}^C D_b^{\beta_j}y(t)})(t)+$$

$$\sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}(\frac{\partial L}{\partial y^{(p)}(t)})(t)-$$

$$\sum_{j=1}^{m}\frac{1}{\Gamma(\beta_j)}\ {}_{b-\tau}D^{\beta_j}(\int_a^{b-\tau}(\ {}_aD^{\beta_j}(\frac{\partial L}{\partial\ {}^C D_b^{\beta_j}y(t)})(s)(t-s)^{\beta_j-1}ds)=0 \tag{34}$$

for $b-\tau\leq t\leq b$, *and the transversality conditions*

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\frac{\partial L}{\partial y^{(p)}(t)})(t)\eta^{p-q-1}(t)|_a^{b-\tau}+$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\frac{\partial L}{\partial y^{(p)}(t-\tau)}(t+\tau)\eta^{p-q-1}(t+\tau)|_a^{b-\tau}+$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\frac{\partial L}{\partial y^{(p)}(t)})(t)\eta^{p-q-1}(t)|_{b-\tau}^{b}=0. \tag{35}$$

for any admissible function η *satisfying* $\eta(t)\equiv 0\ t\in[a-\tau,a]$,
$\eta^{(l)}(b)=0,\ l=0,1,2,...,k-1.$

Theorem 3.1 can be generalized as follows

Theorem 3.2. *Consider the functional of the form*

$$J(y_1,y_2,...,y_d)=\int_a^b L[t,\ {}_a^C D^{\alpha_1}y_1(t),\ {}_a^C D^{\alpha_2}y_1(t),...,\ {}_a^C D^{\alpha_n}y_1(t),$$

$${}_a^C D^{\alpha_1}y_2(t),\ {}_a^C D^{\alpha_2}y_2(t),...,\ {}_a^C D^{\alpha_n}y_2(t),...,$$

$${}_a^C D^{\alpha_1}y_d(t),\ {}_a^C D^{\alpha_2}y_d(t),...,\ {}_a^C D^{\alpha_n}y_d(t),$$

$${}^C D_b^{\beta_1}y_1(t),\ {}^C D_b^{\beta_2}y_1(t),...,\ {}^C D_b^{\beta_m}y_1(t),$$

$${}^C D_b^{\beta_1}y_2(t),\ {}^C D_b^{\beta_2}y_2(t),...,\ {}^C D_b^{\beta_m}y_2(t),...,$$

$${}^C D_b^{\beta_1}y_d(t),\ {}^C D_b^{\beta_2}y_d(t),...,\ {}^C D_b^{\beta_m}y_d(t),$$

$$y_1(t),y_1'(t),...,y_1^{(k)}(t),y_2(t),y_2'(t),...,y_2^{(k)}(t),...,$$

$$y_d(t), y_d'(t), ..., y_d^{(k)}(t), y_1(t-\tau), y_1'(t-\tau), ..., y_1^{(k)}(t-\tau),$$

$$y_2(t-\tau), y_2'(t-\tau), ..., y_2^{(k)}(t-\tau), ..., y_d(t-\tau), y_d'(t-\tau), ..., y_d^{(k)}(t-\tau)]dt, \quad (36)$$

,defined on sets of continuous functions $y_i(x),\ i = 1, 2, ..., d$ that have left Caputo fractional derivatives of order $\alpha_i \in \mathbb{R},\ i = 1, 2, ..., n$ and right Caputo fractional derivatives of order $\beta_j \in \mathbb{R},\ j = 1, 2, ...m$ in the interval $[a, b]$ and satisfy the conditions

$$y_i^{(l)}(b) = c_{il},\ l = 0, 1, ..., k,\ y_i(t) = \phi_i(t)\ i = 1, 2..., d,\ t \in [a-\tau, a]$$

$$a < b,\ \tau > 0, \quad \tau < b - a\ , \quad (37)$$

where $k - 1 \leq \alpha_{\max} < k,\ \ \alpha_{\max} = \max\{\alpha_i, \beta_j\}_{1\leq i\leq n, 1\leq j\leq m}$, c_{il}'s are constant and ϕ_i's are smooth functions and
$L : [a-\tau, b] \times \mathbb{R}^{d(m+n+2k+2)} \to \mathbb{R}$ is a function with continuous first and second partial derivatives with respect to all of its arguments. For $y_i(x),\ i = 1, 2, ..., d$, satisfying (37) to be a minimum of (36), it is necessary that

$$\sum_{i=1}^{n} D_{b-\tau}^{\alpha_i}\left(\frac{\partial L}{\partial\ a^C D^{\alpha_i} y_z(t)}\right)(t) + \sum_{j=1}^{m}\ {}_aD^{\beta_j}\left(\frac{\partial L}{\partial\ {}^C D_b^{\beta_j} y_z(t)}\right)(t) +$$

$$\sum_{p=0}^{k}(-1)^p \frac{d^p}{dt^p}\left(\frac{\partial L}{\partial y_z^{(p)}(t)}\right)(t) + \sum_{p=0}^{k}(-1)^p \frac{d^p}{dt^p}\left(\frac{\partial L}{\partial y_z^{(p)}(t-\tau)}\right)(t+\tau) -$$

$$\sum_{i=1}^{n} \frac{1}{\Gamma(\alpha_i)} D_{b-\tau}^{\alpha_i}\left(\int_{b-\tau}^{b} (D_b^{\alpha_i}\left(\frac{\partial L}{\partial\ {}_a^C D^{\alpha_i} y_z(t)}\right)(s)(s-t)^{\alpha_i - 1} ds\right) = 0 \quad (38)$$

for $a \leq t \leq b - \tau,\ z = 1, 2, ..., d$

$$\sum_{i=1}^{n} D_b^{\alpha_i}\left(\frac{\partial L}{\partial\ a^C D^{\alpha_i} y_z(t)}\right)(t) + \sum_{j=1}^{m} D_{b-\tau}^{\beta_j}\left(\frac{\partial L}{\partial\ {}^C D_b^{\beta_j} y_z(t)}\right)(t) +$$

$$\sum_{p=0}^{k}(-1)^p \frac{d^p}{dt^p}\left(\frac{\partial L}{\partial y_z^{(p)}(t)}\right)(t) -$$

$$\sum_{j=1}^{m} \frac{1}{\Gamma(\beta_j)}\ {}_{b-\tau}D^{\beta_j}\left(\int_{a}^{b-\tau} (\ {}_aD^{\beta_j}\left(\frac{\partial L}{\partial\ {}^C D_b^{\beta_j} y_z(t)}\right)(s)(t-s)^{\beta_j - 1} ds\right) = 0 \quad (39)$$

for $b - \tau \leq t \leq b,\ z = 1, 2, ..., d$ and the transversality conditions

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q \frac{d^q}{dt^q}\ \left(\frac{\partial L}{\partial y_z^{(p)}(t)}\right)(t)\eta_z^{p-q-1}(t)|_a^{b-\tau} +$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\frac{\partial L}{\partial y_z^{(p)}(t-\tau)}(t+\tau)\eta_z^{p-q-1}(t+\tau)|_a^{b-\tau}+$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\frac{\partial L}{\partial y_z^{(p)}(t)})(t)\eta_z^{p-q-1}(t)|_{b-\tau}^{b}=0. \tag{40}$$

for any admissible vector function $\eta=(\eta_1,\eta_2,...,\eta_d)$ satisfying $\eta(t)\equiv(0,0,...,0)\ t\in[a-\tau,a],\ \eta^{(l)}(b)=(0,0,...,0),\ d=0,1,2,...,k-1.$

4 The Fractional Optimal Control Problem

Find the optimal control variable $u(t)$ which minimizes the performance index

$$J(y,u)=\int_a^b F[t,u(t),\ {}_a^CD^{\alpha_1}y(t),\ {}_a^CD^{\alpha_2}y(t),...,\ {}_a^CD^{\alpha_n}y(t),$$

$${}^CD_b^{\beta_1}y(t),\ {}^CD_b^{\beta_2}y(t),...,\ {}^CD_b^{\beta_m}y(t),y(t),y'(t),...,y^{(k)}(t),$$

$$y(t-\tau),y'(t-\tau),...,y^{(k)}(t-\tau)]dt, \tag{41}$$

subject to the constraint

$$G[t,u(t),\ {}_a^CD^{\alpha_1}y(t),\ {}_a^CD^{\alpha_2}y(t),...,\ {}_a^CD^{\alpha_n}y(t),$$

$${}^CD_b^{\beta_1}y(t),\ {}^CD_b^{\beta_2}y(t),...,\ {}^CD_b^{\beta_m}y(t),y(t),y'(t),...,y^{(k)}(t),$$

$$y(t-\tau),y'(t-\tau),...,y^{(k)}(t-\tau)]=0 \tag{42}$$

such that

$$y^{(l)}(b)=c_l,\ l=0,1,2,...,k-1,\ y(t)=\phi(t)\ t\in[a-\tau,a],\ a<b,\tau>0$$

$$\tau<b-a,\ \alpha_i\in\mathbb{R},\ i=1,2,...,n,\ \beta_j\in\mathbb{R},\ j=1,2,...,m, \tag{43}$$

where c_l are constant and F and G are functions $[a-\tau,b]\times\mathbb{R}^{n+m+2k+3}\to\mathbb{R}$ with continuous first and second partial derivatives with respect to all of their arguments arguments.

To find the optimal control, one defines a modified performance index as

$$\hat{J}(y,u)=\int_a^b F+\lambda(t)G\ dt, \tag{44}$$

where λ is a Lagrange multiplier or an adjoint variable. Using the conditions (38), (39) and (40) in Theorem 3.2, the following necessary equations for optimal control are found: Euler-Lagrange equations

$$\sum_{i=1}^{n}D_{b-\tau}^{\alpha_i}(\frac{\partial F}{\partial\ a^CD^{\alpha_i}y(t)})(t)+\sum_{i=1}^{n}D_{b-\tau}^{\alpha_i}(\lambda\frac{\partial G}{\partial\ a^CD^{\alpha_i}y(t)})(t)+$$

$$\sum_{j=1}^{m} {}_{a}D^{\beta_j}(\frac{\partial F}{\partial\ {}^{C}D_b^{\beta_j}y(t)})(t)+\sum_{j=1}^{m} {}_{a}D^{\beta_j}(\lambda\frac{\partial G}{\partial\ {}^{C}D_b^{\beta_j}y(t)})(t)+$$

$$\sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}(\frac{\partial F}{\partial y^{(p)}(t)})(t)+\sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}(\lambda\frac{\partial G}{\partial y^{(p)}(t)})(t)+$$

$$+\sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}(\frac{\partial F}{\partial y^{(p)}(t-\tau)})(t+\tau)++\sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}(\lambda\frac{\partial G}{\partial y^{(p)}(t-\tau)})(t+\tau)-$$

$$\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}D_{b-\tau}^{\alpha_i}(\int_{b-\tau}^{b}(D_b^{\alpha_i}(\frac{\partial F}{\partial\ {}_a^C D^{\alpha_i}y(t)})(s)(s-t)^{\alpha_i-1}ds)-$$

$$\sum_{i=1}^{n}\frac{1}{\Gamma(\alpha_i)}D_{b-\tau}^{\alpha_i}(\int_{b-\tau}^{b}(D_b^{\alpha_i}(\lambda\frac{\partial G}{\partial\ {}_a^C D^{\alpha_i}y(t)})(s)(s-t)^{\alpha_i-1}ds)+$$

$$\frac{\partial F}{\partial u(t)}(t)+\lambda(t)\frac{\partial G}{\partial u(t)}(t)=0, \tag{45}$$

for $a\le t\le b-\tau$,

$$\sum_{i=1}^{n}D_b^{\alpha_i}(\frac{\partial F}{\partial\ a^C D^{\alpha_i}y(t)})(t)+\sum_{i=1}^{n}D_b^{\alpha_i}(\lambda\frac{\partial G}{\partial\ a^C D^{\alpha_i}y(t)})(t)+$$

$$\sum_{j=1}^{m}D_{b-\tau}^{\beta_j}(\frac{\partial F}{\partial\ {}^{C}D_b^{\beta_j}y(t)})(t)+\sum_{j=1}^{m}D_{b-\tau}^{\beta_j}(\lambda\frac{\partial G}{\partial\ {}^{C}D_b^{\beta_j}y(t)})(t)+$$

$$\sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}(\frac{\partial F}{\partial y^{(p)}(t)})(t)+\sum_{p=0}^{k}(-1)^p\frac{d^p}{dt^p}(\lambda\frac{\partial G}{\partial y^{(p)}(t)})(t)+$$

$$\sum_{j=1}^{m}\frac{1}{\Gamma(\beta_j)}\ {}_{b-\tau}D^{\beta_j}(\int_a^{b-\tau}(\ {}_aD^{\beta_j}(\frac{\partial F}{\partial\ {}^{C}D_b^{\beta_j}y(t)})(s)(t-s)^{\beta_j-1}ds)+$$

$$\sum_{j=1}^{m}\frac{1}{\Gamma(\beta_j)}\ {}_{b-\tau}D^{\beta_j}(\int_a^{b-\tau}(\ {}_aD^{\beta_j}(\lambda\frac{\partial L}{\partial\ {}^{C}D_b^{\beta_j}y(t)})(s)(t-s)^{\beta_j-1}ds)+$$

$$\frac{\partial F}{\partial u(t)}(t)+\lambda(t)\frac{\partial G}{\partial u(t)}(t)=0, \tag{46}$$

for $b-\tau\le t\le b$, and the transversality conditions

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\frac{\partial F}{\partial y^{(p)}(t)})(t)\eta^{p-q-1}(t)|_a^{b-\tau}+$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\lambda\frac{\partial G}{\partial y^{(p)}(t)})(t)\eta^{p-q-1}(t)|_a^{b-\tau}+$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\frac{\partial F}{\partial y^{(p)}(t-\tau)}(t+\tau)\eta^{p-q-1}(t+\tau)|_a^{b-\tau}+$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\lambda\frac{\partial G}{\partial y^{(p)}(t-\tau)}(t+\tau)\eta^{p-q-1}(t+\tau)|_a^{b-\tau}+$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\frac{\partial F}{\partial y^{(p)}(t)})(t)\eta^{p-q-1}(t)|_{b-\tau}^{b}$$

$$\sum_{p=1}^{k}\sum_{q=0}^{p-1}(-1)^q\frac{d^q}{dt^q}\ (\lambda\frac{\partial G}{\partial y^{(p)}(t)})(t)\eta^{p-q-1}(t)|_{b-\tau}^{b}=0, \qquad (47)$$

where η is any admissible function satisfying $\eta(t)\equiv 0\ t\in[a-\tau,a]$,
$\eta^{(l)}(b)=0,\ l=0,1,2,...,k-1$.

5 Conclusion

In this manuscript we have developed a fractional control problem in the presence of both left and right Caputo fractional derivatives of any order and delay in the state variables and their derivatives. The results were applied in order to find the necessary conditions for the optimal control problems. When $\alpha_i\to 1,\ \forall i=1,...,n$ and $\beta_j\to 1,\ \forall j=1,...,m$ the classical problem is recovered.

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