Mert *Advances in Difference Equations* 2012, **2012**:68
http://www.advancesindifferenceequations.com/content/2012/1/68

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Oscillation of higher-order neutral dynamic equations on time scales

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Abstract

In this article, using comparison with second-order dynamic equations, we establish sufficient conditions for oscillatory solutions of an *n*th-order neutral dynamic equation with distributed deviating arguments. The arguments are based on Taylor monomials on time scales.

2000 Mathematics Subject Classification: 34K11; 39A10; 39A99.

Keywords: time scale, higher order, oscillation, Taylor's formula

1. Introduction

In this article, we investigate the oscillatory and asymptotic behavior of solutions of higher-order neutral dynamic equations with forcing term of the form

$$[x^{\alpha}(t)+p(t)x(\tau(t))]^{\Delta^n}+\int_c^d\sum_{i=1}^{2}\lambda_i q_i(t,\xi)f_i(x(\eta_i(t,\xi)))\Delta\xi=g(t),\quad t\in[t_0,\infty)_{\mathbb{T}},\tag{1.1}$$

where $\alpha \geq 1$ is the quotient of two odd positive integers, $\lambda_1, \lambda_2 \in \{-1, 0, 1\}$, $c, d, t_0 \in \mathbb{T}$, and $[t_0,\infty)_{\mathbb{T}} := [t_0,\infty)\cap\mathbb{T}$ denotes a time scale interval with $\sup\mathbb{T}=\infty$.

In recent years, there has been much research activity concerning the oscillation and non-oscillation of solutions of dynamic equations on time scales. We refer the reader to the monographs [1-3], the articles [4-11], and the references cited therein. However, most of the obtained results are concerned with second-order dynamic equations whereas for higher order equations results are very seldom.

Motivated by Candan and Dahiya [12], the main purpose of this article is to derive some oscillation and asymptotic criteria for Equation (1.1) via comparison with second-order dynamic equations whose oscillatory character are known.

Recall that a *time scale* $\mathbb{T}$ is an arbitrary non-empty closed subset of the real numbers $\mathbb{R}$. The most well-known examples are $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{Z}$, and $\mathbb{T}=\overline{q^{\mathbb{Z}}}:=\{q^n : n\in\mathbb{Z}\}\cup\{0\}$, where $q > 1$. The forward and backward jump operators are defined by

$$\sigma(t) := \inf\{s\in\mathbb{T} : s > t\}\quad\text{and}\quad \rho(t) := \sup\{s\in\mathbb{T} : s < t\},$$

respectively, where $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$. A point $t\in\mathbb{T}$ is said to be left-dense if $t > \inf\mathbb{T}$ and $\rho(t) = t$, right-dense if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. A function f that is defined on a time scale is called rd-continuous if it is continuous at every right-dense point and if the left-sided limit exists (finite) at every left-dense point. The set of all rd-continuous functions is denoted by C_{rd}. For details, see the monographs [1,2].

We assume throughout that the following conditions are satisfied:

(i) $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [0, \infty))$ with $p(t) < 1$;
(ii) $g \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$;
(iii) $q_i \in C_{rd}([t_0, \infty)_{\mathbb{T}} \times [c, d]_{\mathbb{T}}, (0, \infty))$ with $\int_c^d q_i(t, \xi)\Delta\xi > t^{-n}$ as $t \to \infty$ for $i = 1, 2$;
(iv) $f_i \in C(\mathbb{R}, \mathbb{R})$ and $xf_i(x) \geq L_i x^2$ for some constants $L_i > 0$ for $i = 1, 2$;
(v) $\tau \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\tau(t) \leq t$, and $\lim_{t \to \infty}\tau(t) = \infty$;
(vi) $\eta_1 \in C_{rd}(\mathbb{T} \times [c, d]_{\mathbb{T}}, \mathbb{T})$, $\eta_1(t, \xi) \geq t$ for all $\xi \in [c, d]_{\mathbb{T}}$;
(vii) $\eta_2 \in C_{rd}(\mathbb{T} \times [c, d]_{\mathbb{T}}, \mathbb{T})$, $\eta_2(t, \xi) \leq t$ for all $\xi \in [c, d]_{\mathbb{T}}$, nondecreasing in ξ, and $\lim_{t \to \infty} \eta_2(t, \xi) = \infty$.

By the solution of Equation (1.1), we mean a function $x : [T_x, \infty)_{\mathbb{T}} \to \mathbb{R}$, where $T_x \geq t_0$ depends on the particular solution, which satisfies (1.1) for sufficiently large t, and $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq T_x$. As usual, such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called non-oscillatory. We note that the substitution $y := -x$ transforms (1.1) into an equation of the same form, hence we will consider only eventually positive solutions of Equation (1.1) whenever a non-oscillatory solution is concerned.

We recall the definition of the Taylor monomials as follows:

Definition 1.1. [1] *The Taylor monomials are the functions* $g_k, h_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in \mathbb{N}_0$, *which are defined recursively as follows:*

$$g_0(t, s) = h_0(t, s) \equiv 1 \quad \text{for all} \quad t, s \in \mathbb{T},$$

and for $k \in \mathbb{N}_0$,

$$g_{k+1}(t, s) = \int_s^t g_k(\sigma(\tau), s)\Delta\tau \quad \text{for all} \quad t, s \in \mathbb{T},$$

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\Delta\tau \quad \text{for all} \quad t, s \in \mathbb{T}.$$

Definition 1.2. [1] *If* $\mathbb{T}$ *has a left-scattered maximum m, then* $\mathbb{T}^\kappa := \mathbb{T} - \{m\}$. *Otherwise,* $\mathbb{T}^\kappa := \mathbb{T}$. *And* $\mathbb{T}^{\kappa^k} := (\mathbb{T}^\kappa)^{\kappa^{k-1}}$, $k \geq 2$.

Theorem 1.3. [1] *The functions* h_k *and* g_k *satisfy*

$$h_k(t, s) = (-1)^k g_k(s, t) \quad \text{for all} \quad t \in \mathbb{T}, s \in \mathbb{T}^{\kappa^k}.$$

Finding g_k, h_k for $k > 1$ is not easy in general. But for a particular given time scale, for example for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, one can easily find the functions g_k and h_k. We have for $k \in \mathbb{N}_0$,

$$h_k(t,s) = g_k(t,s) = \frac{(t-s)^k}{k!} \quad \text{for all} \quad t,s \in \mathbb{R} \tag{1.2}$$

and

$$h_k(t,s) = \frac{(t-s)^k}{k!} \quad \text{and} \quad g_k(t,s) = \frac{(t-s+k-1)^{\underline{k}}}{k!} \quad \text{for all} \quad t,s \in \mathbb{Z}, \tag{1.3}$$

where $t^{\underline{m}}$, $m \in \mathbb{N}_0$, is the usual falling (factorial) function; $t^{\underline{m}} := (t-m+1)t^{\underline{m-1}}$, $t^{\underline{0}} := 1\cdot$

For completeness, we recall the following:

Theorem 1.4. [1]*Let* $a \in \mathbb{T}^\kappa$, $b \in \mathbb{T}$, *and assume* $f : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ *is continuous at* (t, t), *where* $t \in \mathbb{T}^\kappa$ *with* $t > a$. *Also assume that* $f^\Delta(t, \cdot)$ *is* rd*-continuous on* $[a, \sigma(t)]_\mathbb{T}$. *Suppose that for each* $\varepsilon > 0$, *there exists a neighborhood U of t, independent of* $\tau \in [a, \sigma(t)]_\mathbb{T}$, *such that*

$$|f(\sigma(t),\tau) - f(s,\tau) - f^\Delta(t,\tau)(\sigma(t)-s)| \le \varepsilon|\sigma(t)-s| \quad \textit{for all} \quad s \in U,$$

where f^Δ *denotes the derivative of f with respect to the first variable. Then*

(1) $g(t) := \int_a^t f(t, \tau)\Delta\tau$ *implies* $g^\Delta(t) = \int_a^t f^\Delta(t,\tau)\Delta\tau + f(\sigma(t), t)$.

(2) $h(t) := \int_t^b f(t, \tau)\Delta\tau$ *implies* $h^\Delta(t) = \int_t^b f^\Delta(t, \tau)\Delta\tau - f(\sigma(t), t)$.

2. Preparatory lemmas

The following lemmas will be a crucial tool in obtaining the main results in this article. The first one is the well-known lemma due to Kiguradze and Kneser in the case $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively (see [13]).

Lemma 2.1. [13]*Let* $n \in \mathbb{N}$ *and f be n-times differentiable on* $\mathbb{T}$. *Assume* $\sup \mathbb{T} = \infty$ *and for any* $\varepsilon > 0$, *the set* $L_\varepsilon(\infty) := \{t \in \mathbb{T} : t > \frac{1}{\varepsilon}\}$. *Suppose there exists* $\varepsilon > 0$ *such that*

$$f(t) > 0,\ sgn(f^{\Delta^n}(t)) \equiv s \in \{-1, 1\} \quad \textit{for all} \quad t \in L_\varepsilon(\infty)$$

and $f^{\Delta^n}(t) \not\equiv 0$ *on* $L_\delta(\infty)$ *for any* $\delta > 0$. *Then there exists* $l \in [0, n] \cap \mathbb{N}_0$ *such that* $n + l$ *is even for* $s = 1$ *and odd for* $s = -1$ *with*

$$f^{\Delta^i}(t) > 0 \textit{ for all } t \in L_{\delta_i}(\infty) \ (\textit{with } \delta_i \in (0,\varepsilon)),\ i \in [1, l-1] \cap \mathbb{N}_0,$$
$$(-1)^{l+i} f^{\Delta^i}(t) > 0 \textit{ for all } t \in L_\varepsilon(\infty),\ i \in [l, n-1] \cap \mathbb{N}_0.$$

The following result provides an explicit formula for the Taylor monomials $h_k(t, s)$ on time scales $\mathbb{T}$ unbounded from above and for which the forward jump operator has a certain explicit form given by $\sigma(t) = at + b$, where $a \ge 1$, $b \ge 0$ are constants. In addition to the fact that it unifies the formulas (1.2) and (1.3), it can also be applied to time scales $\mathbb{T}$ that are different from $\mathbb{R}$ and $\mathbb{Z}$; for example, $\mathbb{T} = h\mathbb{Z} := \{hn : n \in \mathbb{Z}\}$ with $h > 0$, or

$$\mathbb{T} = \overline{q^\mathbb{Z}} := \{q^n : n \in \mathbb{Z}\} \cup \{0\} \quad \text{with} \quad q > 1$$

(see [[1], Example 1.104]).

Lemma 2.2. [14]*Let* $\mathbb{T}$ *be a time scale which is unbounded above with* $\sigma(t) = at + b$, *where* $a \geq 1$, $b \geq 0$ *are constants. Then the Taylor monomials* $h_k(t, s)$ *on* $\mathbb{T}$ *are given by the formula*

$$h_k(t,s) = \prod_{i=0}^{k-1} \frac{(t - \sigma^i(s))}{\beta_i}, t, s \in \mathbb{T}, k \in \mathbb{N}_0,$$

where $\beta_i := \sum_{j=0}^{i} a^j$ *and* $\sigma^0(s) := s$.

The analog of the Kiguradze's second lemma is the following.

Lemma 2.3. [14]*Let* $\mathbb{T}$ *be a time scale which is unbounded above with* $\sigma(t) = at + b$, *where* $a \geq 1$, $b \geq 0$ *are constants. If x is an* $n + 1$*-times differentiable function on* $[t_0, \infty)_{\mathbb{T}}$ *with* $x^{\Delta^i}(t) \geq 0$, $i = 1, 2, \ldots, n$, *and* $x^{\Delta^{n+1}}(t) \leq 0$, *then*

$$x(t) \geq \frac{h_{n-1}(t, \sigma(t_0))}{\beta_{n-1}} x^{\Delta^{n-1}}(t), \quad t \geq \sigma^{n-1}(t_0),$$

where, as earlier, $\beta_i := \sum_{j=0}^{i} a^j$.

We will also use the time scale version of a lemma due to Onose [15] for $\mathbb{T} = \mathbb{R}$. The result here is for an arbitrary time scale and since the proof of this result is similar to the proof of Lemma 2.4 in Erbe et al. [14] we state it without proof.

Lemma 2.4. *Let n be even and consider the delay dynamic inequalities*

$$x^{\Delta^n}(t) + f(t, x(\phi(t))) \leq 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{2.1}$$

$$x^{\Delta^n}(t) + f(t, x(\phi(t))) \geq 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{2.2}$$

and the equation

$$x^{\Delta^n}(t) + f(t, x(\phi(t))) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.3}$$

Here, we assume $f : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ *is a function with the property* $f(\cdot, w(\cdot)) \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *for any function* $w \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$, $f(t, u)$ *is continuous and non-decreasing in u, and* $uf(t, u) > 0$ *for all* $u \neq 0$. *Also, the delay function* $\phi \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\varphi(t) \leq t$, *and satisfy* $\lim_{t\to\infty} \varphi(t) = \infty$.

If inequality (2.1) ((2.2)) *has an eventually positive (negative) solution, then Equation* (2.3) *has an eventually positive (negative) solution.*

3. Main results

In this section, we give the main results of the article.

Theorem 3.1. *Let* $\lambda_1 = 0$ *and* $\lambda_2 = (-1)^{n+1}$. *Assume that there exists an oscillatory function* $h \in C_{rd}^n([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *with*

$$h^{\Delta^n}(t) = g(t), \quad \lim_{t\to\infty} h(t) = \lim_{t\to\infty} h^{\Delta}(t) = 0 \tag{3.1}$$

and that the second-order delay dynamic equation

$$u^{\Delta\Delta}(t) + \frac{L_2 \lambda}{(\beta_{l-1})^{1/\alpha} \beta_{n-l-1}} g_{n-l-1}(\eta_2(t), T)(h_{l-1}(\eta_2(t), \sigma(T)))^{1/\alpha} \phi_1(t) u^{1/\alpha}(\eta_2(t)) = 0, \tag{3.2}$$

where $\phi_1(t) := \int_c^d (1-p(\eta_2(t,\xi)))^{1/\alpha} q_2(t,\xi)\Delta\xi$ *is oscillatory for every constant* λ, $0 < \lambda < 1$, *every integer* l, $1 \le l \le n-1$ *and every* T *sufficiently large.*

(i) If n *is odd, then every solution of Equation (1.1) is either oscillatory or satisfies*

$$\liminf_{t\to\infty} |x(t)| = 0.$$

(ii) If n *is even, then every solution of Equation (1.1) is either oscillatory or else*

$$\lim_{t\to\infty} |x(t)| = \infty \quad \text{or} \quad \liminf_{t\to\infty} |x(t)| = 0.$$

Proof. Assume to the contrary that there exists a non-oscillatory solution x of Equation (1.1). Without loss of generality, we may assume that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\eta_2(t,\xi)) > 0$ for $t \ge t_1$ for some sufficiently large $t_1 \ge t_0$ and $\xi \in [c,d]_{\mathbb{T}}$. For the sake of convenience, set $z(t) := x^{\alpha}(t) + p(t)x(\tau(t))$. Then $z(t) > 0$ for $t \ge t_1$. Define the function $y(t) := z(t) - h(t)$. From Equation (1.1), it follows that

$$\lambda_2 y^{\Delta^n}(t) = -\int_c^d q_2(t,\xi) f_2(x(\eta_2(t,\xi)))\Delta\xi, \quad t \ge t_1,$$

so that $y^{\Delta^n}(t)$ is eventually of one-signed. Hence, lower-order derivatives $y^{\Delta^i}(t)$, $0 \le i \le n-1$, are monotone and one-signed eventually. We must have $y(t) > 0$, since otherwise we obtain a contradiction with the fact that h is oscillatory. By Lemma 2.1, there exists $T \ge t_1$ and an integer $0 \le l \le n$ with $(-1)^{n-l-1}\lambda_2 = 1$ (l is even) such that

$$\begin{aligned} y^{\Delta^i}(t) > 0,\ i = 0,1,\ \dots,\ l-1,\ t \ge T, \\ (-1)^{i-l} y^{\Delta^i}(t) > 0,\ i = l,\ l+1,\ \dots,\ n,\ t \ge T. \end{aligned} \tag{3.3}$$

Suppose that $0 < l < n$. By the time scales Taylor's formula with remainder, we may write

$$\begin{aligned} y^{\Delta^l}(t) &= \sum_{k=0}^{n-l-1} y^{\Delta^{l+k}}(\tau) h_k(t,\tau) + \lambda_2 \int_t^\tau h_{n-l-1}(t,\sigma(s))(-\lambda_2 y^{\Delta^n}(s))\Delta s \\ &= \sum_{k=0}^{n-l-1} (-1)^k y^{\Delta^{l+k}}(\tau) g_k(\tau,t) + (-1)^{n-l-1}\lambda_2 \int_t^\tau g_{n-l-1}(\sigma(s),t)(-\lambda_2 y^{\Delta^n}(s))\Delta s. \end{aligned}$$

Now, using (3.3), we have

$$y^{\Delta^l}(t) \ge \int_t^\tau g_{n-l-1}(\sigma(s),t)(-\lambda_2 y^{\Delta^n}(s))\Delta s, \quad T \le t \le \tau. \tag{3.4}$$

Letting $\tau \to \infty$ and integrating from T to t, we obtain

$$\begin{aligned} y^{\Delta^{l-1}}(t) &\ge y^{\Delta^{l-1}}(T) + \int_T^t \int_r^\infty g_{n-l-1}(\sigma(s),r)(-\lambda_2 y^{\Delta^n}(s))\Delta s \Delta r \\ &= y^{\Delta^{l-1}}(T) + \int_T^t \left[\int_T^{\sigma(s)} g_{n-l-1}(\sigma(s),r)\Delta r\right](-\lambda_2 y^{\Delta^n}(s))\Delta s \\ &\quad + \int_t^\infty \left[\int_T^t g_{n-l-1}(\sigma(s),r)\Delta r\right](-\lambda_2 y^{\Delta^n}(s))\Delta s, \quad t \ge T. \end{aligned}$$

In the above equation, to change the order of integration, we have used the following equalities:

$$\left[\int_t^\infty \int_T^t g_{n-l-1}(\sigma(s),r)(-\lambda_2 y^{\Delta^n}(s))\Delta r\Delta s\right]^\Delta = \int_t^\infty g_{n-l-1}(\sigma(s),t)(-\lambda_2 y^{\Delta^n}(s))\Delta s - \int_T^{\sigma(t)} g_{n-l-1}(\sigma(t),r)(-\lambda_2 y^{\Delta^n}(t))\Delta r,$$

$$\int_t^\infty g_{n-l-1}(\sigma(s),t)(-\lambda_2 y^{\Delta^n}(s))\Delta s = \left[\int_T^t \int_r^\infty g_{n-l-1}(\sigma(s),r)(-\lambda_2 y^{\Delta^n}(s))\Delta s\Delta r\right]^\Delta,$$

$$\int_T^{\sigma(t)} g_{n-l-1}(\sigma(t),r)(-\lambda_2 y^{\Delta^n}(t))\Delta r = \left[\int_T^t \int_T^{\sigma(s)} g_{n-l-1}(\sigma(s),r)(-\lambda_2 y^{\Delta^n}(s))\Delta r\Delta s\right]^\Delta,$$

all of which follow from Theorem 1.4. Now, by direct integration, it follows that

$$y^{\Delta^{l-1}}(t) \ge y^{\Delta^{l-1}}(T) + \int_T^t g_{n-1}(\sigma(s),T)(-\lambda_2 y^{\Delta^n}(s))\Delta s + \int_t^\infty [g_{n-1}(\sigma(s),T) - g_{n-l}(\sigma(s),t)](-\lambda_2 y^{\Delta^n}(s))\Delta s, \quad t \ge T.$$

It can easily be verified that we have for $s \ge t \ge T$,

$$g_{n-l}(\sigma(s),T) - g_{n-l}(\sigma(s),t) \ge (t-T)\frac{\prod_{i=2}^{n-l}(\sigma^i(s)-T)}{\prod_{i=1}^{n-l-1}\beta_i}.$$

By virtue of this inequality and the definition of $g_{n-l}(\sigma(s), T)$, we obtain

$$y^{\Delta^{l-1}}(t) \ge y^{\Delta^{l-1}}(T) + \frac{1}{\prod_{i=1}^{n-l-1}\beta_i}\int_T^t \prod_{i=1}^{n-l}(\sigma^i(s)-T)(-\lambda_2 y^{\Delta^n}(s))\Delta s + \frac{(t-T)}{\prod_{i=1}^{n-l-1}\beta_i}\int_t^\infty \prod_{i=1}^{n-l}(\sigma^i(s)-T)(-\lambda_2 y^{\Delta^n}(s))\Delta s, \quad t \ge T. \tag{3.5}$$

Let us denote the right-hand side of (3.5) by $u(t)$. It is clear that $u(t) > 0$, and it can easily be verified that it satisfies the second-order dynamic equation

$$u^{\Delta\Delta}(t) + \frac{g_{n-l-1}(\sigma^2(t),T)}{\beta_{n-l-1}}(-\lambda_2 y^{\Delta^n}(t)) = 0, \quad t \ge T. \tag{3.6}$$

On the other hand, from $\lim_{t\to\infty} h^\Delta(t) = 0$, $y^\Delta(t) > 0$, and $y^{\Delta\Delta}(t) > 0$ with $z(t) = y(t) + h(t)$, we see that $z^\Delta(t) > 0$ eventually. Hence,

$$z(t) = x^\alpha(t) + p(t)x(\tau(t)) \le x^\alpha(t) + p(t)z(t)$$

or

$$(1-p(t))z(t) \le x^\alpha(t).$$

Since $y^{\Delta}(t) > 0$, $y^{\Delta\Delta}(t) > 0$, and $\lim_{t\to\infty} h(t) = 0$, there exists a constant $0 < \lambda < 1$ such that for T sufficiently large,

$$z(t) \geq \lambda^{\alpha} y(t), \quad t \geq T.$$

Then

$$x^{\alpha}(t) \geq \lambda^{\alpha}(1 - p(t))y(t), \quad t \geq T. \tag{3.7}$$

By Lemma 2.3,

$$y(t) \geq \frac{h_{l-1}(t, \sigma(T))}{\beta_{l-1}} y^{\Delta^{l-1}}(t), \quad t \geq \sigma^{l-1}(T). \tag{3.8}$$

Combining the inequalities (3.7) and (3.8) with the fact that $y^{\Delta^{l-1}}(t) \geq u(t)$, we get

$$\begin{aligned} x(\eta_2(t,\xi)) &\geq [\lambda^{\alpha}(1 - p(\eta_2(t,\xi)))y(\eta_2(t,\xi))]^{1/\alpha} \\ &\geq \frac{\lambda}{(\beta_{l-1})^{1/\alpha}}(1 - p(\eta_2(t,\xi)))^{1/\alpha}(h_{l-1}(\eta_2(t), \sigma(T)))^{1/\alpha} u^{1/\alpha}(\eta_2(t)), \quad t \geq T_1, \end{aligned} \tag{3.9}$$

where $\eta_2(t) := \eta_2(t, c)$, for $T_1 > T$ sufficiently large. Using $xf_2(x) \geq L_2x^2$, multiplying both sides of (3.9) by $L_2q_2(t, \xi)$ and integrating from c to d, it follows from (3.6) that

$$u^{\Delta\Delta}(t) + \frac{L_2\lambda}{(\beta_{l-1})^{1/\alpha}\beta_{n-l-1}} g_{n-l-1}(\eta_2(t), T)(h_{l-1}(\eta_2(t), \sigma(T)))^{1/\alpha}\phi_1(t)u^{1/\alpha}(\eta_2(t)) \leq 0, \quad t \geq T_1,$$

where $\phi_1(t) = \int_c^d (1 - p(\eta_2(t,\xi)))^{1/\alpha} q_2(t,\xi)\Delta\xi$. Applying Lemma 2.4, we see that Equation (3.2) has an eventually positive solution, which is a contradiction.

It is clear that the case $l = n$ is only possible when n is even. In this case,

$$\lim_{t\to\infty} y^{\Delta^i}(t) = \infty \quad \text{for all} \quad 0 \leq i \leq n-2,$$

which implies that

$$\lim_{t\to\infty} x(t) = \infty,$$

since $z(t) = y(t) + h(t) \leq x^{\alpha}(t) + x(\tau(t))$ and $\lim_{t\to\infty} h(t) = 0$.

If $l = 0$, it follows from (3.4) that $\liminf_{t\to\infty} x(t) = 0$, since

$$\int^{\infty}\int_c^d g_{n-1}(\sigma(s), T)q_2(s,\xi)\Delta\xi\Delta s = \infty.$$

□

Theorem 3.2. *Let $\lambda_2 = 0$ and $\lambda_1 = (-1)^{n+1}$. Assume that there exists an oscillatory function $h \in C_{rd}^n([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ satisfying (3.1) and that the second-order dynamic equation*

$$u^{\Delta\Delta}(t) + \frac{L_1\beta}{(\beta_{l-1})^{1/\alpha}\beta_{n-l-1}} g_{n-l-1}(t, T)(h_{l-1}(t, \sigma(T)))^{1/\alpha}\phi_2(t)u^{1/\alpha}(t) = 0, \tag{3.10}$$

where $\phi_2(t) := \int_c^d (1 - p(\eta_1(t,\xi)))^{1/\alpha} q_1(t,\xi)\Delta\xi$ is oscillatory for every constant β, $0 < \beta < 1$, every integer l, $1 \leq l \leq n-1$ and every T sufficiently large.

(i) If n is odd, then every solution of Equation (1.1) is either oscillatory or satisfies

$$\liminf_{t\to\infty}|x(t)| = 0.$$

(ii) If n is even, then every solution of Equation (1.1) is either oscillatory or else

$$\lim_{t\to\infty}|x(t)| = \infty \quad or \quad \liminf_{t\to\infty}|x(t)| = 0.$$

Proof. Assume to the contrary that there exists a non-oscillatory solution x of Equation (1.1). Without loss of generality, assume that $x(t) > 0$ eventually. Define $z(t) := x^{\alpha}(t) + p(t)x(\tau(t))$. Proceeding as in the proof of Theorem 3.1 until we reach inequality (3.9) with λ_2 and λ replaced by λ_1 and β, respectively, and combining the inequalities (3.7) and (3.8) with the fact that $y^{\Delta^{l-1}}(t) \geq u(t)$, we obtain

$$\begin{aligned} x(\eta_1(t,\ \xi)) &\geq \left[\frac{\beta^{\alpha}}{\beta_{l-1}}(1-p(\eta_1(t,\xi)))h_{l-1}(t,\sigma(T))u(t)\right]^{1/\alpha} \\ &\geq \frac{\beta}{(\beta_{l-1})^{1/\alpha}}(1-p(\eta_1(t,\xi)))^{1/\alpha}(h_{l-1}(t,\sigma(T)))^{1/\alpha}u^{1/\alpha}(t), \quad t \geq T_1. \end{aligned} \tag{3.11}$$

Multiplying both sides of (3.11) by $L_1q_1(t, \xi)$, using the fact that $xf_1(x) \geq L_1x^2$, and integrating from c to d, we get from (3.6) that

$$u^{\Delta\Delta}(t) + \frac{L_1\beta}{(\beta_{l-1})^{1/\alpha}\beta_{n-l-1}}g_{n-l-1}(t,\ T)(h_{l-1}(t,\sigma(T)))^{1/\alpha}\phi_2(t)u^{1/\alpha}(t) \leq 0, \quad t \geq T_1,$$

where $\phi_2(t) := \int_c^d (1-p(\eta_1(t,\xi)))^{1/\alpha}q_1(t,\xi)\Delta\xi$. Applying Lemma 2.4, we see that Equation (3.10) has an eventually positive solution, which is a contradiction. The rest is similar to that of Theorem 3.1 and hence is omitted. □

Theorem 3.3. *Let n be odd and* $\lambda_1 = \lambda_2 = 1$. *Assume that there exists an oscillatory function* $h \in C_{rd}^{n}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *satisfying (3.1). Further, assume also that either Equation (3.2) is oscillatory for every constant* λ, $0 < \lambda < 1$, *or Equation (3.10) is oscillatory for every constant* β, $0 < \beta < 1$, *and every integer* l, $1 \leq l \leq n - 1$, *and every T sufficiently large. Then every solution of Equation (1.1) is either oscillatory or satisfies*

$$\liminf_{t\to\infty}|x(t)| = 0.$$

Proof. Assume to the contrary that there exists a non-oscillatory solution x of Equation (1.1). Without loss of generality, assume that $x(t) > 0$ eventually. In case $0 < l < n$, proceeding as in the proof of Theorem 3.1, we get

$$\begin{aligned} u^{\Delta\Delta}(t) &+ \frac{g_{n-l-1}(\sigma^2(t),T)}{\beta_{n-l-1}}\int_c^d q_1(t,\xi)f_1(x(\eta_1(t,\xi)))\Delta\xi \\ &+ \frac{g_{n-l-1}(\sigma^2(t),T)}{\beta_{n-l-1}}\int_c^d q_2(t,\xi)f_2(x(\eta_2(t,\xi)))\Delta\xi = 0, \quad t \geq T. \end{aligned}$$

It is clear that the above equation leads to the dynamic inequalities

$$u^{\Delta\Delta}(t) + \frac{g_{n-l-1}(\sigma^2(t),T)}{\beta_{n-l-1}}\int_c^d q_2(t,\xi)f_2(x(\eta_2(t,\xi)))\Delta\xi \leq 0, \quad t \geq T \tag{3.12}$$

and

$$u^{\Delta\Delta}(t) + \frac{g_{n-l-1}(\sigma^2(t), T)}{\beta_{n-l-1}} \int_c^d q_1(t,\xi) f_1(x(\eta_1(t,\xi)))\Delta\xi \le 0, \quad t \ge T. \tag{3.13}$$

If we assume that (3.2) is oscillatory, then, in view of (3.12), the rest of the proof follows as in the proof of Theorem 3.1. Similarly, if (3.10) is oscillatory, then, because of (3.13), it follows from the proof of Theorem 3.2. The rest is similar to that of Theorems 3.1 and 3.2, and hence it is omitted. □

4. An application

Now several criteria for Equation (1.1) can now be obtained from known oscillation criteria that already exist for second-order dynamic equations. At this stage, we will give an example to illustrate the extent of the use of the main results.

The following result deals with the second-order delay dynamic equation

$$x^{\Delta\Delta}(t) + q(t)x^{\gamma}(\phi(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.1}$$

where $0 < \gamma < 1$ is the quotient of two odd positive integers, $q \in \mathbb{C}_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$, $q(t) \ge 0$, $q(t) ⊠ 0$ on $[T, \infty)_{\mathbb{T}}$ for any $T \ge t_0$, and the delay function φ is as in Lemma 2.4.

Theorem 4.1. [10]*Suppose that* $0 < \gamma < 1$. *If*

$$\int^{\infty} \phi^{\gamma}(t) q(t)\Delta t = \infty, \tag{4.2}$$

then Equation (4.1) is oscillatory.

From Theorem 4.1, we obtain the following corollaries:

Corollary 4.2. *Let* $\alpha > 1$, $\lambda_1 = 0$, *and* $\lambda_2 = (-1)^{n+1}$. *Assume that there exists an oscillatory function* $h \in C^n_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ *satisfying (3.1) and that*

$$\int^{\infty} (\eta_2(t))^{\frac{n-1}{\alpha}} \phi_1(t)\Delta t = \infty, \tag{4.3}$$

where $\phi_1(t) = \int_c^d (1 - p(\eta_2(t,\xi)))^{1/\alpha} q_2(t,\xi)\Delta\xi$.

(i) If n is odd, then every solution of Equation (1.1) is either oscillatory or satisfies

$$\liminf_{t\to\infty} |x(t)| = 0.$$

(ii) If n is even, then every solution of Equation (1.1) is either oscillatory or else

$$\lim_{t\to\infty} |x(t)| = \infty \quad or \quad \liminf_{t\to\infty} |x(t)| = 0.$$

Proof. Condition (4.3) is sufficient for (4.2) to hold with

$$q(t) = \frac{L_2\lambda}{(\beta_{l-1})^{1/\alpha}\beta_{n-l-1}} g_{n-l-1}(\eta_2(t), T)(h_{l-1}(\eta_2(t), \sigma(T)))^{1/\alpha}\phi_1(t).$$

Note that if (4.2) is satisfied for $l = n - 1$, then it holds for all $1 \le l \le n - 1$. Hence, Equation (3.2) is oscillatory. □

Corollary 4.3. *Let* $\alpha > 1$, $\lambda_2 = 0$, *and* $\lambda_1 = (-1)^{n+1}$. *Assume that there exists an oscillatory function* $h \in C^n_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ *satisfying (3.1) and that*

$$\int^{\infty} t^{\frac{n-1}{\alpha}} \phi_2(t)\Delta t = \infty, \tag{4.4}$$

where $\phi_2(t) := \int_c^d (1 - p(\eta_1(t, \xi)))^{1/\alpha} q_1(t, \xi)\Delta\xi$.

(i) If n is odd, then every solution of Equation (1.1) is either oscillatory or satisfies

$$\liminf_{t\to\infty} |x(t)| = 0.$$

(ii) If n is even, then every solution of Equation (1.1) is either oscillatory or else

$$\lim_{t\to\infty} |x(t)| = \infty \quad or \quad \liminf_{t\to\infty} |x(t)| = 0.$$

Proof. Condition (4.4) is sufficient for (4.2) to hold with

$$q(t) = \frac{L_1\beta}{(\beta_{l-1})^{1/\alpha}\beta_{n-l-1}} g_{n-l-1}(t, T)(h_{l-1}(t, \sigma(T)))^{1/\alpha}\phi_2(t).$$

Note that if (4.2) is satisfied for $l = n - 1$, then it holds for all $1 \le l \le n - 1$. Hence, Equation (3.10) is oscillatory. □

Corollary 4.4. *Let n be odd,* $\alpha > 1$, *and* $\lambda_1 = \lambda_2 = 1$. *Assume that there exists an oscillatory function* $h \in C_{rd}^n([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *satisfying (3.1) and that either (4.3) or (4.4) holds. Then every solution of Equation (1.1) is either oscillatory or satisfies*

$$\liminf_{t\to\infty} |x(t)| = 0.$$

Proof. The proof follows as in the proof of Corollaries 4.2 and 4.3. □

Additional criteria for Equation (1.1) may also be given using other known conditions for oscillation of second-order dynamic equations. We leave this to the interested reader.

Competing interests

The author declares that they have no competing interests.

Received: 1 March 2012 Accepted: 24 May 2012 Published: 24 May 2012

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doi:10.1186/1687-1847-2012-68
Cite this article as: Mert: Oscillation of higher-order neutral dynamic equations on time scales. *Advances in Difference Equations* 2012 **2012**:68.