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On coupled fixed point theorems on partially ordered G-metric spaces

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Abstract

In this manuscript, we extend, generalize and enrich some recent coupled fixed point theorems in the framework of partially ordered G-metric spaces in a way that is essentially more natural.
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Keywords: coupled fixed point; coincidence point; mixed g-monotone property; ordered set; G-metric space

1 Introduction and preliminaries

In [1] Aydi *et al.* established coupled coincidence and coupled common fixed point results for a mixed g-monotone mapping satisfying nonlinear contractions in partially ordered G-metric spaces. These results generalize those of Choudhury and Maity [2].

Here we generalize, improve, enrich and extend the above mentioned coupled fixed point results of Aydi *et al.*

Throughout this paper, let $\mathbb{N}$ denote the set of nonnegative integers, and $\mathbb{N}^*$ be the set of positive integers.

Definition 1.1 (See [3]) Let X be a non-empty set, and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or, more specially, a G-metric on X, and the pair (X, G) is called a G-metric space.

Every G-metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \text{for all } x, y \in X. \tag{1.1}$$

Example 1.2 Let (X, d) be a metric space. The function $G : X \times X \times X \to [0, +\infty)$, defined by

$$G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},$$

or

$$G(x, y, z) = d(x, y) + d(y, z) + d(z, x),$$

for all $x, y, z \in X$, is a G-metric on X.

Definition 1.3 (See [3]) Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. We say that (x_n) is G-convergent to $x \in X$ if $\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n \to +\infty} x_n = x$.

Proposition 1.4 (See [3]) *Let (X, G) be a G-metric space. The following are equivalent:*
  (1) $\{x_n\}$ is G-convergent to x,
  (2) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$,
  (3) $G(x_n, x, x) \to 0$ as $n \to +\infty$,
  (4) $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.

Definition 1.5 (See [3]) Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 1.6 (See [3]) *Let (X, G) be a G-metric space. Then the following are equivalent:*
  (1) the sequence $\{x_n\}$ is G-Cauchy,
  (2) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq N$.

Proposition 1.7 (See [3]) *Let (X, G) be a G-metric space. A mapping $f : X \to X$ is G-continuous at x_0 if and only if it is G-sequentially continuous at x_0, that is, whenever (x_n) is G-convergent to x_0, the sequence $(f(x_n))$ is G-convergent to $f(x_0)$.*

Definition 1.8 (See [3]) A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Definition 1.9 (See [2]) Let (X, G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x, y respectively, $\{F(x_n, y_n)\}$ is G-convergent to $F(x, y)$.

Let $(X, \leq)$ be a partially ordered set and (X, G) be a G-metric space, $g : X \to X$ be a mapping. A partially ordered G-metric space, $(X, G, \preceq)$, is called g-ordered complete if for each convergent sequence $\{x_n\}_{n=0}^{\infty} \subset X$, the following conditions hold:

(OC_1) if $\{x_n\}$ is a non-increasing sequence in X such that $x_n \to x^*$ implies $gx^* \preceq gx_n$, $\forall n \in \mathbb{N}$,

(OC_2) if $\{y_n\}$ is a non-decreasing sequence in X such that $y_n \to y^*$ implies $gy^* \succeq gy_n$, $\forall n \in \mathbb{N}$.

Moreover, a partially ordered G-metric space, $(X, G, \preceq)$, is called ordered complete when g is equal to identity mapping in the above conditions (OC_1) and (OC_2).

Definition 1.10 (See [4]) An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \quad \text{and} \quad F(y, x) = y.$$

Definition 1.11 (See [5]) An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x), \qquad F(y, x) = g(y).$$

Moreover, $(x, y) \in X \times X$ is called a common coupled coincidence point of F and g if

$$F(x, y) = g(x) = x, \qquad F(y, x) = g(y) = y.$$

Definition 1.12 Let $F : X \times X \to X$ and $g : X \to X$ be mappings. The mappings F and g are said to commute if

$$g\big(F(x, y)\big) = F\big(g(x), g(y)\big), \quad \text{for all } x, y \in X.$$

Definition 1.13 (See [4]) Let $(X, \le)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. Then F is said to have mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1 \le x_2 \quad \Rightarrow \quad F(x_1, y) \le F(x_2, y), \quad \text{for } x_1, x_2 \in X,$$

and

$$y_1 \le y_2 \quad \Rightarrow \quad F(x, y_2) \le F(x, y_1), \quad \text{for } y_1, y_2 \in X.$$

Definition 1.14 (See [5]) Let $(X, \le)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then F is said to have mixed g-monotone property if $F(x, y)$ is monotone g-non-decreasing in x and is monotone g-non-increasing in y, that is, for any $x, y \in X$,

$$g(x_1) \le g(x_2) \quad \Rightarrow \quad F(x_1, y) \le F(x_2, y), \quad \text{for } x_1, x_2 \in X, \tag{1.2}$$

and

$$g(y_1) \le g(y_2) \quad \Rightarrow \quad F(x, y_2) \le F(x, y_1), \quad \text{for } y_1, y_2 \in X. \tag{1.3}$$

Let Φ denote the set of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying
(a) $\phi^{-1}(\{0\}) = \{0\}$,
(b) $\phi(t) < t$ for all $t > 0$,
(c) $\lim_{r \to t^+} \phi(r) < t$ for all $t > 0$.

Lemma 1.15 (See [5]) *Let $\phi \in \Phi$. For all $t > 0$, we have $\lim_{n \to \infty} \phi^n(t) = 0$.*

Aydi *et al.* [1] proved the following theorems.

Theorem 1.16 *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that there exist $\phi \in \Phi$, $F : X \times X \to X$ and $g : X \to X$ such that*

$$G\big(F(x, y), F(u, v), F(w, z)\big) \leq \phi\left(\frac{G(gx, gu, gw) + G(gy, gv, gz)}{2}\right) \tag{1.4}$$

for all $x, y, u, v, w, z \in X$ with $gw \preceq gu \preceq gx$ and $gy \preceq gv \preceq gz$. Suppose also that F is continuous and has the mixed g-monotone property, $F(X \times X) \subseteq g(X)$ and g is continuous and commutes with F. If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then F and g have a coupled coincidence point, that is, there exists $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Theorem 1.17 *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that $(X, G, \preceq)$ is regular. Suppose that there exist $\phi \in \Phi$ and mappings $F : X \times X \to X$ and $g : X \to X$ such that*

$$G\big(F(x, y), F(u, v), F(w, z)\big) \leq \phi\left(\frac{G(gx, gu, gw) + G(gy, gv, gz)}{2}\right) \tag{1.5}$$

for all $x, y, u, v, w, z \in X$ with $gw \preceq gu \preceq gx$ and $gy \preceq gv \preceq gz$. Suppose also that $(g(X), G)$ is complete, F has the mixed g-monotone property and $F(X \times X) \subseteq g(X)$. If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then F and g have a coupled coincidence point.

In this manuscript, we generalize, improve, enrich and extend the above coupled fixed point results. We also state some examples to illustrate our results. This paper can be considered as a continuation of the remarkable works of Berinde [6, 7].

2 Main results

We begin with an example to illustrate the weakness of Theorem 1.16 and Theorem 1.17 above.

Example 2.1 Let $X = \mathbb{R}$. Define $G : X \times X \times X \to [0, \infty)$ by

$$G(x, y, z) = |x - y| + |x - z| + |y - z|$$

for all $x, y, z \in X$. Then (X, G) is a G-metric space. Define a map $F : X \times X \to X$ by $F(x, y) = \frac{1}{12}x + \frac{7}{12}y$ and $g : X \to X$ by $g(x) = \frac{x}{2}$ for all $x, y \in X$. Suppose $x = u = z$

$$
\begin{aligned}
&G\big(F(x, y), F(u, v), F(z, w)\big) \\
&\quad = G\left(\frac{1}{12}x + \frac{7}{12}y, \frac{1}{12}u + \frac{7}{12}v, \frac{1}{12}z + \frac{7}{12}w\right) \\
&\quad = \frac{7}{12}|v - y| + \frac{7}{12}|w - y| + \frac{7}{12}|w - v|
\end{aligned} \tag{2.1}
$$

and

$$G(gx, gu, gz) + G(gy, gv, gw) = G\left(\frac{x}{2}, \frac{u}{2}, \frac{z}{2}\right) + G\left(\frac{y}{2}, \frac{v}{2}, \frac{w}{2}\right)$$

$$= \frac{1}{2}\big[|y - v| + |y - w| + |v - w|\big]. \tag{2.2}$$

It is clear that there is no $\phi \in \Phi$ that provides the statement (1.4) of Theorem 1.16.

Notice that $(0, 0)$ is the unique common coincidence point of F and g. In fact, $F(0, 0) = g(0) = 0$.

For some coupled fixed point and coupled coincidence point theorems, we refer the reader to [8–34].

We now state our first result which successively guarantees a coupled fixed point.

Theorem 2.2 *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that there exist $\phi \in \Phi$, $F : X \times X \to X$ and $g : X \to X$ such that*

$$\big[G\big(F(x, y), F(u, v), F(w, z)\big) + G\big(F(y, x), F(v, u), F(z, w)\big)\big]$$

$$\leq \phi\big(G(gx, gu, gw) + G(gy, gv, gz)\big) \tag{2.3}$$

for all $x, y, u, v, w, z \in X$ with $gw \preceq gu \preceq gx$ and $gy \preceq gv \preceq gz$. Suppose also that F is continuous and has the mixed g-monotone property, $F(X \times X) \subseteq g(X)$ and g is continuous and commutes with F. If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then F and g have a coupled coincidence point, that is, there exists $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Proof Given $x_0, y_0 \in X$ satisfying $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, we shall construct iterative sequences (x_n) and (y_n) in the following way: Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Analogously, we choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$ due to the same reasoning. Since F has the mixed g-monotone property, we conclude that $gx_0 \preceq gx_1 \preceq gx_2$ and $gy_2 \preceq gy_1 \preceq gy_0$. By repeating this process, we derive the iterative sequence

$$gx_n = F(x_{n-1}, y_{n-1}) \preceq gx_{n+1} = F(x_n, y_n)$$

and

$$gy_{n+1} = F(y_n, x_n) \preceq gy_n = F(y_{n-1}, x_{n-1}).$$

If for some n_0 we have $(gx_{n_0+1}, gy_{n_0+1}) = (gx_{n_0}, gy_{n_0})$, then $F(x_{n_0}, y_{n_0}) = gx_{n_0}$ and $F = (y_{n_0}, x_{n_0}) = gy_{n_0}$, that is, F and g have a coincidence point. So, we assume that $(gx_{n+1}, gy_{n+1}) \neq (gx_n, gy_n)$ for all $n \in \mathbb{N}$. Thus, we have either $gx_{n+1} = F(x_n, y_n) \neq gx_n$ or $gy_{n+1} = F(y_n, x_n) \neq gy_n$. We set

$$t_n = G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n) \tag{2.4}$$

for all $n \in \mathbb{N}$. Due to the property (G2), we have $t_n > 0$ for all $n \in \mathbb{N}$. By using inequality (2.3), we obtain

$$
\begin{aligned}
G(gx_{n+1}, & gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n) \\
&= G\big(F(x_n, y_n), F(x_n, y_n), F(gx_{n-1}, gy_{n-1})\big) \\
&\quad + G\big(F(y_n, x_n), F(y_n, x_n), F(gy_{n-1}, gx_{n-1})\big) \\
&\leq \phi\big(G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})\big).
\end{aligned}
\tag{2.5}
$$

Taking (2.4) into account, (2.5) becomes

$$
t_n \leq \phi(t_{n-1}).
\tag{2.6}
$$

Since $\phi(t) < t$ for all $t > 0$, it follows that t_n is monotone decreasing. Therefore, there is some $L \geq 0$ such that $\lim_{n \to +\infty} t_n = L$.

Now, we assert that $L = 0$. Suppose, on the contrary, that $L > 0$. Letting $n \to +\infty$ in (2.6) and using the properties of the map ϕ, we get

$$
L = \lim_{n \to +\infty} t_n \leq \lim_{n \to +\infty} \phi(t_{n-1}) < L,
$$

which is contradiction. Thus $L = 0$. Hence

$$
\lim_{n \to +\infty} t_n = \lim_{n \to +\infty} G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_n, gy_n, gy_{n-1}) = 0.
\tag{2.7}
$$

Next, we prove that (gx_n) and (gy_n) are Cauchy sequences in the G-metric space (X, G). Suppose, on the contrary, that at least one of (gx_n) and (gy_n) is not a Cauchy sequence in (X, G). Then there exist $\varepsilon > 0$ and sequences of natural numbers $(m(k))$ and $(l(k))$ such that for every natural number k, $m(k) > l(k) \geq k$ and

$$
r_k = G(gx_{m(k)}, gx_{m(k)}, gx_{l(k)}) + G(gy_{m(k)}, gy_{m(k)}, gy_{l(k)}) \geq \varepsilon.
\tag{2.8}
$$

Now, corresponding to $l(k)$, we choose $m(k)$ to be the smallest for which (2.8) holds. Hence

$$
G(gx_{m(k)-1}, gx_{m(k)-1}, gx_{l(k)}) + G(gy_{m(k)-1}, gy_{m(k)-1}, gy_{l(k)}) < \varepsilon.
$$

Using the rectangle inequality (property (G5)), we get

$$
\begin{aligned}
\varepsilon \leq r_k \\
&\leq G(gx_{m(k)}, gx_{m(k)}, gx_{m(k)-1}) + G(gx_{m(k)-1}, gx_{m(k)-1}, gx_{l(k)}) \\
&\quad + G(gy_{m(k)}, gy_{m(k)}, gy_{m(k)-1}) + G(gy_{m(k)-1}, gy_{m(k)-1}, gy_{l(k)}) \\
&= G(gx_{m(k)-1}, gx_{m(k)-1}, gx_{l(k)}) + G(gy_{m(k)-1}, gy_{m(k)-1}, gy_{l(k)}) + t_{m(k)-1} \\
&< \varepsilon + t_{m(k)-1}.
\end{aligned}
\tag{2.9}
$$

Letting $k \to +\infty$ in the above inequality and using (2.7) yields

$$
\lim_{k \to +\infty} r_k = \varepsilon^+.
\tag{2.10}
$$

Again, by the rectangle inequality, we have

$$
\begin{aligned}
r_k &= G(gx_{m(k)}, gx_{m(k)}, gx_{l(k)}) + G(gy_{m(k)}, gy_{m(k)}, gy_{l(k)}) \\
&\leq G(gx_{m(k)}, gx_{m(k)}, gx_{m(k)+1}) + G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{l(k)+1}) \\
&\quad + G(gx_{l(k)+1}, gx_{l(k)+1}, gx_{l(k)}) + G(gy_{m(k)}, gy_{m(k)}, gy_{m(k)+1}) \\
&\quad + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{l(k)+1}) + G(gy_{l(k)+1}, gy_{l(k)+1}, gy_{l(k)}) \\
&= t_{l(k)} + G(gx_{m(k)}, gx_{m(k)}, gx_{m(k)+1}) + G(gy_{m(k)}, gy_{m(k)}, gy_{m(k)+1}) \\
&\quad + G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{l(k)+1}) + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{l(k)+1}).
\end{aligned}
$$

Using the fact that $G(x, x, y) \leq 2G(x, y, y)$ for any $x, y \in X$, we obtain from properties (G2)-(G4)

$$
\begin{aligned}
r_k &\leq t_{l(k)} + 2G(gx_{m(k)}, gx_{m(k)}, gx_{m(k)+1}) + 2G(gy_{m(k)}, gy_{m(k)}, gy_{m(k)+1}) \\
&\quad + G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{l(k)+1}) + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{l(k)+1}) \\
&= t_{l(k)} + 2t_{m(k)} + G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{l(k)+1}) + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{l(k)+1}).
\end{aligned}
$$

Next, using inequality (2.3), we have

$$
\begin{aligned}
G(gx_{m(k)+1}, &gx_{m(k)+1}, gx_{l(k)+1}) + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{l(k)+1}) \\
&= G\big(F(x_{m(k)}, y_{m(k)}), F(x_{m(k)}, y_{m(k)}), F(x_{l(k)}, y_{l(k)})\big) \\
&\quad + G\big(F(y_{l(k)}, x_{l(k)}), F(y_{m(k)}, x_{m(k)}), F(y_{m(k)}, x_{m(k)})\big) \\
&\leq \phi\big(G(gx_{m(k)}, gx_{m(k)}, gx_{l(k)}) + G(gy_{m(k)}, gy_{m(k)}, gy_{l(k)})\big) \\
&\leq \phi(r_k).
\end{aligned} \tag{2.11}
$$

Now, using (2.7),(2.10), the properties of the function ϕ, and letting $k \to +\infty$ in (2.11), we get

$$
\varepsilon \leq \lim_{k \to +\infty} \phi(r_k) = \lim_{t \to (\varepsilon)^+} \phi(t) < \varepsilon,
$$

which is a contradiction. Thus, we have proven that (gx_n) and (gy_n) are Cauchy sequences in the G-metric space (X, G). Now, since (X, G) is complete, there are $x, y \in X$ such that (gx_n) and (gy_n) are respectively G-convergent to x and y. That is from Proposition 1.4, we have

$$
\begin{aligned}
&\lim_{n \to +\infty} G(gx_n, gx_n, x) = \lim_{n \to +\infty} G(gx_n, x, x) = 0, \\
&\lim_{n \to +\infty} G(gy_n, gy_n, y) = \lim_{n \to +\infty} G(gy_n, y, y) = 0.
\end{aligned}
$$

Using the continuity of g, we get from Proposition 1.7

$$
\begin{aligned}
&\lim_{n \to +\infty} G\big(g(gx_n), g(gx_n), gx\big) = \lim_{n \to +\infty} G\big(g(gx_n), gx, gx\big) = 0, \\
&\lim_{n \to +\infty} G\big(g(gy_n), g(gy_n), gy\big) = \lim_{n \to +\infty} G\big(g(gy_n), gy, gy\big) = 0.
\end{aligned} \tag{2.12}
$$

Since $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$, employing the commutativity of F and g yields

$$
\begin{aligned}
g(gx_{n+1}) &= g\big(F(x_n, y_n)\big) = F(gx_n, gy_n), \\
g(gy_{n+1}) &= g\big(F(y_n, x_n)\big) = F(gy_n, gx_n).
\end{aligned}
\tag{2.13}
$$

Now, we shall show that $F(x, y) = gx$ and $F(y, x) = gy$.

The mapping F is continuous, and since the sequences (gx_n) and (gy_n) are respectively G-convergent to x and y, using Definition 1.9, the sequence $(F(gx_n, gy_n))$ is G-convergent to $F(x, y)$. Therefore, from (2.13), $(g(gx_{n+1}))$ is G-convergent to $F(x, y)$. By uniqueness of the limit and using (2.12), we have $F(x, y) = gx$. Similarly, we can show that $F(y, x) = gy$. Hence, (x, y) is a coupled coincidence point of F and g. This completes the proof. $\qquad\square$

The following example illustrates that Theorem 2.2 is an extension of Theorem 1.16.

Example 2.3 Let us reconsider Example 2.1. Define a map $F : X \times X \to X$ by

$$
F(x, y) = \frac{1}{12}x + \frac{7}{12}y
$$

and $g : X \to X$ by $g(x) = \frac{x}{2}$ for all $x, y \in X$. Then $F(X \times X) = [0, \infty) = g(X) = X$. We observe that

$$
\begin{aligned}
& G\big(F(x, y), F(u, v), F(z, w)\big) + G\big(F(y, x), F(v, u), F(w, z)\big) \\
&= G\left(\frac{1}{12}x + \frac{7}{12}y, \frac{1}{12}u + \frac{7}{12}v, \frac{1}{12}z + \frac{7}{12}w\right) \\
&\quad + G\left(\frac{1}{12}y + \frac{7}{12}x, \frac{1}{12}v + \frac{7}{12}u, \frac{1}{12}w + \frac{7}{12}z\right) \\
&= \frac{7}{12}|v - y| + \frac{7}{12}|w - y| + \frac{7}{12}|w - v| \\
&\quad + \frac{1}{12}|u - x| + \frac{1}{12}|z - x| + \frac{1}{12}|z - u| \\
&\quad + \frac{1}{12}|v - y| + \frac{1}{12}|w - y| + \frac{1}{12}|w - v| \\
&\quad + \frac{7}{12}|u - x| + \frac{7}{12}|z - x| + \frac{7}{12}|z - u| \\
&= \frac{8}{12}\big(|v - y| + |w - y| + |w - v|\big) \\
&\quad + \frac{8}{12}\big(|u - x| + |z - x| + |z - u|\big)
\end{aligned}
$$

and

$$
\begin{aligned}
G(gx, gu, gz) + G(gy, gv, gw) &= G\left(\frac{1}{2}x, \frac{1}{2}u, \frac{1}{2}z\right) + G\left(\frac{1}{2}y, \frac{1}{2}v, \frac{1}{2}w\right) \\
&= \frac{1}{2}\Big[\big(|x - u| + |x - z| + |u - z|\big) \\
&\quad + \big(|y - v| + |y - w| + |v - w|\big)\Big].
\end{aligned}
\tag{2.14}
$$

Then, the statement (2.3) of Theorem 2.2 is satisfied for $\phi(t) = \frac{2}{3}t$ and $(0,0)$ is the desired coupled coincidence point.

In the next theorem, we omit the continuity hypothesis of F.

Theorem 2.4 *Let* $(X, \preceq)$ *be a partially ordered set and* G *be a* G-*metric on* X *such that* $(X, G, \preceq)$ *is* g-*ordered complete. Suppose that there exist* $\phi \in \Phi$ *and mappings* $F : X \times X \to X$ *and* $g : X \to X$ *such that*

$$\big[G\big(F(x,y), F(u,v), F(w,z)\big) + G\big(F(y,x), F(v,u), F(z,w)\big) \big]$$
$$\leq \phi\big(G(gx, gu, gw) + G(gy, gv, gz)\big) \tag{2.15}$$

for all $x, y, u, v, w, z \in X$ *with* $gw \preceq gu \preceq gx$ *and* $gy \preceq gv \preceq gz$. *Suppose also that* $(g(X), G)$ *is complete,* F *has the mixed* g-*monotone property and* $F(X \times X) \subseteq g(X)$. *If there exist* $x_0, y_0 \in X$ *such that* $gx_0 \preceq F(x_0, y_0)$ *and* $F(y_0, x_0) \preceq gy_0$, *then* F *and* g *have a coupled coincidence point.*

Proof Proceeding exactly as in Theorem 2.2, we have that (gx_n) and (gy_n) are Cauchy sequences in the complete G-metric space $(g(X), G)$. Then, there exist $x, y \in X$ such that $gx_n \to gx$ and $gy_n \to gy$. Since (gx_n) is non-decreasing and (gy_n) is non-increasing, using the regularity of $(X, G, \preceq)$, we have $gx_n \preceq gx$ and $gy \preceq gy_n$ for all $n \geq 0$. If $gx_n = gx$ and $gy_n = gy$ for some $n \geq 0$, then $gx = gx_n \preceq gx_{n+1} \preceq gx = gx_n$ and $gy \preceq gy_{n+1} \preceq gy_n = gy$, which implies that $gx_n = gx_{n+1} = F(x_n, y_n)$ and $gy_n = gy_{n+1} = F(y_n, x_n)$, that is, (x_n, y_n) is a coupled coincidence point of F and g. Then, we suppose that $(gx_n, gy_n) \neq (gx, gy)$ for all $n \geq 0$. Using the rectangle inequality, (2.15) and property $\phi(t) < t$ for all $t > 0$, we get

$$G\big(F(x,y), gx, gx\big) + G\big(F(y,x), gy, gy\big)$$
$$\leq G\big(F(x,y), gx_{n+1}, gx_{n+1}\big) + G(gx_{n+1}, gx, gx)$$
$$\quad + G\big(F(y,x), gy_{n+1}, gy_{n+1}\big) + G(gy_{n+1}, gy, gy)$$
$$= G\big(F(x,y), F(x_n, y_n), F(x_n, y_n)\big) + G(gx_{n+1}, gx, gx)$$
$$\quad + G\big(F(y,x), F(y_n, x_n), F(y_n, x_n)\big) + G(gy_{n+1}, gy, gy)$$
$$\leq \phi\big(G(gx, gx_n, gx_n) + G(gy, gy_n, gy_n)\big)$$
$$\quad + G(gx_{n+1}, gx, gx) + G(gy_{n+1}, gy, gy)$$
$$< G(gx, gx_n, gx_n) + G(gy, gy_n, gy_n)$$
$$\quad + G(gx_{n+1}, gx, gx) + G(gy_{n+1}, gy, gy).$$

Letting $n \to +\infty$ in the above inequality, we obtain

$$G\big(F(x,y), gx, gx\big) + G\big(F(y,x), gy, gy\big) = 0,$$

which implies that $gx = F(x,y)$ and $gy = F(y,x)$. Thus we proved that (x, y) is a coupled coincidence point of F and g. □

Corollary 2.5 *Let* $(X, \preceq)$ *be a partially ordered set and G be a G-metric on X such that* (X, G) *is a complete G-metric space. Suppose that there exist $k \in [0, 1)$, $F : X \times X \to X$ and* $g : X \to X$ *such that*

$$\left[G\big(F(x, y), F(u, v), F(w, z)\big) + G\big(F(y, x), F(v, u), F(z, w)\big) \right]$$
$$\leq k \left[G(gx, gu, gw) + G(gy, gv, gz) \right]$$

for all $x, y, u, v, w, z \in X$ with $gw \preceq gu \preceq gx$ and $gy \preceq gv \preceq gz$. Suppose also that F is continuous, has the mixed g-monotone property, $F(X \times X) \subseteq g(X)$ and g is continuous and commutes with F. If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then F and g have a coupled coincidence point.

Proof Taking $\phi(t) = kt$ with $k \in [0, 1)$ in Theorem 2.4, we obtain Corollary 2.5. □

Corollary 2.6 *Let* $(X, \preceq)$ *be a partially ordered set and G be a G-metric on X such that* $(X, G, \preceq)$ *is g-ordered complete. Suppose that there exist $k \in [0, 1)$, $F : X \times X \to X$ and* $g : X \to X$ *such that*

$$\left[G\big(F(x, y), F(u, v), F(w, z)\big) + G\big(F(y, x), F(v, u), F(z, w)\big) \right]$$
$$\leq k \left[G(gx, gu, gw) + G(gy, gv, gz) \right]$$

for all $x, y, u, v, w, z \in X$ with $gw \preceq gu \preceq gx$ and $gy \preceq gv \preceq gz$. Suppose also that $(g(X), G)$ is complete, F has the mixed g-monotone property, $F(X \times X) \subseteq g(X)$. If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq gy_0$, then F and g have a coupled coincidence point.

Proof Taking $\phi(t) = kt$ with $k \in [0, 1)$ in Theorem 2.4, we obtain Corollary 2.6 □

Remark 2.7 Taking $g = l_x$ (the identity mapping) in Corollary 2.5, we obtain [2, Theorem 3.1]. Taking $g = I_x$ in Corollary 2.6, we obtain [2, Theorem 3.2].

Now we shall prove the existence and uniqueness theorem of a coupled common fixed point. If $(X, \preceq)$ is a partially ordered set, we endow the product set $X \times X$ with the partial order ∇ defined by

$$(x, y) \nabla (u, v) \iff x \preceq u, \qquad v \preceq y.$$

Theorem 2.8 *In addition to the hypothesis of Theorem 2.2, suppose that for all (x, y), $(x^*, y^*) \in (X \times X)$, there exists $(u, v) \in X \times X$ such that $(F(x, y), F(u, v))$ is comparable with $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Suppose also that ϕ is a non-decreasing function. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x, y) \in X \times X$ such that*

$$x = gx = F(x, y) \quad and \quad y = gy = F(y, x).$$

Proof From Theorem 2.2, the set of coupled coincidences is non-empty. We shall show that if (x, y) and (x^*, y^*) are coupled coincidence points, that is, if $gx = F(x, y)$, $g(y) = F(y, x)$,

$gx^* = F(x^*, y^*)$ and $gy^* = F(y^*, x^*)$, then

$$gx = gx^* \quad \text{and} \quad gy = gy^*. \tag{2.16}$$

By assumption, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Without loss of generality, we can assume that

$$\bigl(F(x, y), F(y, x)\bigr) \nabla \bigl(F(u, v), F(v, u)\bigr)$$

and

$$\bigl(F(x^*, y^*), F(y^*, x^*)\bigr) \nabla \bigl(F(u, v), F(v, u)\bigr).$$

Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ such that $gu_1 = F(u_0, v_0)$ and $gv_1 = F(v_0, u_0)$. Then, similarly as in the proof of Theorem 2.2, we can inductively define sequences (gu_n) and (gv_n) in X by $gu_{n+1} = F(u_n, v_n)$ and $gv_{n+1} = F(v_n, u_n)$.

Further, set $x_0 = x$, $y_0 = y$, $x_0^* = x^*$, $y_0^* = y^*$ and, in the same way, define the sequences (gx_n), (gy_n), (gx_n^*) and (gy_n^*). Since

$$\bigl(F(x, y), F(y, x)\bigr) = (gx_1, gy_1) = (gx, gy) \nabla \bigl(F(u, v), F(v, u)\bigr) = (gu_1, gv_1),$$

then $gx \preceq gu_1$ and $gv_1 \preceq gy$. Using that F is a mixed g-monotone mapping, one can show easily that $gx \preceq gu_n$ and $gv_n \preceq gy$ for all $n \geq 1$. Thus from (2.15), we get

$$
\begin{aligned}
G(gu_{n+1}, gx, gx) + G(gy, gy, gv_{n+1}) &= G\bigl(F(u_n, v_n), F(x, y), F(x, y)\bigr) \\
&\quad + G\bigl(F(y, x), F(y, x), F(v_n, u_n)\bigr) \\
&\leq \phi\bigl(G(gu_n, gx, gx) + G(gv_n, gy, gy)\bigr).
\end{aligned}
$$

Without loss of generality, we can suppose that $(gu_n, gv_n) \neq (gx, gy)$ for all $n \geq 1$. Since ϕ is non-decreasing, from the previous inequality, we get

$$G(gu_{n+1}, gx, gx) + G(gv_{n+1}, gy, gy) \leq \phi^n \bigl(G(gu_1, gx, gx) + G(gv_1, gy, gy)\bigr)$$

for each $n \geq 1$. Letting $n \to +\infty$ in the above inequality and using Lemma 1.15, we obtain

$$\lim_{n \to +\infty} G(gu_{n+1}, gx, gx) = 0 \quad \text{and} \quad \lim_{n \to +\infty} G(gv_{n+1}, gy, gy) = 0. \tag{2.17}$$

Analogously, we derive that

$$\lim_{n \to +\infty} G\bigl(gu_{n+1}, gx^*, gx^*\bigr) = 0 \quad \text{and} \quad \lim_{n \to +\infty} G\bigl(gv_{n+1}, gy^*, gy^*\bigr) = 0. \tag{2.18}$$

Hence, from (2.17), (2.18) and the uniqueness of the limit, we get $gx = gx^*$ and $gy = gy^*$. Hence the equalities in (2.16) are satisfied. Since $gx = F(x, y)$ and $gy = F(y, x)$, by commutativity of F and g, we have

$$g(gx) = g\bigl(F(x, y)\bigr) = F(gx, gy) \quad \text{and} \quad g(gy) = g\bigl(F(y, x)\bigr) = F(gy, gx). \tag{2.19}$$

Denote $gx = z$ and $gy = w$, then by (2.19), we get

$$gz = F(z, w) \quad \text{and} \quad gw = F(w, z). \tag{2.20}$$

Thus, (z, w) is a coincidence point. Then, from (2.16) with $x^* = z$ and $y^* = w$, we have $gx = gz$ and $gy = gw$, that is,

$$gz = z \quad \text{and} \quad gw = w. \tag{2.21}$$

From (2.20), (2.21), we get

$$z = gz = F(z, w) \quad \text{and} \quad w = gw = F(w, z).$$

Then, (z, w) is a coupled common fixed point of F and g.

To prove the uniqueness, assume that (p, q) is another coupled common fixed point. Then by (2.16), we have $p = gp = gz = z$ and $q = gq = gw = w$. □

Theorem 2.9 *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space and $(X, G, \preceq)$ is regular. Suppose that there exist $\phi \in \Phi$ and $F : X \times X \to X$ having the mixed monotone property such that*

$$G\big(F(x, y), F(u, v), F(w, z)\big) + G\big(F(y, x), F(v, u), F(z, w)\big) \leq \phi\big(G(x, u, w) + G(y, v, z)\big),$$

for all $x, y, u, v, w, z \in X$ with $w \preceq u \preceq x$ and $y \preceq v \preceq z$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then F has a coupled fixed point. Furthermore, if $y_0 \preceq x_0$, then $x = y$, that is, $x = F(x, x)$.

Proof Following the proof of Theorem 2.4 with $g = I_x$, we have only to show that $x = F(x, x)$. Since $y_0 \preceq x_0$, we get $y \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0 \preceq x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x$.

Thus, we have $y \preceq x$. Suppose that $G(x, x, y) > 0$. Using inequality (2.15), we have

$$
\begin{aligned}
G(x, x, y) + G(x, y, y) &= G\big(F(x, y), F(x, y), F(y, x)\big) + G\big(F(x, y), F(y, x), F(y, x)\big) \\
&\leq \phi\big(G(x, x, y) + G(y, y, x)\big) \\
&< G(x, x, y) + G(x, y, y),
\end{aligned}
$$

a contradiction. Thus, $G(x, x, y) = 0$ and $x = y = F(x, x)$. □

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