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On the existence of solutions for a fractional finite difference inclusion via three points boundary conditions

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Abstract
In this paper, we discussed the existence of solutions for the fractional finite difference inclusion $\Delta^\nu x(t) \in F(t, x(t), \Delta x(t), \Delta^2 x(t))$ via the boundary value conditions $\xi x(\nu-3) + \beta \Delta x(\nu-3) = 0$, $x(\eta) = 0$, and $\gamma x(b+\nu) + \delta \Delta x(b+\nu) = 0$, where $\eta \in \mathbb{N}_{\nu-2}^{b+\nu-1}$, $2 < \nu < 3$, and $F : \mathbb{N}_{\nu-3}^{b+\nu+1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.

Keywords: fixed point; fractional finite difference inclusion; three points boundary conditions

1 Introduction

There are many works concerned with the existence of solutions for some fractional finite difference equations from different views by using the fixed point theory techniques (see for example, [1–7]). The readers can find more details as regards elementary notions and definitions of fractional finite difference equations in [8–15]. Also, much attention was devoted to the fractional differential inclusions (see for example, [9, 10, 16–24]). To the best of our knowledge, there is no published research work about fractional finite difference inclusions.

In 2011, Goodrich [25] investigated the general discrete fractional boundary problem, namely

$$\begin{cases} -\Delta^\nu y(t) = f(t+\nu-1, y(t+\nu-1)), \\ \alpha y(\nu-2) - \beta \Delta y(\nu-2) = 0, \\ \gamma y(\nu+b) - \delta \Delta y(\nu+b) = 0, \end{cases}$$

where $t \in [0, b]_{\mathbb{N}_0}$, $\nu \in (1, 2]$, and $\alpha\gamma + \alpha\delta + \beta\gamma \neq 0$ with $\alpha, \beta, \gamma, \delta \geq 0$. In this paper, with this thought and motivation in our minds, we investigate the existence of solution for the fractional finite difference inclusion

$$\begin{cases} \Delta^\nu x(t) \in F(t, x(t), \Delta x(t), \Delta^2 x(t)), \\ \xi x(\nu-3) + \beta \Delta x(\nu-3) = 0, \\ x(\eta) = 0, \\ \gamma x(b+\nu) + \delta \Delta x(b+\nu) = 0, \end{cases}$$

where $\eta \in \mathbb{N}_{\nu-2}^{b+\nu-1}$, $2 < \nu < 3$ and $F : \mathbb{N}_{\nu-3}^{b+\nu+1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.

2 Preliminaries

As is well known, the Gamma function has some properties as $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. Define

$$t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$$

for all $t, \nu \in \mathbb{R}$ whenever the right-hand side is defined. If $t+1-\nu$ is a pole of the gamma function and $t+1$ is not a pole, then we define $t^{\underline{\nu}} = 0$. One can verify that $\nu^{\underline{\nu}} = \nu^{\underline{\nu-1}} = \Gamma(\nu+1)$ and $t^{\underline{\nu+1}} = (t-\nu)t^{\underline{\nu}}$. We use the notations $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ for all $a \in \mathbb{R}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\}$ for all real numbers a and b whenever $b-a$ is a natural number.

Let $\nu > 0$ be such that $m-1 < \nu \leq m$ for some natural number m. Then the νth fractional sum of f based at a is defined by

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{k=a}^{t-\nu} \bigl(t - \sigma(k)\bigr)^{\underline{\nu-1}} f(k)$$

for all $t \in \mathbb{N}_{a+\nu}$. Similarly, we define

$$\Delta_a^{\nu} f(t) = \frac{1}{\Gamma(-\nu)} \sum_{k=a}^{t+\nu} \bigl(t - \sigma(k)\bigr)^{\underline{-\nu-1}} f(k)$$

for all $t \in \mathbb{N}_{a+m-\nu}$.

Lemma 2.1 [1] *Let $h : \mathbb{N}_{\nu-3}^{b+\nu+1} \to \mathbb{R}$ be a mapping and $2 < \nu \leq 3$. The general solution of the equation $\Delta_{\nu-3}^{\nu} x(t) = h(t)$ is given by*

$$x(t) = \sum_{i=1}^{3} c_i t^{\underline{\nu-i}} + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} \bigl(t - \sigma(s)\bigr)^{\underline{\nu-1}} h(s), \tag{1}$$

where c_1, c_2, c_3 are arbitrary constants.

Since $\Delta t^{\underline{\mu}} = \mu t^{\underline{\mu-1}}$, we have

$$\Delta x(t) = \sum_{i=1}^{3} c_i (\nu - i) t^{\underline{\nu-i-1}} + \frac{1}{\Gamma(\nu-1)} \sum_{s=0}^{t-\nu+1} \bigl(t - \sigma(s)\bigr)^{\underline{\nu-2}} h(s) \tag{2}$$

for more information see [12].

Let (X, d) be a metric space. Denote by 2^X, $CB(X)$, and $P_{\mathrm{cp}}(X)$ the class of all nonempty subsets, the class of all closed and bounded subsets, and the class of all compact subsets of X, respectively. A mapping $Q : X \to 2^X$ is called a multifunction on X and $u \in X$ is called a fixed point of Q whenever $u \in Qu$.

Consider the Hausdorff metric $H_d : 2^X \times 2^X \to [0, \infty)$ by

$$H_d(A, B) = \max\Bigl\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\Bigr\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$. Let (X,d) be a metric space, $\alpha : X \times X \to [0,\infty)$ a map, and $T : X \to 2^X$ a multifunction.

We say that X obeys the condition (C_α) whenever for each sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x) \geq 1$ for all k. The map T is said to be α-admissible whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$, we have $\alpha(y,z) \geq 1$ for all $z \in Ty$ [26]. Suppose that Ψ is the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$ (for more on this please see [26]).

By using the following fixed point result, we review the existence of solutions for the fractional finite difference inclusion

$$\Delta^{\nu}_{\nu-3}x(t) \in F\bigl(t, x(t), \Delta x(t), \Delta^2 x(t)\bigr)$$

via the boundary conditions $\xi x(\nu - 3) + \beta\Delta x(\nu - 3) = 0$, $\gamma x(b + \nu) + \delta\Delta x(b + \nu) = 0$, and $x(\eta) = 0$, where $\eta \in \mathbb{N}^{b+\nu-1}_{\nu-2}$, $2 < \nu < 3$, and $F : \mathbb{N}^{b+\nu}_{\nu-3} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.

Lemma 2.2 [26] *Let (X,d) be a complete metric space, $\psi \in \Psi$ a strictly increasing map, $\alpha : X \times X \to [0,\infty)$ a map and $T : X \to CB(X)$ an α-admissible multifunction such that $\alpha(x,y)H(Tx,Ty) \leq \psi(d(x,y))$ for all $x,y \in X$ and there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0,x_1) \geq 1$. If X obeys the condition (C_α), then T has a fixed point.*

3 Main result

In this section, we consider the fractional finite difference inclusion

$$\Delta^{\nu}_{\nu-3}x(t) \in F\bigl(t, x(t), \Delta x(t), \Delta^2 x(t)\bigr) \tag{3}$$

via the boundary value conditions $\xi x(\nu - 3) + \beta\Delta x(\nu - 3) = 0$, $\gamma x(b + \nu) + \delta\Delta x(b + \nu) = 0$, and $x(\eta) = 0$, where ξ, β, γ, δ are non-zero numbers, $\eta \in \mathbb{N}^{b+\nu-1}_{\nu-2}$, $2 < \nu < 3$, $x : \mathbb{N}^{b+\nu+1}_{\nu-3} \to \mathbb{R}$ and $F : \mathbb{N}^{b+\nu+1}_{\nu-3} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction.

Lemma 3.1 *Let $y : \mathbb{N}^{b+1}_0 \to \mathbb{R}$ and $2 < \nu < 3$. Then x_0 is a solution for the fractional finite difference equation $\Delta^{\nu}_{\nu-3}x(t) = y(t)$ via the boundary conditions $\xi x(\nu - 3) + \beta\Delta x(\nu - 3) = 0$, $x(\eta) = 0$, and $\gamma x(b + \nu) + \delta\Delta x(b + \nu) = 0$ if and only if x_0 is a solution of the fractional sum equation $x(t) = \sum_{s=0}^{b+1} G(t,s,\eta)y(s)$, where*

$$\begin{aligned} G(t,s,\eta) = {} & \biggl[\frac{[\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu - 1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \\ & - \frac{[\xi - \beta(\nu - 3)][\gamma + \delta(\nu - 1)][(\eta + 2 - \nu)(\eta + 3 - \nu)]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta\beta_0\mu\Gamma(\nu)(b + \nu)^{\underline{\nu-4}}} \biggr] \\ & \times (b - s + 2)\bigl(b + \nu - \sigma(s)\bigr)^{\underline{\nu-2}} + \biggl[\frac{[(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\ & + \frac{[-\xi + \beta(\nu - 3)][(\eta + 2 - \nu)(\eta + 3 - \nu) - \theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu - 2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \biggr] \bigl(\eta - \sigma(s)\bigr)^{\underline{\nu-1}} \\ & + \frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}, \end{aligned}$$

whenever $0 \leq s \leq t-\nu \leq b+1$ *and* $0 \leq s \leq \eta-\nu \leq b+1$,

$$\begin{aligned} G(t,s,\eta) = {} & \biggl[\frac{[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}} - \theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\ & - \frac{[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \biggr] \\ & \times (b-s+2)\bigl(b+\nu-\sigma(s)\bigr)^{\underline{\nu-2}} + \biggl[\frac{[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\ & + \frac{[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \biggr] \bigl(\eta-\sigma(s)\bigr)^{\underline{\nu-1}}, \end{aligned}$$

whenever $0 \leq t-\nu < s \leq \eta-\nu \leq b+1$,

$$\begin{aligned} G(t,s,\eta) = {} & \biggl[\frac{[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}} - \theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\ & - \frac{[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \biggr] \\ & \times (b-s+2)\bigl(b+\nu-\sigma(s)\bigr)^{\underline{\nu-2}} + \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}, \end{aligned}$$

whenever $0 \leq \eta-\nu < s \leq t-\nu \leq b+1$ *and*

$$\begin{aligned} G(t,s,\eta) = {} & \biggl[\frac{[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}} - \theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\ & - \frac{[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \biggr] \\ & \times (b-s+2)\bigl(b+\nu-\sigma(s)\bigr)^{\underline{\nu-2}}, \end{aligned}$$

whenever $0 \leq t-\nu < s \leq b+1$ *and* $0 \leq \eta-\nu < s \leq b+1$. *Here,*

$$\begin{aligned} \theta &= \frac{\eta\beta\nu - \eta\xi - 3\eta\beta - 2\xi + \xi\nu - \beta\nu^2 + 6\beta\nu - 8\beta}{\beta(\nu-2)}, \\ \mu &= \frac{b\xi\delta\nu - 2b\delta\xi + \gamma\xi b^2 + 3b\gamma\xi + \beta b\nu^2\delta + \delta b^2\beta\nu + \beta b\delta\nu - 6\beta\delta b + 3\beta\delta b^2 + 4\xi\delta\nu}{\beta(\nu-2)} \\ &\quad + \frac{-8\delta\xi + 4\gamma\xi b + 12\gamma\xi + 4\beta\nu^2\delta + 7\gamma\beta\nu b + 12\gamma\beta\nu + 4\beta\delta\nu - 24\beta\delta + 21\beta\gamma b + 36\beta\gamma}{\beta(\nu-2)} \end{aligned}$$

and

$$\beta_0 = \frac{\theta[\delta(\nu-1)+\gamma(b+2)](b+3)(b+4) + \mu(\eta+2-\nu)(\eta+3-\nu)}{\theta\mu}.$$

Proof Let x_0 be a solution for the equation $\Delta^{\nu}_{\nu-3}x(t) = y(t)$ via the boundary conditions $\xi x(\nu-3) + \beta\Delta x(\nu-3) = 0$, $x(\eta) = 0$, and $\gamma x(b+\nu) + \delta\Delta x(b+\nu) = 0$. Then by using (2) and Lemma 2.1, we get

$$x_0(t) = c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}} + c_3 t^{\underline{\nu-3}} + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} \bigl(t-\sigma(s)\bigr)^{\underline{\nu-1}} y(s)$$

and

$$\Delta x_0(t) = c_1(\nu-1)t^{\underline{\nu-2}} + c_2(\nu-2)t^{\underline{\nu-3}} + c_3(\nu-3)t^{\underline{\nu-4}} + \frac{1}{\Gamma(\nu-1)}\sum_{s=0}^{t-\nu+1}\bigl(t-\sigma(s)\bigr)^{\underline{\nu-2}}y(s),$$

where $c_1, c_2, c_3 \in \mathbb{R}$ are arbitrary constants. Now, by using the boundary condition

$$\xi x(\nu-3) + \beta\Delta x(\nu-3) = 0,$$

we get $\xi c_3 + \beta[c_2(\nu-2) + c_3(\nu-3)] = 0$. Also, by using the condition $x(\eta) = 0$ we obtain

$$c_3 = -(\eta+2-\nu)(\eta+3-\nu)c_1 - (\eta+2-\nu)c_2 - \frac{1}{\eta^{\underline{\nu-3}}\Gamma(\nu)}\sum_{s=0}^{\eta-\nu}\bigl(\eta-\sigma(s)\bigr)^{\underline{\nu-1}}y(s).$$

Moreover, by using the boundary condition $\gamma x(b+\nu) + \delta\Delta x(b+\nu) = 0$, we get

$$\begin{aligned} &c_1\bigl[\delta(\nu-1) + \gamma(b+2)\bigr](b+\nu)^{\underline{\nu-2}} + c_2\bigl[\delta(\nu-2) + \gamma(b+3)\bigr](b+\nu)^{\underline{\nu-3}} \\ &\quad + c_3\bigl[\delta(\nu-3) + \gamma(b+4)\bigr](b+\nu)^{\underline{\nu-4}} \\ &= -\frac{\delta}{\Gamma(\nu-1)}\sum_{s=0}^{b+1}\bigl(b+\nu-\sigma(s)\bigr)^{\underline{\nu-2}}y(s) - \frac{\gamma}{\Gamma(\nu)}\sum_{s=0}^{b}\bigl(b+\nu-\sigma(s)\bigr)^{\underline{\nu-1}}y(s). \end{aligned}$$

Thus, by using a simple calculation, we get

$$\begin{aligned} c_1 &= -\frac{1}{\beta_0\theta\eta^{\underline{\nu-3}}\Gamma(\nu)}\sum_{s=0}^{\eta-\nu}\bigl(\eta-\sigma(s)\bigr)^{\underline{\nu-1}}y(s) \\ &\quad - \frac{\gamma+\delta(\nu-1)}{\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}\sum_{s=0}^{b+1}(b-s+2)\bigl(b+\nu-\sigma(s)\bigr)^{\underline{\nu-2}}y(s), \\ c_2 &= \frac{[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)}\sum_{s=0}^{\eta-\nu}\bigl(\eta-\sigma(s)\bigr)^{\underline{\nu-1}}y(s) \\ &\quad - \frac{[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\ &\quad \times \sum_{s=0}^{b+1}(b-s+2)\bigl(b+\nu-\sigma(s)\bigr)^{\underline{\nu-2}}y(s) \end{aligned}$$

and

$$\begin{aligned} c_3 &= \frac{(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0}{\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)}\sum_{s=0}^{\eta-\nu}\bigl(\eta-\sigma(s)\bigr)^{\underline{\nu-1}}y(s) \\ &\quad + \frac{[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}\sum_{s=0}^{b+1}(b-s+2)\bigl(b+\nu-\sigma(s)\bigr)^{\underline{\nu-2}}y(s). \end{aligned}$$

Hence,

$$\begin{aligned} x_0(t) = {} & \Bigg[\frac{[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}}-\theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\ & - \frac{[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}\Bigg] \\ & \times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}}y(s) \\ & + \Bigg[\frac{[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}}-\theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\ & + \frac{[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)}\Bigg]\sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}}y(s) \\ & + \sum_{s=0}^{t-\nu}\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}y(s) = \sum_{s=0}^{b+1}G(s,t,\eta)y(s). \end{aligned}$$

Now, let x_0 be a solution for the equation $x(t)=\sum_{s=0}^{b+1}G(s,t,\eta)y(s)$. Then we have

$$\begin{aligned} x_0(t) = {} & \Bigg[\frac{[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}}-\theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\ & - \frac{[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}\Bigg] \\ & \times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}}y(s) \\ & + \Bigg[\frac{[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}}-\theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\ & + \frac{[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)}\Bigg] \\ & \times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}}y(s) + \sum_{s=0}^{t-\nu}\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}y(s). \end{aligned}$$

Since $(\nu-3)^{\underline{\nu-1}}=(\nu-3)^{\underline{\nu-2}}=0$, $(\nu-3)^{\underline{\nu-3}}=(\nu-3)^{\underline{\nu-4}}=\Gamma(\nu-2)$, and

$$\sum_{s=0}^{-3}\big(\nu-3-\sigma(s)\big)^{\underline{\nu-1}}y(s)=\sum_{s=0}^{-2}\big(\nu-3-\sigma(s)\big)^{\underline{\nu-2}}y(s)=0,$$

we get $\xi x_0(\nu-3)+\beta\Delta x_0(\nu-3)=0$. A simple calculation shows us $\gamma x_0(b+\nu)+\delta\Delta x_0(b+\nu)=0$ and $x_0(\eta)=0$. On the other hand,

$$x_0(t)=c_1t^{\underline{\nu-1}}+c_2t^{\underline{\nu-2}}+c_3t^{\underline{\nu-3}}+\frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}\big(t-\sigma(s)\big)^{\underline{\nu-1}}y(s)$$

is a solution for the equation $\Delta^{\nu}_{\nu-3}x(t)=y(t)$ and so $\Delta^{\nu}_{\nu-3}x_0(t)=y(t)$. □

A function $x : \mathbb{N}_{\nu-3}^{b+\nu+1} \to \mathbb{R}$ is a solution of the problem (3) whenever it satisfies the boundary conditions and there exists a function $y : \mathbb{N}_0^{b+1} \to \mathbb{R}$ such that

$$y(t) \in F\bigl(t, x(t), \Delta x(t), \Delta^2 x(t)\bigr)$$

for all $t \in \mathbb{N}_0^{b+1}$ and

$$\begin{aligned} x(t) = {} & \biggl[\frac{[\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}} - \theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\ & - \frac{[\xi - \beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}\biggr] \\ & \times \sum_{s=0}^{b+1}(b-s+2)\bigl(b+\nu-\sigma(s)\bigr)^{\underline{\nu-2}}y(s) \\ & + \biggl[\frac{[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\ & + \frac{[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)}\biggr] \\ & \times \sum_{s=0}^{\eta-\nu}\bigl(\eta-\sigma(s)\bigr)^{\underline{\nu-1}}y(s) + \sum_{s=0}^{t-\nu}\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}y(s). \end{aligned}$$

Let $\mathcal{X}$ be the set of all functions $x : \mathbb{N}_{\nu-3}^{b+\nu+1} \to \mathbb{R}$ endowed with the norm

$$\|x\| = \max_{t\in\mathbb{N}_{\nu-3}^{b+\nu+1}}\bigl|x(t)\bigr| + \max_{t\in\mathbb{N}_{\nu-3}^{b+\nu+1}}\bigl|\Delta x(t)\bigr| + \max_{t\in\mathbb{N}_{\nu-3}^{b+\nu+1}}\bigl|\Delta^2 x(t)\bigr|.$$

We show that $(\mathcal{X}, \|\cdot\|)$ is a Banach space. Let $\{x_n\}$ be a Cauchy sequence in $\mathcal{X}$ and $\epsilon > 0$ be given. Choose a natural number N such that $\|x_n - x_m\| < \epsilon$ for all $m, n > N$. This implies that $\max_{t\in\mathbb{N}_{\nu-3}^{b+\nu+1}} |x_n(t) - x_m(t)| < \epsilon$, $\max_{t\in\mathbb{N}_{\nu-3}^{b+\nu+1}} |\Delta x_n(t) - \Delta x_m(t)| < \epsilon$ and

$$\max_{t\in\mathbb{N}_{\nu-3}^{b+\nu+1}}\bigl|\Delta^2 x_n(t) - \Delta^2 x_m(t)\bigr| < \epsilon.$$

Choose $x(t), z(t), w(t) \in \mathbb{R}$ such that $x_n(t) \to x(t)$, $\Delta x_n(t) \to z(t)$, and $\Delta^2 x_n(t) \to w(t)$ for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$. Note that $\Delta x_n(t) = x_n(t+1) - x_n(t)$ and so $\Delta x(t) = x(t+1) - x(t) = z(t)$. Similarly, we get $\Delta^2 x(t) = w(t)$. This implies that $|x_n(t) - x(t)| < \frac{\epsilon}{3}$, $|\Delta x_n(t) - \Delta x(t)| < \frac{\epsilon}{3}$, and $|\Delta^2 x_n(t) - \Delta^2 x(t)| < \frac{\epsilon}{3}$ for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$ and $n > M$ for some natural number M. Thus,

$$\|x_n - x\| = \max_{t\in\mathbb{N}_{\nu-3}^{b+\nu+1}}\bigl|x_n(t) - x(t)\bigr| + \max_{t\in\mathbb{N}_{\nu-3}^{b+\nu+1}}\bigl|\Delta x_n(t) - \Delta x(t)\bigr| + \max_{t\in\mathbb{N}_{\nu-3}^{b+\nu+1}}\bigl|\Delta^2 x(t) - \Delta^2 x(t)\bigr| < \epsilon.$$

Hence, $(\mathcal{X}, \|\cdot\|)$ is a Banach space.

Let $x \in \mathcal{X}$. Define the set of selections of F by

$$S_{F,x} = \bigl\{y : \mathbb{N}_0^{b+1} \to \mathbb{R} \mid y(t) \in F\bigl(t, x(t), \Delta x(t), \Delta^2 x(t)\bigr) \text{ for all } t \in \mathbb{N}_0^{b+1}\bigr\}.$$

Since $F(t, x(t), \Delta x(t), \Delta^2 x(t)) \neq \emptyset$, the selection principle implies that $S_{F,x}$ is nonempty.

Theorem 3.2 *Suppose that* $\psi \in \Psi$ *and* $F : \mathbb{N}_{\nu-3}^{b+\nu+1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to P_{\mathrm{cp}}(\mathbb{R})$ *is a multifunction such that*

$$H_d\bigl(F(t,x_1,x_2,x_3) - F(t,z_1,z_2,z_3)\bigr) \le \psi\bigl(|x_1 - z_1| + |x_2 - z_2| + |x_3 - z_3|\bigr)$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$ *and* $x_1, x_2, x_3, z_1, z_2, z_3 \in \mathbb{R}$. *Then the boundary value inclusion* (3) *has a solution.*

Proof Choose $y \in S_{F,x}$ and put $h(t) = \sum_{s=0}^{b+1} G(t,s,\eta)y(s)$ for all $t \in \mathbb{N}_{\nu-3}^{\nu+b+1}$. Then $h \in \mathcal{X}$ and so the set

$$\left\{ h \in \mathcal{X} : \text{there exists } y \in S_{F,x} \text{ such that } h(t) = \sum_{s=0}^{b+1} G(t,s,\eta)y(s) \text{ for all } t \in \mathbb{N}_{\nu-3}^{b+\nu+1} \right\}$$

is nonempty. Now define $\mathcal{F} : \mathcal{X} \to 2^{\mathcal{X}}$ by

$$\mathcal{F}(x) = \left\{ h \in \mathcal{X} : \text{there exists } y \in S_{F,x} \text{ such that } h(t) = \sum_{s=0}^{b+1} G(t,s,\eta)y(s) \right. \\ \left. \text{for all } t \in \mathbb{N}_{\nu-3}^{b+\nu+1} \right\}.$$

We show that the multifunction $\mathcal{F}$ has a fixed point. First, we show that $\mathcal{F}(x)$ is closed subset of $\mathcal{X}$ for all $x \in \mathcal{X}$. Let $x \in \mathcal{X}$ and $\{u_n\}_{n\ge 1}$ be a sequence in $\mathcal{F}(x)$ with $u_n \to u$. For each n, choose $y_n \in S_{F,x}$ such that

$$\begin{aligned} u_n(t) = {} & \left[\frac{[\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}} - \theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right. \\ & \left. - \frac{[\xi - \beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right] \\ & \times \sum_{s=0}^{b+1} (b-s+2)\bigl(b+\nu-\sigma(s)\bigr)^{\underline{\nu-2}} y_n(s) \\ & + \left[\frac{[(\eta+2-\nu)(\eta+3-\nu) - \theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \right. \\ & \left. + \frac{[-\xi + \beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu) - \theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \right] \\ & \times \sum_{s=0}^{\eta-\nu} \bigl(\eta - \sigma(s)\bigr)^{\underline{\nu-1}} y_n(s) + \sum_{s=0}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} y_n(s) \end{aligned}$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$. Since F has compact values, $\{y_n\}_{n\ge 1}$ has a subsequence which converges to some $y \in S_{F,x}$. We denote this subsequence again by $\{y_n\}_{n\ge 1}$. So

$$\begin{aligned} u_n(t) \to u(t) & \\ = {} & \left[\frac{[\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}} - \theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \right. \end{aligned}$$

$$- \frac{[\xi - \beta(\nu-3)][\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}\Bigg]$$
$$\times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}}y(s)$$
$$+ \Bigg[\frac{[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)}$$
$$+ \frac{[-\xi + \beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)}\Bigg]$$
$$\times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}}y(s) + \sum_{s=0}^{t-\nu}\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}y(s)$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$. This implies that $u \in \mathcal{F}(x)$. Thus, the multifunction $\mathcal{F}$ has closed values. Since F is a compact multifunction, it is easy to check that $\mathcal{F}(x)$ is bounded set in $\mathcal{X}$ for all $x \in \mathcal{X}$. Let $x, z \in \mathcal{X}$, $h_1 \in \mathcal{F}(x)$, and $h_2 \in \mathcal{F}(z)$. Choose $y_1 \in S_{F,x}$ and $y_2 \in S_{F,z}$ such that

$$h_1(t) = \Bigg[\frac{[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}} - \theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}$$
$$- \frac{[\xi - \beta(\nu-3)][\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}\Bigg]$$
$$\times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}}y_1(s)$$
$$+ \Bigg[\frac{[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)}$$
$$+ \frac{[-\xi + \beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)}\Bigg]$$
$$\times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}}y_1(s) + \sum_{s=0}^{t-\nu}\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}y_1(s)$$

and

$$h_2(t) = \Bigg[\frac{[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}} - \theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}$$
$$- \frac{[\xi - \beta(\nu-3)][\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}\Bigg]$$
$$\times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}}y_2(s)$$
$$+ \Bigg[\frac{[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)}$$
$$+ \frac{[-\xi + \beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)}\Bigg]$$
$$\times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}}y_2(s) + \sum_{s=0}^{t-\nu}\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}y_2(s)$$

for all $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$. Since

$$\begin{aligned} &H_d\big(F\big(t, x(t), \Delta x(t), \Delta^2 x(t)\big) - F\big(t, z(t), \Delta z(t), \Delta^2 z(t)\big)\big) \\ &\quad \le \psi\big(\big|x(t) - z(t)\big| + \big|\Delta x(t) - \Delta z(t)\big| + \big|\Delta^2 x(t) - \Delta^2 z(t)\big|\big) \end{aligned}$$

for all $x, z \in \mathcal{X}$ and $t \in \mathbb{N}_{\nu-3}^{b+\nu+1}$, we get

$$\big|y_1(t) - y_2(t)\big| \le \psi\big(\big|x(t) - z(t)\big| + \big|\Delta x(t) - \Delta z(t)\big| + \big|\Delta^2 x(t) - \Delta^2 z(t)\big|\big).$$

Now, put

$$\begin{aligned} G_1 = \max_{t \in \mathbb{N}_{\nu-3}^{b+1+\nu}} \Bigg\{ &\Bigg| \frac{[\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}} - \theta[\gamma + \delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\ &- \frac{[\xi - \beta(\nu-3)][\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \Bigg| \\ &\times \sum_{s=0}^{b+1} (b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}} + \Bigg| \frac{[(\eta+2-\nu)(\eta+3-\nu) - \theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\ &+ \frac{[-\xi + \beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu) - \theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \Bigg| \\ &\times \sum_{s=0}^{\eta-\nu} \big(\eta - \sigma(s)\big)^{\underline{\nu-1}} + \sum_{s=0}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} \Bigg\}, \\ G_2 = \max_{t \in \mathbb{N}_{\nu-3}^{b+1+\nu}} \Bigg\{ &\Bigg| \frac{(\nu-3)[\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-4}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\ &- \frac{(\nu-1)\theta[\gamma + \delta(\nu-1)]t^{\underline{\nu-2}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\ &- \frac{[\xi - \beta(\nu-3)][\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}}}{\beta\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \Bigg| \\ &\times \sum_{s=0}^{b+1} (b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}} \\ &+ \Bigg| \frac{(\nu-3)[(\eta+2-\nu)(\eta+3-\nu) - \theta\beta_0]t^{\underline{\nu-4}} - \theta(\nu-1)t^{\underline{\nu-2}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\ &+ \frac{[-\xi + \beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu) - \theta\beta_0]t^{\underline{\nu-3}}}{\beta\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \Bigg| \\ &\times \sum_{s=0}^{\eta-\nu} \big(\eta - \sigma(s)\big)^{\underline{\nu-1}} + \sum_{s=0}^{t-\nu+1} \frac{(t-\sigma(s))^{\underline{\nu-2}}}{\Gamma(\nu-1)} \Bigg\} \end{aligned}$$

and

$$\begin{aligned} G_3 = \max_{t \in \mathbb{N}_{\nu-3}^{b+1+\nu}} \Bigg\{ &\Bigg| \frac{(\nu-3)(\nu-4)[\gamma + \delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-5}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\ &- \frac{(\nu-1)(\nu-2)\theta[\gamma + \delta(\nu-1)]t^{\underline{\nu-3}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \end{aligned}$$

$$
\begin{aligned}
&- \frac{(\nu-3)[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-4}}}{\beta\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}\Bigg| \\
&\times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}} \\
&+ \Bigg|\frac{(\nu-3)(\nu-4)[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-5}}-\theta(\nu-1)(\nu-2)t^{\underline{\nu-3}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\
&+ \frac{(\nu-3)[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-4}}}{\beta\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)}\Bigg| \\
&\times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}} + \sum_{s=0}^{t-\nu+2}\frac{(t-\sigma(s))^{\underline{\nu-3}}}{\Gamma(\nu-2)}\Bigg\}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\big|h_1(t)-h_2(t)\big| \\
&= \Bigg|\Bigg[\frac{[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}}-\theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\
&\quad - \frac{[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}\Bigg] \\
&\quad \times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}}(y_1-y_2)(s) \\
&\quad + \Bigg[\frac{[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}}-\theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\
&\quad + \frac{[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)}\Bigg] \\
&\quad \times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}}(y_1-y_2)(s) + \sum_{s=0}^{t-\nu}\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}(y_1-y_2)(s)\Bigg| \\
&\le \Bigg|\frac{[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}}-\theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\
&\quad - \frac{[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}\Bigg| \\
&\quad \times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}}\big|y_1(s)-y_2(s)\big| \\
&\quad + \Bigg|\frac{[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}}-\theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\
&\quad + \frac{[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)}\Bigg| \\
&\quad \times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}}\big|y_1(s)-y_2(s)\big| + \sum_{s=0}^{t-\nu}\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}\big|y_1(s)-y_2(s)\big| \\
&\le \max_{t\in\mathbb{N}_0^{b+1}}\big|y_1(t)-y_2(t)\big|
\end{aligned}
$$

$$
\begin{aligned}
&\times \max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}} \Bigg\{ \bigg| \frac{[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}} - \theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-1}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\
&- \frac{[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \bigg| \\
&\times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}} + \bigg| \frac{[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}} - \theta t^{\underline{\nu-1}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\
&+ \frac{[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-2}}}{\beta(\nu-2)\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \bigg| \\
&\times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}} + \sum_{s=0}^{t-\nu}\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} \Bigg\} \\
\le{}& \psi\big(\big|x(t)-z(t)\big| + \big|\Delta x(t)-\Delta z(t)\big| + \big|\Delta^2 x(t)-\Delta^2 z(t)\big|\big) \times G_1.
\end{aligned}
$$

Since

$$
\begin{aligned}
\Delta h_1(t) ={}& \bigg[\frac{(\nu-3)[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-4}} - (\nu-1)\theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-2}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\
&- \frac{[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}}}{\beta\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \bigg] \\
&\times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}} y_1(s) \\
&+ \bigg[\frac{(\nu-3)[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-4}} - \theta(\nu-1)t^{\underline{\nu-2}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\
&+ \frac{[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}}}{\beta\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \bigg] \\
&\times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}} y_1(s) + \sum_{s=0}^{t-\nu+1}\frac{(t-\sigma(s))^{\underline{\nu-2}}}{\Gamma(\nu-1)} y_1(s),
\end{aligned}
$$

we get

$$
\begin{aligned}
&\big|\Delta h_1(t) - \Delta h_2(t)\big| \\
&\quad\le \bigg| \frac{(\nu-3)[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-4}} - (\nu-1)\theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-2}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\
&\qquad - \frac{[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}}}{\beta\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \bigg| \\
&\qquad \times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}} \big|y_1(s)-y_2(s)\big| \\
&\qquad + \bigg| \frac{(\nu-3)[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-4}} - \theta(\nu-1)t^{\underline{\nu-2}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\
&\qquad + \frac{[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}}}{\beta\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \bigg| \\
&\qquad \times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}} \big|y_1(s)-y_2(s)\big| + \sum_{s=0}^{t-\nu+1}\frac{(t-\sigma(s))^{\underline{\nu-2}}}{\Gamma(\nu-1)} \big|y_1(s)-y_2(s)\big|
\end{aligned}
$$

$$\begin{aligned}
&\le \max_{t\in\mathbb{N}_0^{b+1}} \bigl|y_1(t)-y_2(t)\bigr| \\
&\quad\times \max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}} \Biggl\{ \biggl| \frac{(\nu-3)[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-4}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\
&\quad- \frac{(\nu-1)\theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-2}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\
&\quad- \frac{[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-3}}}{\beta\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \biggr| \\
&\quad\times \sum_{s=0}^{b+1} (b-s+2)\bigl(b+\nu-\sigma(s)\bigr)^{\underline{\nu-2}} \\
&\quad+ \biggl| \frac{(\nu-3)[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-4}}-\theta(\nu-1)t^{\underline{\nu-2}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\
&\quad+ \frac{[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-3}}}{\beta\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \biggr| \\
&\quad\times \sum_{s=0}^{\eta-\nu} \bigl(\eta-\sigma(s)\bigr)^{\underline{\nu-1}} + \sum_{s=0}^{t-\nu+1} \frac{(t-\sigma(s))^{\underline{\nu-2}}}{\Gamma(\nu-1)} \Biggr\} \\
&\le \psi\bigl(\bigl|x(t)-z(t)\bigr| + \bigl|\Delta x(t)-\Delta z(t)\bigr| + \bigl|\Delta^2 x(t)-\Delta^2 z(t)\bigr|\bigr) \times G_2.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\bigl|\Delta^2 h_1(t)-\Delta^2 h_2(t)\bigr| \\
&\quad\le \biggl| \frac{(\nu-3)(\nu-4)[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-5}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\
&\quad- \frac{(\nu-1)(\nu-2)\theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-3}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\
&\quad- \frac{(\nu-3)[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-4}}}{\beta\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \biggr| \\
&\quad\times \sum_{s=0}^{b+1} (b-s+2)\bigl(b+\nu-\sigma(s)\bigr)^{\underline{\nu-2}} \bigl|y_1(s)-y_2(s)\bigr| \\
&\quad+ \biggl| \frac{(\nu-3)(\nu-4)[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-5}}-\theta(\nu-1)(\nu-2)t^{\underline{\nu-3}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\
&\quad+ \frac{(\nu-3)[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-4}}}{\beta\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)} \biggr| \\
&\quad\times \sum_{s=0}^{\eta-\nu} \bigl(\eta-\sigma(s)\bigr)^{\underline{\nu-1}} \bigl|y_1(s)-y_2(s)\bigr| + \sum_{s=0}^{t-\nu+2} \frac{(t-\sigma(s))^{\underline{\nu-3}}}{\Gamma(\nu-2)} \bigl|y_1(s)-y_2(s)\bigr| \\
&\le \max_{t\in\mathbb{N}_0^{b+1}} \bigl|y_1(t)-y_2(t)\bigr| \\
&\quad\times \max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}} \Biggl\{ \biggl| \frac{(\nu-3)(\nu-4)[\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-5}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}} \\
&\quad- \frac{(\nu-1)(\nu-2)\theta[\gamma+\delta(\nu-1)]t^{\underline{\nu-3}}}{\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(\nu-3)[\xi-\beta(\nu-3)][\gamma+\delta(\nu-1)][(\eta+2-\nu)(\eta+3-\nu)]t^{\underline{\nu-4}}}{\beta\theta\beta_0\mu\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}}\bigg| \\
&\times \sum_{s=0}^{b+1}(b-s+2)\big(b+\nu-\sigma(s)\big)^{\underline{\nu-2}} \\
&+ \bigg|\frac{(\nu-3)(\nu-4)[(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-5}}-\theta(\nu-1)(\nu-2)t^{\underline{\nu-3}}}{\beta_0\theta^2\eta^{\underline{\nu-3}}\Gamma(\nu)} \\
&+ \frac{(\nu-3)[-\xi+\beta(\nu-3)][(\eta+2-\nu)(\eta+3-\nu)-\theta\beta_0]t^{\underline{\nu-4}}}{\beta\theta^2\beta_0\eta^{\underline{\nu-3}}\Gamma(\nu)}\bigg| \\
&\times \sum_{s=0}^{\eta-\nu}\big(\eta-\sigma(s)\big)^{\underline{\nu-1}} + \sum_{s=0}^{t-\nu+2}\frac{(t-\sigma(s))^{\underline{\nu-3}}}{\Gamma(\nu-2)}\bigg\} \\
\le{}& \psi\big(\big|x(t)-z(t)\big|+\big|\Delta x(t)-\Delta z(t)\big|+\big|\Delta^2x(t)-\Delta^2z(t)\big|\big)\times G_3.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\|h_1-h_2\| ={}& \max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}}\big|h_1(t)-h_2(t)\big| + \max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}}\big|\Delta h_1(t)-\Delta h_2(t)\big| \\
&+ \max_{t\in\mathbb{N}_{\nu-3}^{b+1+\nu}}\big|\Delta^2h_1(t)-\Delta^2h_2(t)\big| \\
\le{}& \psi\big(\big|x(t)-z(t)\big|+\big|\Delta x(t)-\Delta z(t)\big|+\big|\Delta^2x(t)-\Delta^2z(t)\big|\big)(G_1+G_2+G_3) \\
\le{}& (G_1+G_2+G_3)\psi\big(\|x-z\|\big)
\end{aligned}
$$

for all $x,z\in\mathcal{X}$, $h_1\in\mathcal{F}(x)$, and $h_2\in\mathcal{F}(z)$. So $H_d(\mathcal{F}(x),\mathcal{F}(z))\le(G_1+G_2+G_3)\psi(\|x-z\|)$ for all $x,z\in\mathcal{X}$.

Define the function α on $\mathcal{X}\times\mathcal{X}$ by $\alpha(x,z)=1$ whenever $G_1+G_2+G_3<1$ and $\alpha(x,z)=\frac{1}{G_1+G_2+G_3}$ otherwise. Thus,

$$
\alpha(x,z)H_d\big(\mathcal{F}(x),\mathcal{F}(z)\big)\le\psi\big(\|x-z\|\big)
$$

for all $x,z\in\mathcal{X}$. Let $\{x_n\}$ be a sequence in $\mathcal{X}$ with $\alpha(x_n,x_{n+1})\ge1$ for all n and $x_n\to x$. Then it is easy to check that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x)\ge1$ for all k. This implies that $\mathcal{X}$ obeys the condition (C_α). If $x\in\mathcal{X}$ and $y\in\mathcal{F}(x)$ with $\alpha(x,y)\ge1$, then it is easy to see that $\alpha(y,z)\ge1$ for all $z\in\mathcal{F}(y)$. Thus, $\mathcal{F}$ is an α-admissible α-ψ-contractive multifunction. Hence by using Theorem 2.2, there exists $x^*\in\mathcal{X}$ such that $x^*\in\mathcal{F}(x^*)$. One can check that x^* is a solution for the problem (3). □

Example 3.1 Consider the fractional finite difference inclusion

$$
\Delta_{-0.5}^{2.5}x(t)\in\left[1,e^{t^2}+2+\frac{\sin x(t)}{e^{2|t|}}+\sinh^2t+\frac{|\Delta x(t)|}{4|t|}+\frac{3}{6t^2-1}+\frac{|\Delta^2x(t)|}{\cosh|3t|}\right] \tag{4}
$$

via the boundary value conditions $\xi x(-0.5)+\beta\Delta x(-0.5)=0$, $\gamma x(6.5)+\delta\Delta x(6.5)=0$, and $x(3.5)=0$, where ξ, β, γ, δ are non-zero numbers. In fact, this problem is a special case of the problem (3), where $\nu=2.5$, $\eta=3.5$, $b=4$, and

$$
F(t,x_1,x_2,x_3)=\left[1,e^{t^2}+2+\frac{\sin x_1}{e^{2|t|}}+\sinh^2t+\frac{|x_2|}{4|t|}+\frac{3}{6t^2-1}+\frac{|x_3|}{\cosh|3t|}\right].
$$

Note that $e^{t^2} + 2 + \frac{\sin x_1}{e^{2|t|}} + \sinh^2 t + \frac{|x_2|}{4|t|} + \frac{3}{6t^2-1} + \frac{|x_3|}{\cosh|3t|} > 1$ for all $t \in \mathbb{N}_{-0.5}^{7.5}$ and $x_1, x_2, x_3 \in \mathbb{R}$. Also, $e^{2|t|} \geq 2$, $4|t| \geq 2$, and $\cosh|3t| \geq 2$ for all $t \in \mathbb{N}_{-0.5}^{7.5}$ and F is a compact valued multifunction on $\mathbb{N}_{-0.5}^{7.5} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Now, define $\psi \in \Psi$ by $\psi(z) = \frac{z}{2}$ for all $z \geq 0$. Since

$$
\begin{aligned}
& H_d\bigl(F(t,x_1,x_2,x_3), F(t,z_1,z_2,z_3)\bigr) \\
& \quad \leq \left| \frac{\sin x_1}{e^{2|t|}} - \frac{x_2}{4|t|} + \frac{x_3}{\cosh|3t|} - \frac{\sin z_1}{e^{2|t|}} + \frac{z_2}{4|t|} - \frac{z_3}{\cosh|3t|} \right| \\
& \quad \leq \frac{|x_1 - z_1| + |x_2 - z_2| + |x_3 - z_3|}{2} \\
& \quad = \psi\bigl(|x_1 - z_1| + |x_2 - z_2| + |x_3 - z_3|\bigr)
\end{aligned}
$$

for all $t \in \mathbb{N}_{-0.5}^{7.5}$ and $x_1, x_2, x_3, z_1, z_2, z_3 \in \mathbb{R}$, by using Theorem 3.2 the problem (4) has at least one solution.

4 Conclusions

In this manuscript, based on a fixed point theorem, we provided the existence result for a fractional finite difference inclusion in the presence of the general boundary conditions. An example illustrates our result.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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