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Numerical solutions of fractional differential equations of Lane-Emden type by an accurate technique

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Abstract

In this work, we study the fractional order Lane-Emden differential equations by using the reproducing kernel method. The exact solution is shown in the form of a series in the reproducing kernel Hilbert space. Some numerical examples are given in order to demonstrate the accuracy of the present method. The results obtained from the method are compared with the exact solutions and another method. The obtained numerical results are better than the ones provided by the collocation method. Results of numerical examples show that the presented method is simple, effective, and easy to use.

Keywords: reproducing kernel method; fractional differential equations; Lane-Emden differential equations

1 Introduction

The fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. As is well known, the Lane-Emden differential equations are important for mathematical modeling [1]. Therefore, the goal of our manuscript is to research the effectiveness of reproducing kernel method (RKM) to solve fractional differential equations of Lane-Emden type. To demonstrate this, we solve several examples in the succeeding sections. We consider the following equation:

$$D^{\alpha}y(t)+\frac{k}{t^{\alpha-\beta}}D^{\beta}y(t)+f(t,y)=g(t), \tag{1}$$

with the initial conditions

$$y(0)=A, \qquad y'(0)=B, \tag{2}$$

where $0<t\leq 1$, $k\geq 0$, $1<\alpha\leq 2$, $0<\beta\leq 1$, A, B are constants, $f(t,y)$ is a continuous real valued function and $g(t)\in C[0,1]$ [2].

Lane-Emden differential equations are singular initial value problems relating to second order differential equations (ODEs) utilized to model successfully several real world phenomena in mathematical physics and astrophysics. The Lane-Emden equation describes plenty of phenomena including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres, and thermionic currents. We recall that the ordinary Lane-Emden equation does not always give a correct description of the dynamics of systems in complex media. Thus, in order to bypass this obstacle and to better describe the dynamical processes in a fractal medium, numerous generalizations of Lane-Emden equation were suggested. Thus, taking into account the memory effects are better described within the fractional derivatives, the fractional Lane-Emden equations are extracting hidden aspects for the complex phenomena they described in various field of the applied mathematics, mathematical physics, and astrophysics [2, 3].

Fractional order Lane-Emden differential equations involve multi-term fractional ODEs. The multi-term fractional differential equations have been considered by many authors and some numerical methods have been proposed [4–7].

Fractional calculus has a large variety of implementations in lots of several scientific and engineering disciplines. The main notions of fractional calculus and implementations are given in [8, 9].

We recall that a general solution technique for fractional differential equations has not yet been constituted. Some methods have been enhanced for particular sorts of problems. Consequently, a single standard method for problems regarding fractional calculus has not appeared. Thus, finding credible and affirmative solution methods along with fast application techniques is beneficial and enables examination of the field. The power series method [10], the differential transform [11] and [12], the homotopy analysis method [13], the variational iteration method [14], the homotopy perturbation method [15] and the sinc-Galerkin method [16] are some well known methods for solving fractional differential equations. For more details see [17–19].

The theory of reproducing kernels [20], was utilized for the first time at the beginning of the 20th century by Zaremba in his work on boundary value problems. Recently, much attention was devoted to the further investigations of RKM in order to be applied to various scientific models. Since RKM accurately computes the series solution it is of great interest for applied sciences. The method provides the solution in a rapidly convergent series with components that can easily be calculated. In [21] an overview of RKM is shown. For more details of this method the reader can see [22–29].

The organization of the manuscript is as follows.

Section 2 gives the basic theorems of fractional calculus. Section 3 introduces several reproducing kernel spaces. The representation in ${}^{o}\varpi_2^3[0,1]$ and a related linear operator are presented in Section 4. Section 5 exhibits the main results. The exact and approximate solutions of (1)-(2) are given in this section. We verify that the approximate solution converges uniformly to the exact solution. Two examples are shown in Section 6. Some conclusions are given in the final section.

2 Preliminaries

Definition 1 [16] The left and right Riemann-Liouville fractional derivatives of order α of $h(t)$ are given as

$$ {}_aD_t^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-\nu)^{n-\alpha-1} h(\nu)\,\mathrm{d}\nu \tag{3}$$

and

$$
{}_tD_b^\alpha h(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_t^b (\nu - t)^{n-\alpha-1} h(\nu)\,\mathrm{d}\nu. \tag{4}
$$

The left and right Caputo fractional derivatives of order α of $h(t)$ are

$$
{}_a^{\mathrm{C}}D_t^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\nu)^{n-\alpha-1} h^{(n)}(\nu)\,\mathrm{d}\nu \tag{5}
$$

and

$$
{}_t^{\mathrm{C}}D_b^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)}\int_t^b (-1)^n(\nu-t)^{n-\alpha-1} h^{(n)}(\nu)\,\mathrm{d}\nu, \tag{6}
$$

such that $h : [a,b] \to \mathbb{R}$ is a function, α is a positive real number, n is the integer satisfying $n-1 \leq \alpha \leq n$, and Γ is the Euler gamma function.

Definition 2 [16] If $0 < \alpha < 1$ and h is a function such that $h(a) = h(b) = 0$, we can write

$$
\int_a^b g(t)\,{}_a^{\mathrm{C}}D_t^\alpha h(t)\,\mathrm{d}t = \int_a^b h(t)\,{}_tD_b^\alpha g(t)\,\mathrm{d}t \tag{7}
$$

and

$$
\int_a^b g(t)\,{}_t^{\mathrm{C}}D_b^\alpha h(t)\,\mathrm{d}t = \int_a^b h(t)\,{}_aD_t^\alpha g(t)\,\mathrm{d}t. \tag{8}
$$

3 Reproducing kernel functions

Definition 3 [21] Let $F \neq \emptyset$. A function $R : F \times F \to \mathbb{C}$ is called a *reproducing kernel function* of the Hilbert space H if and only if

(a) $R(\cdot,\nu) \in H$ for all $\nu \in F$,
(b) $\langle \varrho, R(\cdot,\nu)\rangle = \varrho(\nu)$ for all $\nu \in F$ and all $\varrho \in H$.

Definition 4 [21] A Hilbert space H which is defined on a non-empty set F is called a *reproducing kernel Hilbert space* if there exists a reproducing kernel function $R : F \times F \to \mathbb{C}$.

Definition 5 [21] We describe the space $\varpi_2^1[0,1]$ by

$$
\varpi_2^1[0,1] = \{\zeta \in AC[0,1] : \zeta' \in L^2[0,1]\}.
$$

The inner product and the norm in $\varpi_2^1[0,1]$ are defined by

$$
\langle \zeta, \psi\rangle_{\varpi_2^1} = \int_0^1 \zeta(t)\psi(t) + \zeta'(t)\psi'(t)\,\mathrm{d}t, \quad \zeta,\psi \in \varpi_2^1[0,1] \tag{9}
$$

and

$$
\|\zeta\|_{\varpi_2^1} = \sqrt{\langle \zeta,\zeta\rangle_{\varpi_2^1}}, \quad \zeta \in \varpi_2^1[0,1]. \tag{10}
$$

The space $\varpi_2^1[0,1]$ is a reproducing kernel Hilbert space. The reproducing kernel function T_t of this space is given as

$$T_t(y)= \frac{1}{2\sinh(1)}\bigl[\cosh(t+y-1)+\cosh\bigl(|t-y|-1\bigr)\bigr]. \tag{11}$$

Definition 6 [21] We denote the space ${}^o\varpi_2^3[0,1]$ by

$${}^o\varpi_2^3[0,1]=\bigl\{\zeta\in AC[0,1]:\zeta',\zeta''\in AC[0,1],\zeta^{(3)}\in L^2[0,1],\zeta(0)=0=\zeta'(0)\bigr\}.$$

The inner product and the norm in ${}^o\varpi_2^3[0,1]$ are defined as

$$\langle\zeta,\nu\rangle_{{}^o\varpi_2^3}=\sum_{i=0}^{2}\zeta^{(i)}(0)\nu^{(i)}(0)+\int_0^1\zeta^{(3)}(t)\nu^{(3)}(t)\,\mathrm{d}t,\quad \zeta,\nu\in{}^o\varpi_2^3[0,1]$$

and

$$\|\zeta\|_{{}^o\varpi_2^3}=\sqrt{\langle\zeta,\zeta\rangle_{{}^o\varpi_2^3}},\quad \zeta\in{}^o\varpi_2^3[0,1].$$

Theorem 3.1 *The reproducing kernel function V_y of the reproducing kernel Hilbert space ${}^o\varpi_2^3[0,1]$ is obtained as*

$$V_y(t)=\begin{cases}\frac{1}{4}y^2t^2+\frac{1}{12}y^2t^30-\frac{1}{24}yt^4+\frac{1}{120}t^5, & 0\le t\le y\le 1,\\ \frac{1}{4}y^2t^2+\frac{1}{12}t^2y^3-\frac{1}{24}ty^4+\frac{1}{120}y^5, & 0\le y<t\le 1.\end{cases} \tag{12}$$

Proof Let $\zeta\in{}^o\varpi_2^3[0,1]$ and $0\le y\le 1$. By using Definition 6 and integration by parts, we get

$$\begin{aligned}\langle\zeta,V_y\rangle_{{}^o\varpi_2^3}&=\sum_{i=0}^{2}\zeta^{(i)}(0)V_y^{(i)}(0)+\int_0^1\zeta^{(3)}(t)V_y^{(3)}(t)\,\mathrm{d}t\\&=\zeta(0)V_y(0)+\zeta'(0)V_y'(0)\\&\quad+\zeta''(0)V_y''(0)+\zeta''(1)V_y^{(3)}(1)-\zeta''(0)V_y^{(3)}(0)\\&\quad-\zeta'(1)V_y^{(4)}(1)+\zeta'(0)V_y^{(4)}(0)+\int_0^1\zeta'(t)V_y^{(5)}(t)\,\mathrm{d}t.\end{aligned}$$

After substituting the values of $V_y(0)$, $V_y'(0)$, $V_y''(0)$, $V_y^{(3)}(0)$, $V_y^{(4)}(0)$, $V_y^{(3)}(1)$, $V_y^{(4)}(1)$ into the above equation we conclude that

$$\begin{aligned}\langle\zeta,V_y\rangle_{\varpi_2^3}&=\int_0^1\zeta'(t)V_y^{(5)}(t)\,\mathrm{d}t\\&=\int_0^y\zeta'(t)V_y^{(5)}(t)\,\mathrm{d}t+\int_y^1\zeta'(t)V_y^{(5)}(t)\,\mathrm{d}t\\&=\int_0^y\zeta'(t)1\,\mathrm{d}t+\int_y^1\zeta'(t)0\,\mathrm{d}t\\&=\zeta(y)-\zeta(0)=\zeta(y).\end{aligned}$$

This completes the proof. □

4 Bounded linear operator in ${}^{o}\varpi_2^3[0,1]$

The solution of (1)-(2) is presented in the reproducing kernel Hilbert space ${}^{o}\varpi_2^3[0,1]$. Let us describe the linear operator $A : {}^{o}\varpi_2^3[0,1] \to \varpi_2^1[0,1]$ by

$$A\zeta = D^{\alpha}\zeta(t) + \frac{k}{t^{\alpha-\beta}} D^{\beta}\zeta(t), \quad \zeta \in {}^{o}\varpi_2^3[0,1]. \tag{13}$$

Model problem (1)-(2) alters to the problem

$$\begin{cases} A\zeta = z(t,\zeta), \\ \zeta(0) = 0, \qquad \zeta'(0) = 0, \end{cases} \tag{14}$$

after homogenizing the initial conditions.

Theorem 4.1 *The linear operator A is a bounded linear operator.*

Proof We should prove $\|A\zeta\|^2_{\varpi_2^1} \le N\|\zeta\|^2_{{}^{o}\varpi_2^3}$, where $N > 0$ is a positive constant. By making use of (9) and (10), we get

$$\|A\zeta\|^2_{\varpi_2^1} = \langle A\zeta, A\zeta\rangle_{\varpi_2^1} = \int_0^1 \big[A\zeta(t)\big]^2 + \big[A\zeta'(t)\big]^2 \,\mathrm{d}t.$$

By the reproducing property, we conclude that

$$\zeta(t) = \big\langle \zeta(\cdot), V_t(\cdot)\big\rangle_{{}^{o}\varpi_2^3}$$

and

$$A\zeta(t) = \big\langle \zeta(\cdot), AV_t(\cdot)\big\rangle_{{}^{o}\varpi_2^3}.$$

Therefore, we get

$$\big|A\zeta(t)\big| \le \|\zeta\|_{{}^{o}\varpi_2^3}\|AV_t\|_{{}^{o}\varpi_2^3} = N_1\|\zeta\|_{{}^{o}\varpi_2^3},$$

where $N_1 > 0$ is a positive constant. Thus, we obtain

$$\int_0^1 \big[(A\zeta)(t)\big]^2 \,\mathrm{d}t \le N_1^2\|\zeta\|^2_{{}^{o}\varpi_2^3}.$$

Taking into account that $(A\zeta)'(t) = \langle\zeta(\cdot), (AV_t)'(\cdot)\rangle_{{}^{o}\varpi_2^3}$, then we get

$$\big|(A\zeta)'(t)\big| \le \|\zeta\|_{\varpi_2^3}\big\|(AV_t)'\big\|_{{}^{o}\varpi_2^3} = N_2\|\zeta\|_{{}^{o}\varpi_2^3},$$

where $N_2 > 0$ is a positive constant. Thus, we obtain

$$\big[(A\zeta)'(t)\big]^2 \le N_2^2\|\zeta\|^2_{{}^{o}\varpi_2^3}$$

and

$$\int_0^1 \big[(A\zeta)'(t)\big]^2 \,\mathrm{d}t \le N_2^2\|\zeta\|^2_{{}^{o}\varpi_2^3}.$$

Therefore, we get

$$\|A\zeta\|^2_{\varpi_2^1} \le \int_0^1 \left(\left[(A\zeta)(t)\right]^2 + \left[(A\zeta)'(t)\right]^2\right) \mathrm{d}t \le \left(N_1^2 + N_2^2\right)\|\zeta\|^2_{\varpi_2^3} = N\|\zeta\|^2_{\varpi_2^3},$$

where $N = N_1^2 + N_2^2 > 0$. □

5 Exact and approximate solutions

Let us put $\varrho_i(t) = T_{t_i}(t)$ and $\eta_i(t) = A^*\varrho_i(t)$, where A^* is conjugate operator of A. The orthonormal system $\{\hat{\eta}_i(t)\}_{i=1}^{\infty}$ of ${}^o\varpi_2^3[0,1]$ can be obtained from Gram-Schmidt orthogonalization process of $\{\eta_i(t)\}_{i=1}^{\infty}$ and

$$\hat{\eta}_i(t) = \sum_{k=1}^{i} \sigma_{ik}\eta_k(t) \quad (\sigma_{ii} > 0, i = 1, 2, \ldots). \tag{15}$$

Theorem 5.1 *Let $\{t_i\}_{i=1}^{\infty}$ be dense in $[0,1]$ and $\eta_i(t) = A_y V_t(y)|_{y=t_i}$. Then the sequence $\{\eta_i(t)\}_{i=1}^{\infty}$ is a complete system in ${}^o\varpi_2^3[0,1]$.*

Proof We recall that

$$\eta_i(t) = \left(A^*\varrho_i\right)(t) = \left\langle \left(A^*\varrho_i\right)(y), V_t(y)\right\rangle = \left\langle (\varrho_i)(y), AyV_t(y)\right\rangle = A_y V_t(y)|_{y=t_i}.$$

Thus, $\eta_i(t) \in {}^o\varpi_2^3[0,1]$. For each fixed $\zeta(t) \in {}^o\varpi_2^3[0,1]$, let $\langle \zeta(t), \eta_i(t)\rangle = 0$ $(i = 1, 2, \ldots)$, *i.e.*,

$$\left\langle \zeta(t), \left(A^*\varrho_i\right)(t)\right\rangle = \left\langle A\zeta(\cdot), \varrho_i(\cdot)\right\rangle = (A\zeta)(t_i) = 0,$$

$\{t_i\}_{i=1}^{\infty}$ is dense in $[0,1]$. Therefore, $(A\zeta)(t) = 0$ and $\zeta \equiv 0$. This completes the proof. □

Theorem 5.2 *If $\zeta(t)$ is the exact solution of* (14), *then we have*

$$\zeta(t) = A^{-1}z(t,\zeta) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \sigma_{ik} z\left(t_k, \zeta(t_k)\right)\hat{\eta}_i(t), \tag{16}$$

where $\{(t_i)\}_{i=1}^{\infty}$ is dense in $[0,1]$.

Proof We know

$$\begin{aligned}
\zeta(t) &= \sum_{i=1}^{\infty} \left\langle \zeta(t), \hat{\eta}_i(t)\right\rangle_{{}^o\varpi_2^3} \hat{\eta}_i(t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \sigma_{ik}\left\langle \zeta(t), \eta_k(t)\right\rangle_{{}^o\varpi_2^3} \hat{\eta}_i(t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \sigma_{ik}\left\langle \zeta(t), A^*\varrho_k(t)\right\rangle_{{}^o\varpi_2^3} \hat{\eta}_i(t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \sigma_{ik}\left\langle A\zeta(t), \varrho_k(t)\right\rangle_{\varpi_2^1} \hat{\eta}_i(t),
\end{aligned}$$

from (15). By uniqueness of the solution of (14), we obtain

$$\zeta(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}\langle z(t,\zeta), T_{t_k}\rangle_{\varpi_2^1}\hat{\eta}_i(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}z(t_k,\zeta(t_k))\hat{\eta}_i(t).$$

This completes the proof. □

The approximate solution $\zeta_n(t)$ is obtained as

$$\zeta_n(t) = \sum_{i=1}^{n}\sum_{k=1}^{i}\sigma_{ik}z(t_k,\zeta(t_k))\hat{\eta}_i(t). \tag{17}$$

Lemma 5.3 [30] *If* $\|\zeta_n-\zeta\|_{{}^o\varpi_2^3}\to 0$, $t_n\to t$ $(n\to\infty)$ *and* $z(t,\zeta)$ *is continuous for* $t\in[0,1]$, *then*

$$z(t_n,\zeta_{n-1}(t_n))\to z(t,\zeta)\quad \text{as } n\to\infty.$$

Theorem 5.4 *For any fixed* $\zeta_0(t)\in {}^o\varpi_2^3[0,1]$ *assume* $\zeta_n(t)=\sum_{i=1}^{n}A_i\hat{\eta}_i(t)$, $A_i=\sum_{k=1}^{i}\sigma_{ik}z(t_k,\zeta_{k-1}(t_k))$, $\|\zeta_n\|_{{}^o\varpi_2^3}$ *is bounded*, $\{t_i\}_{i=1}^{\infty}$ *is dense in* $[0,1]$, $z(t,\zeta)\in\varpi_2^1[0,1]$ *for any* $\zeta(t)\in {}^o\varpi_2^3[0,1]$. *Then* $\zeta_n(t)$ *converges to the exact solution of* (16) *in* ${}^o\varpi_2^3[0,1]$ *and*

$$\zeta(t)=\sum_{i=1}^{\infty}A_i\hat{\eta}_i(t).$$

Proof Let us prove the convergence of $\zeta_n(t)$. We have

$$\zeta_{n+1}(t)=\zeta_n(t)+A_{n+1}\hat{\eta}_{n+1}(t), \tag{18}$$

from the orthonormality of $\{\hat{\eta}_i\}_{i=1}^{\infty}$. Thus, we get

$$\|\zeta_{n+1}\|^2=\|\zeta_n\|^2+A_{n+1}^2=\|\zeta_{n-1}\|^2+A_n^2+A_{n+1}^2=\cdots=\sum_{i=1}^{n+1}A_i^2, \tag{19}$$

from the boundedness of $\|\zeta_n\|_{{}^o\varpi_2^3}$. Then we obtain

$$\sum_{i=1}^{\infty}A_i^2<\infty,$$

i.e.,

$$\{A_i\}\in l^2\quad (i=1,2,\ldots).$$

Let $m>n$, in view of $(\zeta_m-\zeta_{m-1})\perp(\zeta_{m-1}-\zeta_{m-2})\perp\cdots\perp(\zeta_{n+1}-\zeta_n)$, we get

$$\begin{aligned}\|\zeta_m-\zeta_n\|_{{}^o\varpi_2^3}^2&=\|\zeta_m-\zeta_{m-1}+\zeta_{m-1}-\zeta_{m-2}+\cdots+\zeta_{n+1}-\zeta_n\|_{{}^o\varpi_2^3}^2\\&\le\|\zeta_m-\zeta_{m-1}\|_{{}^o\varpi_2^3}^2+\cdots+\|\zeta_{n+1}-\zeta_n\|_{{}^o\varpi_2^3}^2\\&=\sum_{i=n+1}^{m}A_i^2\to 0,\quad m,n\to\infty.\end{aligned}$$

There exists $\zeta(t) \in {}^{o}\varpi_2^3[0,1]$, such that

$$\zeta_n(t) \rightarrow \zeta(t) \quad \text{as } n \to \infty,$$

by completeness of ${}^{o}\varpi_2^3[0,1]$. We have

$$\zeta(t) = \sum_{i=1}^{\infty} A_i \hat{\eta}_i(t).$$

In virtue of

$$\begin{aligned}(A\zeta)(t_j) &= \sum_{i=1}^{\infty} A_i \bigl\langle A\hat{\eta}_i(t), \varrho_j(t)\bigr\rangle_{\varpi_2^1} = \sum_{i=1}^{\infty} A_i \bigl\langle \hat{\eta}_i(t), A^*\varrho_j(t)\bigr\rangle_{{}^{o}\varpi_2^3} \\ &= \sum_{i=1}^{\infty} A_i \bigl\langle \hat{\eta}_i(t), \eta_j(t)\bigr\rangle_{{}^{o}\varpi_2^3},\end{aligned}$$

we get

$$\begin{aligned}\sum_{j=1}^{n} \sigma_{nj}(A\zeta)(t_j) &= \sum_{i=1}^{\infty} A_i \Biggl\langle \hat{\eta}_i(t), \sum_{j=1}^{n} \sigma_{nj}\eta_j(t)\Biggr\rangle_{{}^{o}\varpi_2^3} \\ &= \sum_{i=1}^{\infty} A_i \bigl\langle \hat{\eta}_i(t), \hat{\eta}_n(t)\bigr\rangle_{{}^{o}\varpi_2^3} = A_n.\end{aligned}$$

If $n = 1$, then we get

$$A\zeta(t_1) = z\bigl(t_1, \zeta_0(t_1)\bigr). \tag{20}$$

If $n = 2$, then we have

$$\sigma_{21}(A\zeta)(t_1) + \sigma_{22}(A\zeta)(t_2) = \sigma_{21} z\bigl(t_1, \zeta_0(t_1)\bigr) + \sigma_{22} z\bigl(t_2, \zeta_1(t_2)\bigr). \tag{21}$$

It is obvious from (20) and (21) that

$$(A\zeta)(t_2) = z\bigl(t_2, \zeta_1(t_2)\bigr).$$

By induction, we conclude that

$$(A\zeta)(t_j) = z\bigl(t_j, \zeta_{j-1}(j)\bigr). \tag{22}$$

Using the convergence of $\zeta_n(t)$ and Lemma 5.3 gives

$$(A\zeta)(y) = z\bigl(y, \zeta(y)\bigr),$$

i.e., $\zeta(t)$ is the solution of (14) and

$$\zeta(t) = \sum_{i=1}^{\infty} A_i \hat{\eta}_i.$$

This completes the proof. □

Theorem 5.5 *If* $\zeta \in {}^{o}\varpi_2^3[0,1]$ *then* $\|\zeta_n - \zeta\|_{{}^{o}\varpi_2^3} \to 0$, $n \to \infty$, *and the sequence* $\|\zeta_n - \zeta\|_{{}^{o}\varpi_2^3}$ *is monotonically decreasing in* n.

Proof We obtain

$$\|\zeta_n - \zeta\|_{{}^{o}\varpi_2^3} = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \sigma_{ik} z\bigl(t_k, \zeta(t_k)\bigr)\hat{\eta}_i \right\|_{{}^{o}\varpi_2^3},$$

by (16) and (17). Therefore, we have

$$\|\zeta_n - \zeta\|_{{}^{o}\varpi_2^3} \to 0, \quad n \to \infty$$

and

$$\begin{aligned} \|\zeta_n - \zeta\|_{{}^{o}\varpi_2^3}^2 &= \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \sigma_{ik} z\bigl(t_k, \zeta(t_k)\bigr)\hat{\eta}_i \right\|_{{}^{o}\varpi_2^3}^2 \\ &= \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \sigma_{ik} z\bigl(t_k, \zeta(x_k)\bigr)\hat{\eta}_i \right)^2. \end{aligned}$$

Consequently, $\|\zeta_n - \zeta\|_{{}^{o}\varpi_2^3}$ is monotonically decreasing in n. □

6 Numerical experiments

Two examples are given in this section. A comparison of the absolute errors is shown in Tables 1 and 2.

Table 1 Comparison of RKM (first line) and collocation method [2] (second line) of the absolute errors for Example 6.1

m**t	**0**	**0.25**	**0.5**	**0.75**	**1.0**
5	0	8.7370×10^{-4}	9.9×10^{-4}	7.6702×10^{-4}	5.4736×10^{-4}
	0	1.3345×10^{-3}	1.5×10^{-3}	5.0673×10^{-3}	3.6339×10^{-3}
10	0	8.4636×10^{-6}	2.9×10^{-6}	8.5754×10^{-6}	5.4345×10^{-6}
	0	1.3232×10^{-5}	2.6×10^{-5}	1.5634×10^{-6}	4.1443×10^{-5}
50	0	9.4673×10^{-8}	1.1×10^{-8}	4.1627×10^{-8}	9.3989×10^{-8}
	0	2.3416×10^{-7}	1.6×10^{-7}	5.1126×10^{-7}	2.1233×10^{-7}
100	0	4.5864×10^{-9}	7.0×10^{-10}	7.1591×10^{-9}	8.4693×10^{-9}
	0	4.9383×10^{-8}	3.4×10^{-8}	5.0347×10^{-8}	6.4332×10^{-7}

Table 2 Comparison of RKM (first line) and collocation method [2] (second line) of absolute errors for Example 6.2

m**t	**0**	**0.25**	**0.5**	**0.75**	**1.0**
5	0	1.1652×10^{-4}	1.66×10^{-4}	3.8729×10^{-3}	5.5282×10^{-4}
	0	1.3323×10^{-3}	1.10×10^{-3}	5.0953×10^{-3}	4.4409×10^{-3}
10	0	7.5343×10^{-6}	4.61×10^{-6}	1.6356×10^{-6}	8.8855×10^{-6}
	0	1.2731×10^{-5}	1.06×10^{-5}	2.5165×10^{-5}	4.4409×10^{-5}
50	0	9.0960×10^{-8}	1.50×10^{-7}	2.6000×10^{-8}	5.1786×10^{-7}
	0	2.0256×10^{-6}	1.66×10^{-7}	5.0926×10^{-7}	2.4573×10^{-6}
100	0	1.6526×10^{-8}	1.19×10^{-8}	1.7952×10^{-8}	4.5451×10^{-9}
	0	9.2963×10^{-8}	1.66×10^{-8}	5.0927×10^{-8}	6.4482×10^{-7}

Table 3 Exact and approximate solutions for Example 6.1 when $m = 5$

t	Exact solutions	Approximate solutions	CPU time (s)
0.13	–0.014703	–0.01458300000	2.480
0.24	–0.043776	–0.04365470000	2.293
$1/\sqrt{2}$	–0.1464466095	–0.1463252639	2.512
0.85	–0.108375	–0.1082536544	2.340
0.999	–0.000998001	–0.0008781340000	2.371

Example 6.1 Let us consider

$$D^\alpha y(t) + \frac{k}{t^{\alpha-\beta}} D^\alpha y(t) + \frac{1}{t^{\alpha-2}} y(t) = g(t), \tag{23}$$

with the initial conditions

$$y(0) = 0 = y'(0), \tag{24}$$

where

$$g(t) = t^{2-\alpha}\left(6t\left(\frac{t^2}{6} + \frac{\Gamma(4-\beta) + k\Gamma(4-\alpha)}{\Gamma(4-\beta)\Gamma(4-\alpha)}\right) - 2\left(\frac{t^2}{2} + \frac{\Gamma(3-\beta) + k\Gamma(3-\alpha)}{\Gamma(3-\beta)\Gamma(3-\alpha)}\right)\right),$$

and $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$. The exact solution of (23)-(24) is given as [2]

$$y(t) = t^3 - t^2.$$

Using the above method, we obtain Tables 1 and 3.

Example 6.2 We regard

$$D^\alpha y(t) + \frac{k}{t^{\alpha-\beta}} D^\alpha y(t) + \frac{1}{t^{\alpha-2}} y(t) = h(t), \tag{25}$$

with the initial conditions

$$y(0) = 0 = y'(0), \tag{26}$$

where

$$h(t) = t^{2-\alpha}\left(-6t\left(\frac{t^2}{6} + \frac{\Gamma(4-\beta) + k\Gamma(4-\alpha)}{\Gamma(4-\beta)\Gamma(4-\alpha)}\right) + 2\left(\frac{t^2}{2} + \frac{\Gamma(3-\beta) + k\Gamma(3-\alpha)}{\Gamma(3-\beta)\Gamma(3-\alpha)}\right)\right),$$

and $\alpha = \frac{3}{2}$, $\beta = 1$. The exact solution of (25)-(26) is given as [2]

$$y(t) = t^2 - t^3.$$

Using the above method, we draw Table 2.

Remark We found numerical results of Examples 6.1 and 6.2 by RKM and we show our results in Tables 1-3 and Figure 1. We used Maple 16 to obtain these results. We mention the time in Table 3. This proves that we can find good results in very short times.

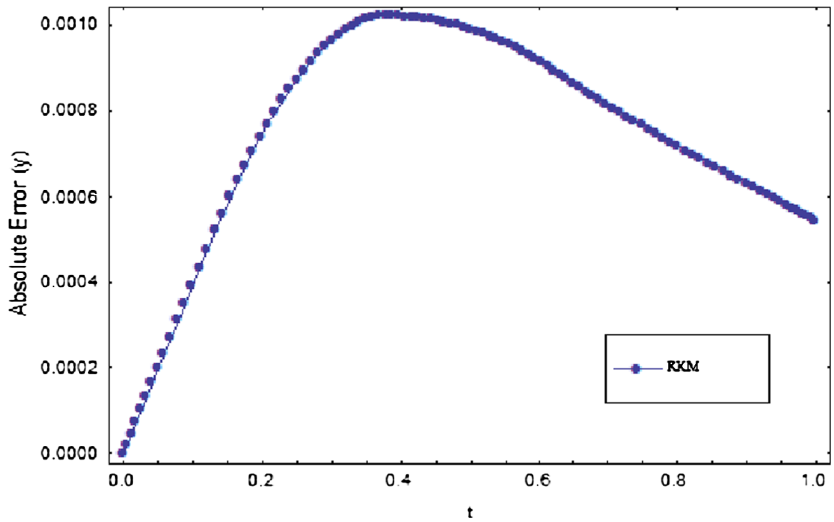

Figure 1 Absolute error of $y(t)$ for Example 6.1.

7 Conclusion

Fractional differential equations of Lane-Emden type were investigated by RKM. We explained the technique and managed it in some illustrative examples. The results obtained indicated that RKM can solve the problem with few computations. Numerical examples demonstrate that our method supports the theoretical results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have made equal contributions. All authors have read and approved the final manuscript.

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References

1. Turkyilmazoglu, M: Effective computation of exact and analytic approximate solutions to singular nonlinear equations of Lane-Emden-Fowler type. Appl. Math. Model. **37**, 7539-7548 (2013)
2. Mechee, MS, Senu, N: Numerical study of fractional differential equations of Lane-Emden type by method of collocation. Appl. Math. **3**, 851-856 (2012)
3. Rahba, WI: Existence of nonlinear Lane-Emden equation of fractional order. Miskolc Math. Notes **13**, 39-52 (2012)
4. Ye, H, Liu, F, Anh, V, Turner, I: Maximum principle and numerical method for the multi-term time-space Riesz-Caputo fractional differential equations. Appl. Math. Comput. **227**, 531-540 (2014)
5. Liu, F, Meerschaert, MM, McGough, R, Zhuang, P, Liu, Q: Numerical methods for solving the multi-term time fractional wave equations. Fract. Calc. Appl. Anal. **16**(1), 9-25 (2013)
6. Ye, H, Liu, F, Turner, I, Anh, V, Burrage, K: Series expansion solutions for the multi-term time and space fractional partial differential equations in two and three dimensions. Eur. Phys. J. Spec. Top. **222**, 1901-1914 (2013)
7. Jiang, H, Liu, F, Meerschaert, MM, McGough, R: Fundamental solutions for the multi-term modified power law wave equations in a finite domain. Electron. J. Math. Anal. Appl. **1**(1), 55-66 (2013)
8. Podlubny, I: Fractional Differential Equations. Mathematics in Science and Engineering, vol. 198. Academic Press, San Diego (1999)
9. Hilfer, R: Fractional calculus and regular variation in thermodynamics. In: Applications of Fractional Calculus in Physics, pp. 429-463. World Scientific, River Edge (2000)

10. Celik, E, Karaduman, E, Bayram, M: Numerical solutions of chemical differential-algebraic equations. Appl. Math. Comput. **139**(2-3), 259-264 (2003)
11. Secer, A, Akinlar, MA, Cevikel, A: Efficient solutions of systems of fractional PDEs by the differential transform method. Adv. Differ. Equ. **2012**, 188 (2012)
12. Kurulay, M, Bayram, M: Approximate analytical solution for the fractional modified KdV by differential transform method. Commun. Nonlinear Sci. Numer. Simul. **15**(7), 1777-1782 (2010)
13. Liao, SJ: Homotopy analysis method: a new analytic method for nonlinear problems. Appl. Math. Mech. **19**(10), 885-890 (1998)
14. Odibat, ZM, Momani, S: Application of variational iteration method to nonlinear differential equations of fractional order. Int. J. Nonlinear Sci. Numer. Simul. **7**(1), 27-34 (2006)
15. Biazar, J, Ghazvini, H: Exact solutions for non-linear Schrodinger equations by He's homotopy perturbation method. Phys. Lett. A **366**(1-2), 79-84 (2007)
16. Secer, A, Alkan, S, Akinlar, MA, Bayram, M: Sinc-Galerkin method for approximate solutions of fractional order boundary value problems. Bound. Value Probl. **2013**, 281 (2013)
17. Baleanu, D, Wu, GC, Duan, JS: Some analytical techniques in fractional calculus: realities and challenges. In: Tenreiro Machado, JA, Baleanu, D, Luo, ACJ (eds.) Discontinuity and Complexity in Nonlinear Physical Systems, pp. 35-62. Springer, New York (2014)
18. Gomez, A, Jose, F, Baleanu, D: Solutions of the telegraph equations using a fractional calculus approach. Proc. Rom. Acad., Ser. A : Math. Phys. Tech. Sci. Inf. Sci. **15**(1), 27-34 (2014)
19. Agarwal, RP, Lupulescu, V, O'Regan, D, Rahman, G: Fractional calculus and fractional differential equations in nonreflexive Banach spaces. Commun. Nonlinear Sci. Numer. Simul. **20**(1), 59-73 (2015)
20. Aronszajn, N: Theory of reproducing kernels. Trans. Am. Math. Soc. **68**, 337-404 (1950)
21. Cui, M, Lin, Y: Nonlinear Numerical Analysis in the Reproducing Kernel Space. Nova Science Publishers, New York (2009)
22. Geng, F, Cui, M: Solving a nonlinear system of second order boundary value problems. J. Math. Anal. Appl. **327**(2), 1167-1181 (2007)
23. Geng, F, Cui, M, Zhang, B: Method for solving nonlinear initial value problems by combining homotopy perturbation and reproducing kernel Hilbert space methods. Nonlinear Anal., Real World Appl. **11**(2), 637-644 (2010)
24. Akgül, A: A new method for approximate solutions of fractional order boundary value problems. Neural Parallel Sci. Comput. **22**(1-2), 223-237 (2014)
25. Abu Arqub, O, Al-Smadi, M: Numerical algorithm for solving two-point, second-order periodic boundary value problems for mixed integro-differential equations. Appl. Math. Comput. **243**(2), 911-922 (2014)
26. Jiang, W, Chen, Z: A collocation method based on reproducing kernel for a modified anomalous subdiffusion equation. Numer. Methods Partial Differ. Equ. **30**, 289-300 (2014)
27. Abu Arqub, O, Al-Smadi, M, Momani, S, Hayat, T: Numerical solutions of fuzzy differential equations using reproducing kernel Hilbert space method. Soft Comput. (2015, to appear)
28. Daniel, A: Reproducing Kernel Spaces and Applications. Springer, Basel (2003)
29. Berlinet, A, Agnan, CT: Reproducing Kernel Hilbert Space in Probability and Statistics. Kluwer Academic, Boston (2004)
30. Inc, M, Akgül, A, Geng, F: Reproducing kernel Hilbert space method for solving Bratu's problem. Bull. Malays. Math. Soc. **38**, 271-287 (2015)