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Performance evaluation of matched asymptotic expansions for fractional differential equations with multi-order

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Abstract

An extension of the concept of the asymptotic expansions method is presented in this paper. The multi-order differential equations of fractional order are investigated and the convergence of the proposed method is proven. The reported results show that the present approach is very effective and accurate and also are in good agreement with the ones in the literature.

Key Words: Asymptotic expansion, Caputo derivative, convergence analysis.
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1 Introduction

Fractional calculus is a field of mathematics that deals with theory of integrals and derivatives of any arbitrary real or complex orders. The number of applications of fractional calculus rapidly grows and many researchers have utilized it to describe the dynamics of various phenomena encountered in electrochemistry, mechanics, acoustic, thermal engineering, finance, hydrology and many other areas. For more applications and an extensive literature of fractional differential equation see for example [1] and the references therein.

We recall that finding the general solution of the fractional differential equations is an important issue. However, the exact solution of the fractional differential equations are often difficult to obtain and therefore we utilize the approximate solutions. Thus, there have been many attempts to develop new methods for obtaining solutions which reasonably approximate the exact ones. In recent years, several techniques have drawn special attention, e.g. the (G'/G)-expansion method [2], the Adomian's decomposition method [3], the homotopy perturbation method [4], the homotopy analysis method [5], and many others.

In this study, the fractional differential equations with multi-order are investigated by means of the systematic methods of perturbations (asymptotic expansions) in terms of a small or a large parameter or coordinates [6]. We discuss about the approximate solutions of this family of equations by these asymptotic expansions within Caputo derivative [1]. The fractional differential operator in the sense of Caputo, defined by

$$D_{*\tau}^{\alpha}\xi(\tau) = I_{\tau}^{m-\alpha}\xi^{(m)}(\tau),\ m-1<\alpha<m, \quad m\in\mathbb{N}, \tag{1}$$

and I_{τ}^{α} is the Riemann-Liouville integral operator of order α, defined by

$$I_{\tau}^{\alpha}\xi(\tau) = \frac{1}{\Gamma(\alpha)}\int_0^s \frac{\xi(\tau)}{(s-\tau)^{1-\alpha}}d\tau,\ \alpha>0,\ s>0, \tag{2}$$

and also, a multi-order fractional differential equations can be presented in the following form

$$\begin{aligned} D_{*\tau}^{\alpha_n}\xi(\tau) &= \digamma(\tau,\xi(\tau),D_{*\tau}^{\alpha_1}\xi(\tau),D_{*\tau}^{\alpha_2}\xi(\tau),\cdots,D_{*\tau}^{\alpha_{n-1}}\xi(\tau)), \\ \xi^{(j)}(0) &= \gamma_j,\ \ n\in\mathbb{N},\ \ j=0,1,2,\cdots, \end{aligned} \tag{3}$$

where $\alpha_i \in \mathbb{N}$, are in ascending order ($0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n$). Daftardar and Jafari [7], proved that Eq. (3) is equivalent to the system of equations in the following form

$$\begin{cases} D_{*\tau}^{\alpha_i}\xi_i(\tau) = \xi_{i+1}(\tau),\ i=1,2,\cdots,n-1, \\ D_{*\tau}^{\alpha_n}\xi_i(\tau) = \digamma(\tau,\xi_1(\tau),\xi_2(\tau),\cdots,\xi_n(\tau)), \\ \xi_i^k(0)=\gamma_k^i,\ k=0,1,2,\cdots,\alpha_i,\ i=1,2,\cdots,n. \end{cases} \tag{4}$$

2 Analysis of the expanding method

Below, we propose a formal construction with the expanding asymptotic method [6]. Our aim is to motivate the definitions of our scheme in order to establish the idea of the suggested method. Let us assume that

$$\xi_i(\tau) = \sum_{m=0}^{\infty}\xi_{i,m}\tau^{\alpha m}, \tag{5}$$

such that $\xi_{i,m}, \alpha \in \mathbb{R}$. If we introduce this formula in the first part of the system (4), we obtain

$$\sum_{m=0}^{\infty}\xi_{i,m}\frac{\Gamma(\alpha m+1)}{\Gamma(\alpha m+1-\alpha_i)}\tau^{\alpha m-\alpha_i} = \sum_{m=0}^{\infty}\xi_{i+1,m}\tau^{\alpha m}, \tag{6}$$

and after that we get

$$\xi_{i,k_i+m}\frac{\Gamma(\alpha m+\alpha_i+1)}{\Gamma(m\alpha+1)} = \xi_{i+1,m},\ m=0,1,2,\cdots,\ k_i=\frac{\alpha_i}{\alpha}. \tag{7}$$

In Eq. (4), let us set

$$F(\tau,\xi_1(\tau),\cdots,\xi_n(\tau)) = g(\tau) + \sum_{j=1}^{n}\phi_j(\tau)\xi_j(\tau) + N(\tau,\xi_1(\tau)\cdots,\xi_n(\tau)), \tag{8}$$

where $g(\tau)$ and $\phi_j(\tau)$ are arbitrary functions. As a result we obtain the following equation

$$g(\tau) = \sum_{m=0}^{\infty} g_m\tau^{\alpha m},\ \phi_j(\tau) = \sum_{m=0}^{\infty}\phi_{j,m}\tau^{\alpha m}. \tag{9}$$

The next step is to assume that

$$N(\tau,\xi_1(\tau),\cdots,\xi_n(\tau)) = N(\tau,\sum_{m=0}^{\infty}\xi_{1,m}\tau^{\alpha m},\cdots,\sum_{m=0}^{\infty}\xi_{n,m}\tau^{\alpha m}) = \sum_{m=0}^{\infty}N_m\tau^{\alpha m}, \tag{10}$$

where

$$\begin{aligned}
N_0 &= N_0(\tau,\xi_{1,0},\xi_{2,0},\cdots,\xi_{n,0}),\\
N_1 &= N_1(\tau,\xi_{1,0},\xi_{2,0},\cdots,\xi_{n,0},\xi_{1,1},\xi_{2,1},\cdots,\xi_{n,1}),\\
N_2 &= N_2(\tau,\xi_{1,0},\xi_{2,0},\cdots,\xi_{n,0},\xi_{1,1},\xi_{2,1},\cdots,\xi_{n,1},\xi_{1,2},\xi_{2,2},\cdots,\xi_{n,2}),\\
&\vdots
\end{aligned}$$

According to the expanding method, we obtain

$$\begin{aligned}
\sum_{m=0}^{\infty}\xi_{n,m}\frac{\Gamma(m\alpha+1)}{\Gamma(\alpha m-\alpha_n+1)}\tau^{\alpha m-\alpha_n} &= \sum_{m=0}^{\infty}g_m\tau^{\alpha m}\\
+\sum_{j=1}^{n}[\sum_{m=0}^{\infty}\phi_{j,m}\tau^{\alpha m}][\sum_{m=0}^{\infty}\xi_{j,m}\tau^{\alpha m}] &+ \sum_{m=0}^{\infty}N_m\tau^{\alpha m}.
\end{aligned} \tag{11}$$

Now, by equating the terms having the identical powers of τ, we get the following relations

$$\begin{aligned}
\xi_{n,k_n} &= \frac{1}{\Gamma(\alpha_n+1)}\{g_0 + \sum_{j=1}^{n}\gamma_0^j\phi_{j,0} + N_0\},\\
\xi_{n,k_n+1} &= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\alpha_n+1)}\{g_1 + \sum_{j=1}^{n}[\phi_{j,0}\xi_{j,1} + \gamma_0^j\phi_{j,1}] + N_1\},\\
\xi_{n,k_n+2} &= \frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha+\alpha_n+1)}\{g_2 + \sum_{j=1}^{n}[\phi_{j,0}\xi_{j,2} + \phi_{j,1}\xi_{j,1} + \phi_{j,2}\xi_{j,0}] + N_2\},\\
&\vdots
\end{aligned}$$

Thus, we get the approximate solution of Eq. (3) as $\xi(\tau) = \sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\xi_{i,m}\tau^{m\alpha}$.

3 A flexible framework for building the matched asymptotic expansions method

Now, as an extended alternative for Eq. (5), again we consider the following problem

$$D_{*\tau}^{\alpha_{n+1}}\xi(\tau) = F(\tau, \xi(\tau), D_{*\tau}^{\alpha_1}\xi(\tau), \cdots, D_{*\tau}^{\alpha_n}\xi(\tau)), \tag{12}$$

under the following initial conditions

$$\xi^{(k)}(0) = \gamma_k,\ k = 0, 1, \cdots, [\alpha_{n+1}] - 1 = m - 1, \tag{13}$$

where $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha_{n+1}$ and F is a given function on $D := [0, T] \times \mathbb{R}^{n+1}$.

Let us set

$$\begin{cases} \xi(\tau) = Y(\tau) + G(\tau),\ Y \in C^\infty, \\ G(\tau) = \sum_{k=0}^{m-1} \gamma_k \frac{\tau^k}{k!}. \end{cases} \tag{14}$$

Consequently, Eq. (12) can be written as

$$\begin{gathered} D_{*\tau}^{\alpha_{n+1}} Y(\tau) = F(\tau, Y(\tau) + G(\tau), L(\tau)), \\ Y^{(k)}(0) = 0,\ k = 0, 1, \cdots, m - 1. \end{gathered} \tag{15}$$

where $L(\tau) = D_{*\tau}^{\alpha_1} Y(\tau) + D_{*\tau}^{\alpha_1} G(\tau), \cdots, D_{*\tau}^{\alpha_n} Y(\tau) + D_{*\tau}^{\alpha_n} G(\tau)$.
Assume now that

$$D_{*\tau}^{\alpha_{n+1}} Y(\tau) = \overline{Y}(\tau). \tag{16}$$

Thus, for $j = 1, 2, \cdots, n + 1$ we have

$$D_{*\tau}^{\alpha_j} Y(\tau) = I_\tau^{\alpha_{n+1}-\alpha_j} D_{*\tau}^{\alpha_{n+1}} Y(\tau) = I_\tau^{\alpha_{n+1}-\alpha_j} D_{*\tau}^{\alpha_{n+1}} (I_\tau^{\alpha_{n+1}} \overline{Y}(\tau)) = I_\tau^{\alpha_{n+1}-\alpha_j} \overline{Y}(\tau). \tag{17}$$

Substituting Eq. (17) into (15), we obtain

$$\overline{Y}(\tau) = F(\tau, I_\tau^{\alpha_{n+1}} \overline{Y}(\tau) + G(\tau), I_\tau^{\alpha_{n+1}-\alpha_1} \overline{Y} + D_{*\tau}^{\alpha_1} G(\tau), \cdots, I_\tau^{\alpha_{n+1}-\alpha_n} \overline{Y} + D_{*\tau}^{\alpha_n} G(\tau)). \tag{18}$$

In Eq. (18), we set

$$\begin{aligned} &F(\tau, I_\tau^{\alpha_{n+1}} \overline{Y} + G(\tau), I_\tau^{\alpha_{n+1}-\alpha_1} \overline{Y} + D_{*\tau}^{\alpha_1} G(\tau), \cdots, I_\tau^{\alpha_{n+1}-\alpha_n} \overline{Y} + D_{*\tau}^{\alpha_n} G(\tau)) \\ &= \widehat{f}(\tau) + \widehat{F}(\tau, I_\tau^{\alpha_{n+1}} \overline{Y}, I_\tau^{\alpha_1} \overline{Y}(\tau), \cdots, I_\tau^{\alpha_n} \overline{Y}(\tau)), \end{aligned} \tag{19}$$

where $\widehat{f}(\tau)$ represents a given function that does not contain $I_\tau^{\alpha_{n+1}} \overline{Y}$ and $I_\tau^{\alpha_i} \overline{Y}(\tau)$, $i = 1, 2, \cdots, n$. We assume that the solution of (18) has the form

$$\overline{Y}(\tau) = \sum_{i=0}^{\infty} \overline{Y}_i(\tau), \tag{20}$$

where $\overline{Y}_i(\tau)$ will be determined upon the following iteration algorithm

$$\overline{Y}_0(\tau) = \widehat{f}(\tau),\ \overline{Y}_1(\tau) = \widehat{F}(\tau, I_\tau^{\alpha_{n+1}} \overline{Y}_0, I_\tau^{\alpha_{n+1}-\alpha_1} \overline{Y}_0, \cdots, I_\tau^{\alpha_{n+1}-\alpha_n} \overline{Y}_0),$$

and

$$\overline{Y}_i(\tau) = \widehat{F}(\tau, I_\tau^{\alpha_{n+1}} \sum_{j=0}^{i-1} \overline{Y}_j, I_\tau^{\alpha_{n+1}-\alpha_1} \sum_{j=0}^{k-1} \overline{Y}_j, \cdots, I_\tau^{\alpha_{n+1}-\alpha_n} \sum_{j=0}^{i-1} \overline{Y}_j),\ i = 2, 3, \cdots.$$

We approximate the solution $\xi(\tau)$ by the truncated series

$$I_\tau^{\alpha_{n+1}} \widehat{f}(\tau) + \sum_{j=0}^{n} I_\tau^{\alpha_{n+1}} \overline{Y}_j(\tau). \tag{21}$$

3.1 Analysis of the convergence

In this section, we provide the sufficient condition for the convergence of the solution series. Let $C([0,T])$ be the space of all continuous functions defined on $[0,T]$ with norm

$$\|\xi(\tau)\|_\infty = \max_{\tau \in [0,T]} |\xi(\tau)|,\ \forall \xi(\tau) \in C([0,T]). \tag{22}$$

Theorem 1. *Let the continuous function F in (12) satisfy*

$$|F(\tau, z_0, z_1, \cdots, z_n) - F(\tau, x_0, x_1, \cdots, x_n)| \le \sum_{j=0}^{k-1} \Lambda_j |z_j - x_j|,\ \Lambda_j \in \mathbb{R}^+. \tag{23}$$

Also, we assume that the orders $\alpha_j,\ j = 1, 2, \cdots, n$ are rational. Then, (12) subject to (13) has a unique continuous solution on an interval $[0,T]$ of the real line.

Theorem 2. *Let F defined in (12) be the continuous functions on $D := [0,T] \times \mathbb{R}^{n+1}$ and there exist nonnegative functions $p_0(\tau), p_1(\tau), \cdots, p_n(\tau) \in L[0,T]$ such that*

$$|F(\tau, z_0, z_1, \xi, z_n) - F(\tau, x_0, x_1, \xi, x_n)| \le p_0(\tau)|z_0 - x_0| + \xi + p_n(\tau)|z_n - x_n|, \tag{24}$$

and assume that $d < 1$, where

$$\begin{aligned} d &= \sum_{i=1}^{n} \frac{1}{\Gamma(\alpha_{n+1} - \alpha_i)} \int_0^\tau (\tau - t)^{(\alpha_{n+1} - \alpha_i - 1)} p_i(\tau) dt \\ &+ \frac{1}{\Gamma(\alpha_{n+1})} \int_0^\tau (\tau - t)^{(\alpha_{n+1} - 1)} p_0(\tau) dt. \end{aligned} \tag{25}$$

Also, suppose that $\overline{Y}_0 \in N_r(\overline{Y})$ where $N_r(\overline{Y}) = \{\overline{\xi} \in C[0,T]; \|\overline{\xi} - \overline{Y}\| < r\}$. Then, the sequence of partial sums of the series solution is absolutely convergent

Proof. Define the sequence $\{S_k\}_{k=0}^{\infty}$ as

$$\begin{cases} S_0(\tau) = \overline{Y}_0(\tau), \\ S_1(\tau) = \overline{Y}_0(\tau) + \overline{Y}_1(\tau), \\ \vdots \\ S_k(\tau) = \overline{Y}_0(\tau) + \overline{Y}_1(\tau) + \cdots + \overline{Y}_k(\tau). \end{cases} \tag{26}$$

Let $S_k(\tau)$ be an arbitrary partial sums. Then for $M = |S_k(\tau) - \overline{Y}(\tau)|$ we get

$$\begin{aligned} M &= \big|\widehat{F}(\tau, I_\tau^{\alpha_{n+1}}\sum_{j=0}^{k-1}\overline{Y}_j, \cdots, I_\tau^{\alpha_{n+1}-\alpha_n}\sum_{j=0}^{k-1}\overline{Y}_j) - \widehat{F}(\tau, I_\tau^{\alpha_{n+1}}\overline{Y}, \cdots, I_\tau^{\alpha_{n+1}-\alpha_n}\overline{Y})\big| \\ &\leq \frac{1}{\Gamma(\alpha_{n+1})}\int_0^\tau (\tau-t)^{(\alpha_{n+1}-1)}p_0(\tau)|S_{k-1}(\tau) - \overline{Y}(\tau)\big|dt \\ &+ \sum_{i=1}^{n}\frac{1}{\Gamma(\alpha_{n+1}-\delta_i)}\int_0^\tau (\tau-t)^{(\alpha_{n+1}-\alpha_i-1)}p_i(\tau)\big|S_{k-1}(\tau) - \overline{Y}(\tau)\big|dt. \\ &\leq \frac{1}{\Gamma(\alpha_{n+1})}\int_0^\tau (\tau-t)^{(\alpha_{n+1}-1)}p_i(\tau)dt\big\|S_{k-1}(\tau) - \overline{Y}(\tau)\big\|_\infty \\ &+ \sum_{i=1}^{n}\frac{1}{\Gamma(\alpha_{n+1}-\alpha_i)}\int_0^\tau (\tau-t)^{(\alpha_{n+1}-\alpha_i-1)}p_i(\tau)dt\big\|S_{k-1}(\tau) - \overline{Y}(\tau)\big\|_\infty. \end{aligned} \tag{27}$$

Hence, we have $\|S_k(\tau) - \overline{Y}(\tau)\|_\infty \leq d\big\|S_{k-1}(\tau) - \overline{Y}(\tau)\big\|_\infty$. Proceeding by induction, we obtain

$$\|S_q - \overline{Y}(\tau)\|_\infty \leq d^q\big\|S_0 - \overline{Y}(\tau)\big\|, \; q = 0, 1, \cdots, k. \tag{28}$$

Since $S_0 = \overline{Y}_0 \in N_r(\overline{Y})$ and $d < 1$, then $\lim_{k\to\infty} d^k = 0$, therefore the sequence $\{S_k\}_{k=0}^{\infty}$ is absolutely convergent. □

Theorem 3. *Let F defined in (12) be continuous functions on $D := [0, T] \times \mathbb{R}^{n+1}$ such that*

$$|F(\tau, z_0, z_1, \cdots, z_n) - F(\tau, x_0, x_1, \cdots, x_n)| \leq \Lambda_0|z_0 - x_0| + \cdots + \Lambda_n|z_n - x_n|, \tag{29}$$

where Λ, is a constant and assume that

$$\frac{T^{\alpha_{n+1}}}{\alpha_{n+1}!} + \sum_{i=1}^{n}\frac{T^{(\alpha_{n+1}-\alpha_i)}}{\Gamma(\alpha_{n+1}-\alpha_i+1)} < \frac{1}{\Lambda}, \Lambda = max\{\Lambda_j\}, \; j = 0, 1, \cdots, n. \tag{30}$$

Also, suppose that $S_0 \in N_r(\overline{Y})$ where $N_r(\overline{Y}) = \{\overline{\xi} \in C[0, T]; |\overline{\xi} - \overline{Y}\| < r\}$. Then, the series $\{S_k\}_{k=0}^{\infty}$ defined in (26) is absolutely convergent.

Proof. We prove that $\{S_k\}_{k=0}^{\infty}$ defined in (26) is absolutely convergent. Let $S_k(\tau)$ be an arbitrary partial sums. Similar to (27) we get

$$|S_k(\tau)-\overline{Y}(\tau)| \leq \frac{\Lambda T^{\alpha_{n+1}}}{\alpha_{n+1}!}\left\|S_{k-1}(\tau)-\overline{Y}(\tau)\right\|_{\infty}+\sum_{i=1}^{n}\frac{\Lambda T^{(\alpha_{n+1}-\alpha_i)}}{\Gamma(\alpha_{n+1}-\alpha_i+1)}\left\|S_{k-1}(\tau)-\overline{Y}(\tau)\right\|_{\infty}. \tag{31}$$

Hence we obtain

$$\|S_k(\tau)-\overline{Y}(\tau)\|_{\infty} \leq \Lambda\left\|S_{k-1}(\tau)-\overline{Y}(\tau)\right\|_{\infty}\left(\frac{T^{\alpha_{n+1}}}{\alpha_{n+1}!}+\sum_{i=1}^{n}\frac{T^{(\alpha_{n+1}-\alpha_i)}}{\Gamma(\alpha_{n+1}-\alpha_i+1)}\right). \tag{32}$$

Proceeding by induction we obtain

$$\|S_q-\overline{Y}(\tau)\|_{\infty} \leq \Lambda^q\left(\frac{T^{\alpha_{n+1}}}{\alpha_{n+1}!}+\sum_{i=1}^{n}\frac{T^{(\alpha_{n+1}-\alpha_i)}}{\Gamma(\alpha_{n+1}-\alpha_i+1)}\right)^q\left\|S_0-\overline{Y}(\tau)\right\|_{\infty}, \tag{33}$$

where $q=0,1,\cdots,k, S_0=\overline{Y}_0 \in N_r(\overline{Y})$ and

$$\lim_{k\to\infty}\Lambda^k\left(\frac{T^{\alpha_{n+1}}}{\alpha_{n+1}!}+\sum_{i=1}^{n}\frac{T^{(\alpha_{n+1}-\alpha_i)}}{\Gamma(\alpha_{n+1}-\alpha_i+1)}\right)^k=0. \tag{34}$$

Consequently, we derive $\lim_{k\to\infty}\|S_k-\overline{Y}(\tau)\|_{\infty}=0$, and this completes the proof. □

4 Applications

In order to demonstrate the performance of the present method as a novel solver for multi order fractional differential equations, some different problems were selected as test examples. In all cases we have $\xi \in C^k[0,T],\ T<\infty,\ k=1,2,\cdots$.

Example 1. *Consider the following initial value problem for the inhomogeneous Bagley-Torvik equation [8]*

$$\begin{aligned} MD^2_{*\tau}\xi(\tau) &+ 2S\sqrt{\mu\rho}\ D^{3/2}_{*\tau}\xi(\tau)+K\xi(\tau)=\vartheta(\tau),\ 0\leq\tau\leq T, \\ \xi(0) &= 1, \quad \xi'(0)=1. \end{aligned} \tag{35}$$

In order to make comparison with the numerical solution of [8] we choose $M=2S\sqrt{\mu\rho}=K=1, T=5$ *and* $\vartheta(\tau)=K(\tau+1)$. *By the same manipulation as Section 2 we set*

$$\begin{aligned} \xi_{1,0}(\tau) &= 1+\tau, \\ \xi_{1,m+1}(\tau) &= I^{1.5}\xi_{2,m},\ m=0,1,\cdots, \\ \xi_{2,0}(\tau) &= 0, \\ \xi_{2,1}(\tau) &= 0, \\ \xi_{2,m+1}(\tau) &= -I^{0.5}(\xi_{1,m}(\tau)+\xi_{2,m}(\tau)),\ m=1,2,\cdots. \end{aligned} \tag{36}$$

Thus, we obtain

$$\begin{cases} \xi_{1,m+1}(\tau) = 0, \\ \xi_{2,m+1}(\tau) = 0, \ m = 0, 1, \cdots . \end{cases} \tag{37}$$

Hence, $\xi_1(\tau) = 1 + \tau$ *and* $\xi_2(\tau) = 0$*. So,* $\xi(\tau) = 1 + \tau$ *is the solution of Eq. (35). It is easily verified that* $\tau + 1$ *is the exact solution of Eq. (35).*
Table (1) shows the resulting error at $\tau = 5$ *obtained by numerical method in [8] and compared with the solution obtained by the proposed scheme.*

Table 1: The resulting error at $\tau = 5$

Error at $\tau = 5$ *by proposed method*	*Error at* $\tau = 5$ *by [8]*	*(step size)*
0	−0.15131473519232	(0.5000)
0	−0.04684102179946	(0.2500)
0	−0.01602947553912	(0.1250)
0	−0.00562770408881	(0.0625)

Example 2. *Consider the following equation*

$$\alpha D^2_{*\tau}\xi(\tau) + \beta D^{\alpha_2}_{*\tau}\xi(\tau) + \theta D^{\alpha_1}_{*\tau}\xi(\tau) + \lambda\xi^3(\tau) = g(\tau), \ \alpha, \beta, \theta, \lambda \in \mathbb{R}. \tag{38}$$

where

$$g(\tau) = 2\alpha\tau + \frac{2\beta\tau^{3-\alpha_2}}{\Gamma(4-\alpha_2)} + \frac{2\theta\tau^{3-\alpha_1}}{\Gamma(4-\alpha_1)} + \frac{\lambda\tau^9}{27}, \ 0 < \alpha_1 < \alpha_2 \leq 1, \ \xi(0) = \xi'(0) = 0. \tag{39}$$

By the same manipulation as Section 3, the first few terms of the series are given by $\overline{Y}_0(\tau) = \frac{1}{\alpha}g(\tau)$ *and*

$$\overline{Y}_i(\tau) = -\frac{1}{\alpha}(\beta I^{2-\alpha_2}_\tau \overline{Y}_{i-1}(\tau) + \theta I^{2-\alpha_1}_\tau \overline{Y}_{i-1}(\tau) + \lambda[(I^2_\tau \sum_{j=0}^{i-1} \overline{Y}_j(\tau))^3 - (I^2_\tau \sum_{j=0}^{i-2} \overline{Y}_j(\tau))^3]).$$

By considering $\alpha = \beta = \theta = \lambda = 1, \alpha_1 = 1$ *and* $\alpha_2 = 0$*, the exact solution of Eq. (38) can be written as* $\xi(\tau) = \frac{\tau^3}{3}$*. Under this assumption, the exact and approximate solution* ($\xi(\tau) \approx I^2_\tau(\overline{Y}_0(\tau) + \overline{Y}_1(\tau))$) *of Eq. (38) are shown in Fig. 1. This example shows that the result of the present method (after only 3 iterations) is in excellent agreement with the exact solution. Furthermore, the approximate solution using* (G'/G)*-expansion method and Adomian decomposition method (ADM) (after 20 iterations) are presented.*

5 Concluding remarks

In this manuscript the matched asymptotic expansions method was successfully applied to compute the approximate solution of the fractional differential equations with multi-orders. The convergence analysis of our solutions was thoroughly reported. The obtained results proved the performance robustness of

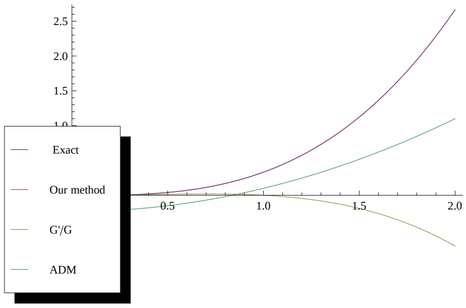

Fig. 1: Numerical convergence of the exact and approximate solution.

the proposed scheme. Furthermore, we notice that a large category of analytical methods such as (G'/G)-expansion method, Adomian decomposition method and etc., are in principle based on the matched asymptotic expansions method. In other words, the matched asymptotic expansions method includes an auxiliary parameter which can control and adjust the convergence region of the series solution. For instance, by using (G'/G)-expansion method we can write the solution of Eq. (38) in the form $\xi(\tau) = \sum_{j=0}^{N}(\frac{G'}{G})^j$ where G satisfies the Eq. (38) but the obtained results are not satisfactory. Similar results are reported for Adomian decomposition method.

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