GENERALIZED LAGUERRE-GAUSS-RADAU SCHEME FOR FIRST ORDER HYPERBOLIC EQUATIONS ON SEMI-INFINITE DOMAINS

A.H. BHRAWY[1,2], R.M. HAFEZ[3], E.O. ALZAHRANI[1], D. BALEANU[4,5], A.A. ALZAHRANI[6]

[1]Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt
[3]Department of Basic Science, Institute of Information Technology, Modern Academy, Cairo, Egypt
[4]Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Cankaya University, Balgat, Ankara, Turkey
[5]Institute of Space Sciences, Magurele-Bucharest, RO-077125, Romania
E-mail: dumitru@cankaya.edu.tr
[6]Department of Chemical and Materials Engineering, Faculty of Engineering, King Abdulaziz University, Jeddah, Saudi Arabia

Received December 15, 2014

In this article, we develop a numerical approximation for first-order hyperbolic equations on semi-infinite domains by using a spectral collocation scheme. First, we propose the generalized Laguerre-Gauss-Radau collocation scheme for both spatial and temporal discretizations. This in turn reduces the problem to the obtaining of a system of algebraic equations. Second, we use a Newton iteration technique to solve it. Finally, the obtained results are compared with the exact solutions, highlighting the performance of the proposed numerical method.

Key words: First-order hyperbolic equations; Two-dimensional hyperbolic equations; Collocation method; Generalized Laguerre-Gauss-Radau quadrature.

PACS: 02.30.Gp, 02.30.Jr, 02.30.Mv, 02.60.-x.

1. INTRODUCTION

A simple example of a hyperbolic partial differential equation (PDE) is the wave equation. Hyperbolic PDEs describe a wide range of problems in various fields of science and engineering such as the phenomena of turbulence and supersonic flow, flow of a shock wave traveling in a non-viscous and viscous fluid [1], traffic and aerofoil flow theory [2], process engineering [3], population based modeling and batch crystallization [4], acoustic transmission [5], hypoelastic solids [6], astrophysics [7], and many other disciplines. More recently, the study of exact and numerical solutions of either hyperbolic or parabolic PDEs has received a lot of interest [8–24].

Several mathematical physics problems are studied on semi-infinite domains. The earthquake engineering field and underwater acoustic problems can be modeled as semi-infinite domain PDEs. Spectral methods (see for instance, [25–31]) based on specific polynomials/functions (Laguerre, Hermite, rational Legendre, rational Jacobi functions, etc.) [32–35] may be utilized to numerically solve problems on semi-infinite domains. The mapping problem in an unbounded domain to that in a bounded domain has been used in [36–38] to approximate the problems in unbounded domains. For more details about numerical solutions for unbounded domain problems, see for example [39–44]. The spectral collocation method [45–54] is a very useful technique for approximating several kinds of equations. Due to its advantages, such as exponential rate convergence, good accuracy and computational efficiency, the collocation method has been used successfully in many different fields of science and engineering. Orthogonal polynomials are usually used as basis functions in the numerical approximation of the collocation method. Here, we use the generalized Laguerre polynomials as basis functions.

The main aim of this article is to extend the application of the collocation method to solve one- and two-dimensional space hyperbolic PDEs of first-order on semi infinite domains. We propose a generalized Laguerre-Gauss-Radau collocation (GLGRC) scheme for both spatial and temporal discretizations. The main advantage of this scheme is to reduce the considered problems to systems of algebraic equations. The algebraic system of equations is solved by Newton's iterative method. Finally, numerical results of both one- and two-dimensional hyperbolic PDEs on semi infinite domains are given to highlight the applicability and validity of the present numerical approach.

This article is arranged as follows. We present few definitions and preliminaries of generalized Laguerre polynomials (GLPs) in Sec. 2. In Sec. 3, we present the GLGRC method for both one- and two-dimensional first-order hyperbolic PDEs on semi-infinite domains. Numerical examples and simulations are presented in Sec. 4 to show the effectiveness and accuracy of the proposed method. Some conclusions are drawn in the last section.

2. PRELIMINARIES

We recall below some revelent properties of the GLPs [55–59]. Now, let $\Lambda = (0, \infty)$ and $w^{(\alpha)}(x) = x^{\alpha} e^{-x}$ be a weight function on Λ in the usual sense. Define the space

$$L^2_{w^{(\alpha)}}(\Lambda) = \{v \mid v \text{ that is measurable on } \Lambda \text{ and is } ||v||_{w^{(\alpha)}} < \infty\},$$

equipped with the following inner product and norm

$$(u, v)_{w^{(\alpha)}} = \int_{\Lambda} u(x)\, v(x)\, w^{(\alpha)}(x)\, dx, \qquad ||v||_{w^{(\alpha)}} = (v, v)^{\frac{1}{2}}_{w^{(\alpha)}}.$$

Next, let $L_i^{(\alpha)}(x)$ be the GLPs of degree i for $\alpha > -1$ that is defined by

$$L_i^{(\alpha)}(x) = \frac{1}{i!} x^{-\alpha} e^x \partial_x^i (x^{i+\alpha} e^{-x}), \quad i = 1, 2, \cdots.$$

For $\alpha > -1$, we obtain [60]

$$\partial_x L_i^{(\alpha)}(x) = -L_{i-1}^{(\alpha+1)}(x),$$

$$L_{i+1}^{(\alpha)}(x) = \frac{1}{i+1}[(2i+\alpha+1-x)L_i^{(\alpha)}(x) - (i+\alpha)L_{i-1}^{(\alpha)}(x)], \quad i = 1, 2, \ldots,$$

where $L_0^{(\alpha)}(x) = 1$ and $L_1^{(\alpha)}(x) = 1+\alpha-x$.
The set of GLPs is the $L^2_{w^{(\alpha)}}(\Lambda)$-orthogonal system

$$\int_0^\infty L_j^{(\alpha)}(x) L_k^{(\alpha)}(x) w^{(\alpha)}(x) dx = h_k \delta_{jk}, \tag{1}$$

where $h_k = \frac{\Gamma(k+\alpha+1)}{k!}$. The analytical form of GLPs, is given by [55]:

$$L_i^{(\alpha)}(x) = \sum_{k=0}^{i} (-1)^k \frac{\Gamma(i+\alpha+1)}{\Gamma(k+\alpha+1)\,(i-k)!\,k!}\, x^k, \quad i = 0, 1, \cdots. \tag{2}$$

The special value is

$$D^q L_i^{(\alpha)}(0) = (-1)^q \sum_{j=0}^{i-q} \frac{(i-j-1)!}{(q-1)!(i-j-q)!} L_j^{(\alpha)}(0), \quad i \geqslant q, \tag{3}$$

where $L_j^{(\alpha)}(0) = \frac{\Gamma(j+\alpha+1)}{\Gamma(\alpha+1)j!}$.
Let $u(x) \in L^2_{w^{(\alpha)}}(\Lambda)$, then $u(x)$ can be expanded by means of GLPs as

$$u(x) = \sum_{j=0}^{\infty} a_j L_j^{(\alpha)}(x), \tag{4}$$

and in numerical approximations, the GLPs up to degree $N+1$ are considered.

We present now the quadratures based on GLPs, including Laguerre-Gauss and Laguerre-Gauss-Radau. Let $\{x_{N,j}^{(\alpha)}, \varpi_{N,j}^{(\alpha)}\}$ be the set of generalized Laguerre-Gauss or generalized Laguerre-Gauss-Radau quadrature nodes and weights (see *e.g.* [55]).

$$\int_\Lambda \phi(x) w^{(\alpha)}(x) dx = \sum_{j=0}^{N} \phi(x_{N,j}^{(\alpha)}) \varpi_{N,j}^{(\alpha)} \tag{5}$$

- For the generalized Laguerre-Gauss quadrature:

$\{x_{N,j}^{(\alpha)}\}$ are the zeros of $L_{i+1}^{(\alpha)}(x)$;

$$\begin{aligned}\varpi_{N,j}^{(\alpha)} &= -\frac{\Gamma(i+\alpha+1)}{(i+1)!L_i^{(\alpha)}(x_{N,j}^{(\alpha)})\partial_x L_{i+1}^{(\alpha)}(x_{N,j}^{(\alpha)})}\\ &= \frac{\Gamma(i+\alpha+1)\, x_{N,j}^{(\alpha)}}{(i+\alpha+1)(i+1)![L_i^{(\alpha)}(x_{N,j}^{(\alpha)})]^2}, \qquad 0 \le j \le i.\end{aligned}$$

- For the generalized Laguerre-Gauss-Radau quadrature:
  $x_{N,0}^{(\alpha)} = 0$, $\{x_{N,j}^{(\alpha)}\}_{j=1}^{i}$ are the zeros of $\partial_x L_{i+1}^{(\alpha)}(x)$;

$$\begin{aligned}\varpi_{N,0}^{(\alpha)} &= \frac{(\alpha+1)\Gamma^2(\alpha+1)\Gamma(i+1)}{\Gamma(i+\alpha+2)},\\ \varpi_{N,j}^{(\alpha)} &= \frac{\Gamma(i+\alpha+1)}{i!(i+\alpha+1)[\partial_x L_i^{(\alpha)}(x_{N,j}^{(\alpha)})]^2}\\ &= \frac{\Gamma(i+\alpha+1)}{i!(i+\alpha+1)(L_i^{(\alpha)}(x_{N,j}^{(\alpha)}))^2}, \qquad 1 \le j \le i.\end{aligned}$$

3. GLGRC METHOD

In this section, we derive the GLGRC method and describe its implementation for solving the one- and two-dimensional hyperbolic PDEs of first-order. The core of the proposed method is to discretize the equation in the spatial and temporal directions and create a system of algebraic equations of the unknown expansion coefficients. The collocation points are selected to be the generalized Laguerre-Gauss-Radau interpolation nodes.

3.1. ONE-DIMENSIONAL HYPERBOLIC PDES OF FIRST-ORDER

The main objective of this subsection is to develop the collocation method to solve numerically the one-dimensional hyperbolic equations of first-order in the following form

$$\partial_t u(x,t) = a_1 \partial_x u(x,t) + a_2 u(x,t) + H(x,t), \quad (x,t) \in [0,\infty) \times [0,\infty), \tag{6}$$

subject to the initial conditions

$$u(x,0) = g_0(x), \quad x \in [0,\infty), \tag{7}$$

$$u(0,t) = g_1(t), \quad t \in [0,\infty), \tag{8}$$

where a_1 and a_2 are constants, while $H(x,t)$, $g_0(x)$ and $g_1(t)$ are given functions. Here, we use the set of generalized Laguerre-Gauss-Radau points for the spatial and temporal approximations. To this end, we approximate the space and time variables using GLGRC method at $x_{N,r}^{(\alpha_1)}$ and $t_{M,s}^{(\alpha_2)}$ nodes. These collocation points are distributed in the semi-infinite interval. Now, we outline the main steps of the collocation method for solving the previous $1+1$ hyperbolic equations of first-order. Assume we approximate the solution as a finite double expansion of the form,

$$\begin{aligned} u_{N,M}(x,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} L_i^{(\alpha_1)}(x) L_j^{(\alpha_2)}(t) \\ &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x,t), \end{aligned} \tag{9}$$

where

$$f_0^{i,j}(x,t) = L_i^{(\alpha_1)}(x) L_j^{(\alpha_2)}(t).$$

Then the spatial and temporal partial derivatives ($\partial_x u_{N,M}(x,t)$ and $\partial_t u_{N,M}(x,t)$) may be written as

$$\begin{aligned} \partial_x u_{N,M}(x,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} \partial_x (L_i^{(\alpha_1)}(x)) L_j^{(\alpha_2)}(t) \\ &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_1^{i,j}(x,t), \end{aligned} \tag{10}$$

$$\begin{aligned} \partial_t u_{N,M}(x,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} L_i^{(\alpha_1)}(x) \partial_t (L_j^{(\alpha_2)}(t)) \\ &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_2^{i,j}(x,t), \end{aligned} \tag{11}$$

where

$$f_1^{i,j}(x,t) = \partial_x (L_i^{(\alpha_1)}(x)) L_j^{(\alpha_2)}(t),$$

and

$$f_2^{i,j}(x,t) = L_i^{(\alpha_1)}(x) \partial_t (L_j^{(\alpha_2)}(t)).$$

Now, adopting (9)-(11), enable one to write (6)-(8) in the form:

$$\begin{aligned} \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_2^{i,j}(x,t) =& a_1 \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_1^{i,j}(x,t) + a_2 \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x,t) \\ &+ H(x,t), \qquad (x,t) \in [0,\infty) \times [0,\infty). \end{aligned} \tag{12}$$

The functions $f_1^{i,j}(x,t)$ and $f_2^{i,j}(x,t)$, can be explicitly obtained by using (3) with $q=1$. The initial conditions immediately give

$$\begin{aligned} u_{N,M}(x,0) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x,0) = g_0(x), \\ u_{N,M}(0,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(0,t) = g_1(t). \end{aligned} \tag{13}$$

In the proposed collocation method the residual of (12) is set to be zero at $(N\,M)$ collocation points; this yields $(M\,N)$ algebraic equations in $(M+1)(N+1)$ unknown expansion coefficients, $a_{i,j},\ i=0,\cdots,N;\ j=0,\cdots,M,$

$$\sum_{i=0}^{N}\sum_{j=0}^{M} F_{r,s}^{i,j}\, a_{i,j} = H(x_{N,r}^{(\alpha_1)}, t_{M,s}^{(\alpha_2)}), \quad r=1,\cdots,N; \quad s=1,\cdots,M, \tag{14}$$

where

$$F_{r,s}^{i,j} = f_2^{i,j}(x_{N,r}^{(\alpha_1)}, t_{M,s}^{(\alpha_2)}) - a_1\, f_1^{i,j}(x_{N,r}^{(\alpha_1)}, t_{M,s}^{(\alpha_2)}) - a_1\, f_0^{i,j}(x_{N,r}^{(\alpha_1)}, t_{M,s}^{(\alpha_2)}),$$

and the initial conditions (13) give

$$\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x_{N,r}^{(\alpha_1)}, 0) = g_0(x_{N,r}^{(\alpha_1)}), \quad r=1,\cdots,N, \tag{15}$$

and

$$\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}^{i,j} f_0^{i,j}(0, t_{M,s}^{(\alpha_2)}) = g_1(t_{M,s}^{(\alpha_2)}), \quad s=0,\cdots,M, \tag{16}$$

and this in turn, yields $(M+1)(N+1)$ algebraic equations, namely

$$\begin{aligned} &\sum_{i=0}^{N}\sum_{j=0}^{M} F_{r,s}^{i,j}\, a_{i,j} = H(x_{N,r}^{(\alpha_1)}, t_{M,s}^{(\alpha_2)}), \quad r=1,\cdots,N; \quad s=1,\cdots,M, \\ &\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j} f_0^{i,j}(x_{N,r}^{(\alpha_1)}, 0) = g_0(x_{N,r}^{(\alpha_1)}), \quad r=1,\cdots,N, \\ &\sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}^{i,j} f_0^{i,j}(0, t_{M,s}^{(\alpha_2)}) = g_1(t_{M,s}^{(\alpha_2)}), \quad s=0,\cdots,M. \end{aligned} \tag{17}$$

The resulting system of algebraic equations (14) is then solved by any standard iterative solver.

3.2. TWO-DIMENSIONAL HYPERBOLIC PDES OF FIRST-ORDER

This subsection is devoted to extend the GLGRC method discussed in the previous subsection to numerically solve the two-dimensional hyperbolic PDEs of first-order of the following form

$$\begin{aligned}\partial_t u(x,y,t) =& a_0\,\partial_x u(x,y,t) + a_1\,\partial_y u(x,y,t) + a_2\,u(x,y,t) + H(x,y,t),\\ &(x,y,t)\in[0,\infty)\times[0,\infty)\times[0,\infty),\end{aligned} \tag{18}$$

subject to the initial conditions

$$\begin{aligned}u(x,y,0) &= g_0(x,y), \quad (x,y)\in[0,\infty)\times[0,\infty),\\ u(0,y,t) &= g_1(y,t), \quad (y,t)\in[0,\infty)\times[0,\infty),\\ u(x,0,t) &= g_2(x,t), \quad (x,t)\in[0,\infty)\times[0,\infty),\end{aligned} \tag{19}$$

where a_0, a_1 and a_2 are constants, while $H(x,y,t)$, $g_0(x,y)$, $g_1(y,t)$ and $g_2(x,t)$ are given functions. Also, we use the set of generalized Laguerre-Gauss-Radau points for the spatial and temporal approximations. Now, we outline the main steps of the collocation method for solving the two-dimensional hyperbolic equations of first-order. Let

$$\begin{aligned}u_{N,M,K}(x,y,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} L_i^{(\alpha_0)}(x) L_j^{(\alpha_1)}(y) L_k^{(\alpha_2)}(t)\\ &= \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{j=0}^{K} a_{i,j,k} f_0^{i,j,k}(x,y,t),\end{aligned} \tag{20}$$

where

$$f_0^{i,j,k}(x,y,t) = L_i^{(\alpha_0)}(x) L_j^{(\alpha_1)}(y) L_k^{(\alpha_2)}(t).$$

Then the spatial and temporal partial derivatives ($\partial_x u_{N,M,K}(x,y,t)$, $\partial_y u_{N,M,K}(x,y,t)$ and $\partial_t u_{N,M,K}(x,y,t)$) can be computed as

$$\begin{aligned}\partial_x u_{N,M,K}(x,y,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} \partial_x(L_i^{(\alpha_0)}(x)) L_j^{(\alpha_1)}(y) L_k^{(\alpha_2)}(t)\\ &= \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{j=0}^{K} a_{i,j,k} f_1^{i,j,k}(x,y,t),\end{aligned} \tag{21}$$

$$\begin{aligned}\partial_y u_{N,M,K}(x,y,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} L_i^{(\alpha_0)}(x)\partial_y(L_j^{(\alpha_1)}(y))L_k^{(\alpha_2)}(t)\\ &= \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{j=0}^{K} a_{i,j,k} f_2^{i,j,k}(x,y,t),\end{aligned} \tag{22}$$

$$\begin{aligned}\partial_t u_{N,M,K}(x,y,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} L_i^{(\alpha_0)}(x)L_j^{(\alpha_1)}(y)\partial_y(L_k^{(\alpha_2)}(t))\\ &= \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{j=0}^{K} a_{i,j,k} f_3^{i,j,k}(x,y,t),\end{aligned} \tag{23}$$

where

$$\begin{aligned} f_1^{i,j,k}(x,y,t) &= \partial_x(L_i^{(\alpha_0)}(x))L_j^{(\alpha_1)}(y)L_k^{(\alpha_2)}(t),\\ f_2^{i,j,k}(x,y,t) &= L_i^{(\alpha_0)}(x)\partial_y(L_j^{(\alpha_1)}(y))L_k^{(\alpha_2)}(t),\\ f_3^{i,j,k}(x,y,t) &= L_i^{(\alpha_0)}(x))L_j^{(\alpha_1)}(y)\partial_t(L_k^{(\alpha_2)}(t)).\end{aligned}$$

Accordingly, adopting (20)-(23), enable one to write (18)-(19) in the form:

$$\begin{aligned}&\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} f_3^{i,j,k}(x,y,t) = a_0\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} f_1^{i,j,k}(x,y,t)\\ &\quad + a_1\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} f_2^{i,j,k}(x,y,t) + a_2\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} f_0^{i,j,k}(x,y,t)\\ &\quad + H(x,y,t), \qquad (x,y,t)\in[0,\infty)\times[0,\infty)\times[0,\infty).\end{aligned} \tag{24}$$

Moreover, the treatment of the initial conditions at the collocation points immediately gives

$$\begin{aligned} u_{N,M,K}(x,y,0) &= \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} f_0^{i,j,k}(x,y,0) = g_0(x,y),\\ u_{N,M,K}(0,y,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} f_0^{i,j,k}(0,y,t) = g_1(y,t),\\ u_{N,M,K}(x,0,t) &= \sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} f_0^{i,j,k}(x,0,t) = g_2(x,t).\end{aligned} \tag{25}$$

In the proposed collocation method the residual of (18) is set to be zero at $(N\,M\,K)$ of the collocation points. Moreover, the initial conditions in (25) will be collocated at these collocation points. First, we have $(N\,M\,K)$ algebraic equations for the unknown expansion coefficients, $a_{i,j,k}$, from

$$\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} F^{i,j,k}_{r,s,\varsigma}\, a_{i,j,k} = H(x^{(\alpha_0)}_{N,r}, y^{(\alpha_1)}_{M,s}, t^{(\alpha_2)}_{K,\varsigma}), \begin{cases} r = 1, \cdots, N; \\ s = 1, \cdots, M; \\ \varsigma = 1, \cdots, K, \end{cases} \tag{26}$$

where

$$\begin{aligned} F^{i,j,k}_{r,s,\varsigma} =& f^{i,j,k}_3(x^{(\alpha_0)}_{N,r}, y^{(\alpha_1)}_{M,s}, t^{(\alpha_2)}_{K,\varsigma}) - a_0\, f^{i,j,k}_1(x^{(\alpha_0)}_{N,r}, y^{(\alpha_1)}_{M,s}, t^{(\alpha_2)}_{K,\varsigma}) - \\ & a_1\, f^{i,j,k}_2(x^{(\alpha_0)}_{N,r}, y^{(\alpha_1)}_{M,s}, t^{(\alpha_2)}_{K,\varsigma}) - a_2\, f^{i,j,k}_0(x^{(\alpha_0)}_{N,r}, y^{(\alpha_1)}_{M,s}, t^{(\alpha_2)}_{K,\varsigma}) \end{aligned} \tag{27}$$

and from the initial conditions, we have $N\,M + (K+1)(M+N+1)$ algebraic equations

$$\begin{aligned} &\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} f^{i,j,k}_0(x^{(\alpha_0}_{N,r}, y^{(\alpha_1)}_{M,s}, 0) = g_0(x^{(\alpha_0)}_{N,r}, y^{(\alpha_1)}_{M,s}), \begin{cases} r = 1, \cdots, N; \\ s = 1, \cdots, M, \end{cases} \\ &\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a^{i,j}_{i,j,k} f^{i,j,k}_0(0, y^{(\alpha_1)}_{M,s}, t^{(\alpha_2)}_{K,\varsigma}) = g_1(, y^{(\alpha_1)}_{M,s}, t^{(\alpha_2)}_{K,\varsigma}), \quad \begin{cases} s = 0, \cdots, M; \\ \varsigma = 0, \cdots, K, \end{cases} \\ &\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a^{i,j}_{i,j,k} f^{i,j,k}_0(x^{(\alpha_0)}_{N,r}, 0, t^{(\alpha_2)}_{K,\varsigma}) = g_2(x^{(\alpha_0)}_{N,r}, t^{(\alpha_2)}_{K,\varsigma}), \quad \begin{cases} r = 1, \cdots, N; \\ \varsigma = 0, \cdots, K, \end{cases} \end{aligned} \tag{28}$$

and this in turn yields $(M+1)(N+1)(K+1)$ algebraic equations

$$\begin{aligned} &\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} F^{i,j,k}_{r,s,\varsigma}\, a_{i,j,k} = H(x^{(\alpha_0)}_{N,r}, y^{(\alpha_1)}_{M,s}, t^{(\alpha_2)}_{K,\varsigma}), \begin{cases} r = 1, \cdots, N; \\ s = 1, \cdots, M; \\ \varsigma = 1, \cdots, K, \end{cases} \\ &\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a_{i,j,k} f^{i,j,k}_0(x^{(\alpha_0)}_{N,r}, y^{(\alpha_1)}_{M,s}, 0) = g_0(x^{(\alpha_0)}_{N,r}, y^{(\alpha_1)}_{M,s}), \quad \begin{cases} r = 1, \cdots, N; \\ s = 1, \cdots, M, \end{cases} \\ &\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a^{i,j}_{i,j,k} f^{i,j,k}_0(0, y^{(\alpha_1)}_{M,s}, t^{(\alpha_2)}_{K,\varsigma}) = g_1(, y^{(\alpha_1)}_{M,s}, t^{(\alpha_2)}_{K,\varsigma}), \quad \begin{cases} s = 0, \cdots, M; \\ \varsigma = 0, \cdots, K, \end{cases} \\ &\sum_{i=0}^{N}\sum_{j=0}^{M}\sum_{k=0}^{K} a^{i,j}_{i,j,k} f^{i,j,k}_0(x^{(\alpha_0)}_{N,r}, 0, t^{(\alpha_2)}_{K,\varsigma}) = g_2(x^{(\alpha_0)}_{N,r}, t^{(\alpha_2)}_{K,\varsigma}), \quad \begin{cases} r = 1, \cdots, N; \\ \varsigma = 0, \cdots, K. \end{cases} \end{aligned} \tag{29}$$

This system of algebraic equations can then be solved by using any suitable solver.

4. NUMERICAL RESULTS

In this section, three examples are considered to show the accuracy of the algorithms presented in the previous section. The obtained results reveal that the present method is very effective and convenient. The difference between the measured value of approximate solution and its actual value (absolute error) is given by

$$E(x,y,t) = |u(x,y,t) - u_{N,M,K}(x,y,t)|, \tag{30}$$

where $u(x,y,t)$ and $u_{N,M,K}(x,y,t)$ are the exact solution and the numerical solution at the point (x,y,t), respectively. The maximum absolute error is given by

$$M_E = \text{Max}\{E(x,y,t) : \forall (x,y,t) \in [0,\infty[\times[0,\infty[\times[0,\infty[\}. \tag{31}$$

Also, we can define the infinity norm as

$$L_\infty = \text{Max}\{E(x,y,t_k) : \forall (x,y) \in [0,\infty[\times[0,\infty[\}. \tag{32}$$

4.1. EXAMPLE 1

Consider the one-dimensional hyperbolic equation of first-order of the form

$$\partial_t u(x,t) = \partial_x u(x,t) + u(x,t) - \sqrt{2}e^{-\sqrt{2}t-x}, \quad (x,t) \in [0,\infty) \times (0,\infty), \tag{33}$$

subject to the initial conditions

$$u(0,t) = e^{-x}, \quad x \in [0,\infty), \quad u(x,0) = e^{-\sqrt{2}t}, \quad t \in [0,\infty), \tag{34}$$

The exact solution of Eq. (33) is

$$u(x,t) = e^{-\left(\sqrt{2}t+x\right)}, \quad (x,t) \in [0,\infty) \times (0,\infty). \tag{35}$$

Table 1 displays the maximum absolute errors (M_E) using GLGRC method with several choices of N, M, α_1 and α_2. Table 2 lists the results obtained by the GLGRC method in terms of absolute errors at $N = M = 16$ for $t = 0.1$, 0.5 and 1.0 and some values of x in the finite interval $[0,1]$. We see in this tables that the results are very accurate for small choice of N and M. Fig. 1, allows us to see the coincidence between the curves of exact and numerical solution at some values of t where $\alpha_1 = \alpha_2 = 3$ and $N = M = 16$. In the case of $\alpha_1 = \alpha_2 = 1$ and $N = M = 16$, the absolute error curve in t-direction of problem (33) is shown in Fig. 2 in the interval $[0,100]$.

4.2. EXAMPLE 2

Consider the following hyperbolic equation of first-order

$$\begin{aligned} \partial_t u(x,t) =& \partial_x u(x,t) + u(x,t) + e^{-t-x}(\cos(t) - \sin(t)), \\ & (x,t) \in [0,\infty) \times [0,\infty), \end{aligned} \tag{36}$$

Table 1

Maximum absolute error using GLGRC method for problem (33)

α_1	α_2	$N = M =$ 4	8	12	16
1	1	1.58×10^{-1}	1.86×10^{-2}	2.06×10^{-3}	7.90×10^{-5}
2	2	1.28×10^{-1}	1.54×10^{-2}	2.12×10^{-3}	2.61×10^{-4}
3	3	1.25×10^{-1}	1.93×10^{-2}	2.82×10^{-3}	3.96×10^{-4}

Table 2

The absolute errors using the GLGRC method for problem (33) at $N = M = 16$ and $\alpha_1 = \alpha_2 = 1$

x	t	E	x	t	E	x	t	E
0.1	0.1	2.84×10^{-7}	0.1	0.5	8.95×10^{-6}	0.1	1	4.87×10^{-5}
0.2		8.79×10^{-6}	0.2		4.10×10^{-6}	0.2		4.89×10^{-5}
0.3		1.20×10^{-5}	0.3		1.39×10^{-5}	0.3		4.17×10^{-5}
0.4		1.12×10^{-5}	0.4		2.07×10^{-5}	0.4		3.037×10^{-5}
0.5		7.95×10^{-6}	0.5		2.47×10^{-5}	0.5		1.75×10^{-5}
0.6		3.29×10^{-6}	0.6		2.62×10^{-5}	0.6		4.74×10^{-6}
0.7		1.78×10^{-6}	0.7		2.55×10^{-5}	0.7		6.68×10^{-6}
0.8		6.53×10^{-6}	0.8		2.31×10^{-5}	0.8		1.61×10^{-5}
0.9		1.04×10^{-5}	0.9		1.93×10^{-5}	0.9		2.32×10^{-5}
1		1.32×10^{-5}	1		1.45×10^{-5}	1		2.79×10^{-5}

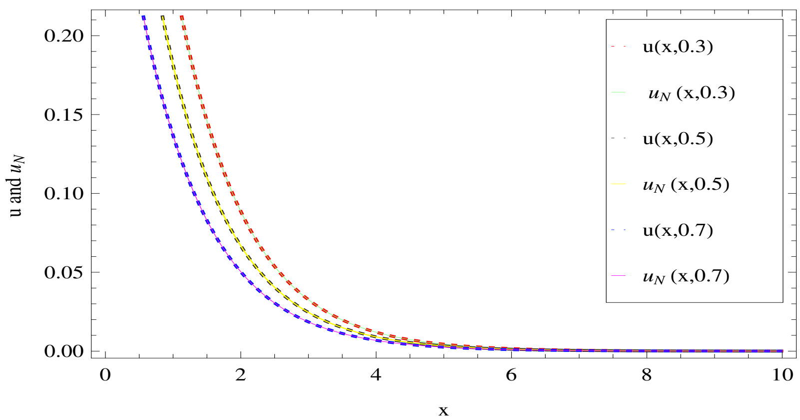

Fig. 1 – x-direction curves of the exact and numerical solutions for problem (33) with $\alpha_1 = \alpha_2 = 3$ and $N = M = 16$.

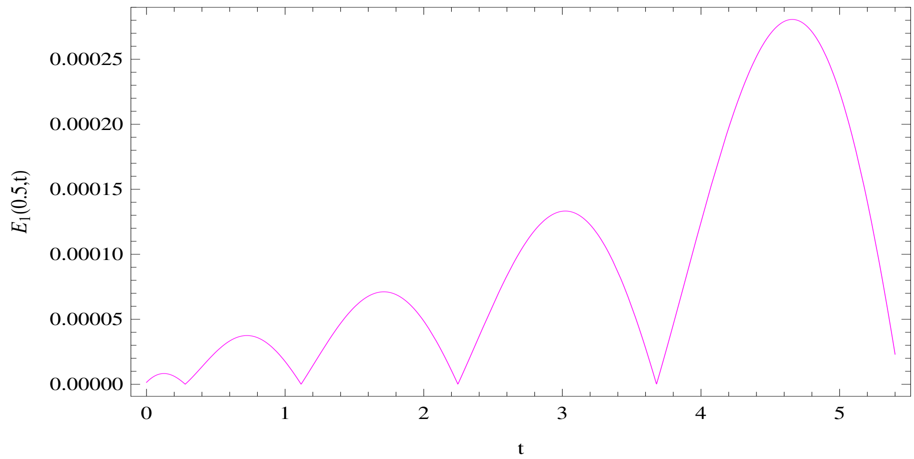

Fig. 2 – t-direction absolute error curve for problem (33) with $\alpha_1 = \alpha_2 = 1$ and $N = M = 16$.

Table 3

The maximum absolute error using the GLGRC method for problem (33)

		$N = M =$			
α_1	α_2	4	8	12	16
1	1	5.76×10^{-2}	1.83×10^{-3}	1.30×10^{-3}	6.00×10^{-5}
2	2	1.10×10^{-1}	2.93×10^{-3}	2.19×10^{-3}	2.56×10^{-4}
3	3	1.93×10^{-1}	1.79×10^{-2}	5.92×10^{-3}	13.51×10^{-4}

with the initial conditions

$$u(0,t) = e^{-t}\sin(t), \quad u(x,0) = 0, \quad (x,t) \in [0,\infty) \times [0,\infty). \tag{37}$$

The exact solution of Eq. (36) is

$$u(x,t) = e^{-(t+x)}\sin(t), \quad (x,t) \in [0,\infty) \times [0,\infty). \tag{38}$$

The maximum absolute errors of $u(x,t)$ related to (36)-(37) are introduced in Table 3 using the GLGRC method with three choices of N and M. In Fig. 3, we present the numerical solution of problem (36) at $\alpha_1 = \alpha_2 = 2$ and $N = M = 16$. Moreover, the three-dimensional graph of absolute error is shown in Fig. 4, with the values listed in its caption.

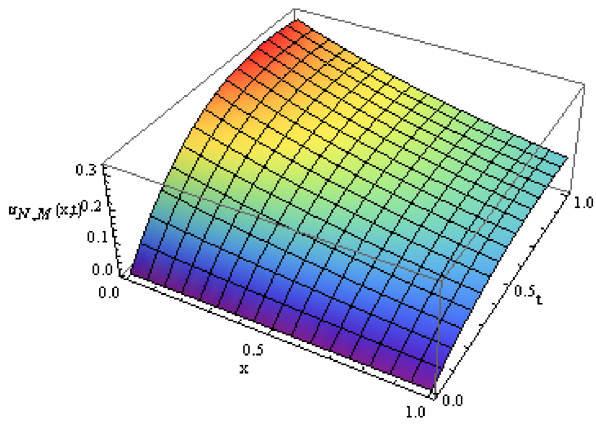

Fig. 3 – Numerical solution $u_{N,M}(x,t)$ of (36) with $\alpha_1 = \alpha_2 = 2$ and $N = M = 16$.

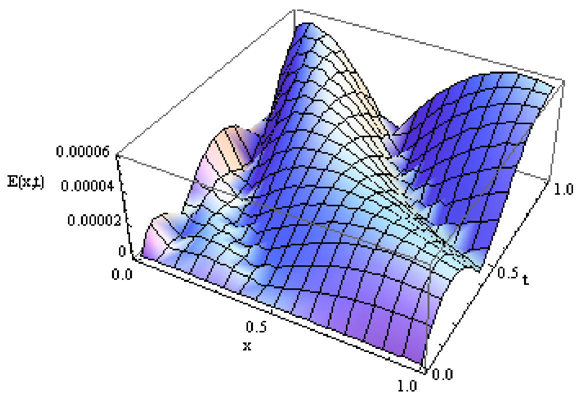

Fig. 4 – Three-dimensional graph of absolute error $E(x,t)$ of (36) with $\alpha_1 = \alpha_2 = 1$ and $N = M = 16$.

Table 4

The maximum absolute error using the GLGRC method for problem (39) with $N = M = K = 4, 6, 8$.

			(1,1,1)		(3,3,3)	
N	M	K	L_∞	M_E	L_∞	M_E
4	4	4	1.90×10^{-2}	3.11×10^{-2}	5.74×10^{-2}	9.17×10^{-2}
6	6	6	1.45×10^{-2}	1.45×10^{-2}	1.51×10^{-2}	4.96×10^{-2}
8	8	8	6.15×10^{-3}	6.92×10^{-3}	2.96×10^{-3}	2.86×10^{-2}

4.3. EXAMPLE 3

Finally, we consider the following two-dimensional hyperbolic equation

$$\partial_t u(x,y,t) = \partial_x u(x,y,t) + \partial_y u(x,y,t) + \frac{t+x+y+2}{(t+x+y+1)^2}, \quad (x,y,t) \in [0,\infty)\times[0,\infty)\times[0,\infty), \tag{39}$$

subject to initial conditions

$$\begin{aligned} u(x,y,0) &= \frac{1}{x+y+1}, \quad u(x,0,t) = \frac{1}{t+x+1}, \\ u(0,y,t) &= \frac{1}{t+y+1}, \quad (x,y,t) \in [0,\infty)\times[0,\infty)\times[0,\infty). \end{aligned} \tag{40}$$

The exact solution of Eq. (39) is

$$u(x,y,t) = \frac{1}{t+x+y+1}, \quad (x,y,t) \in [0,\infty)\times[0,\infty)\times[0,\infty). \tag{41}$$

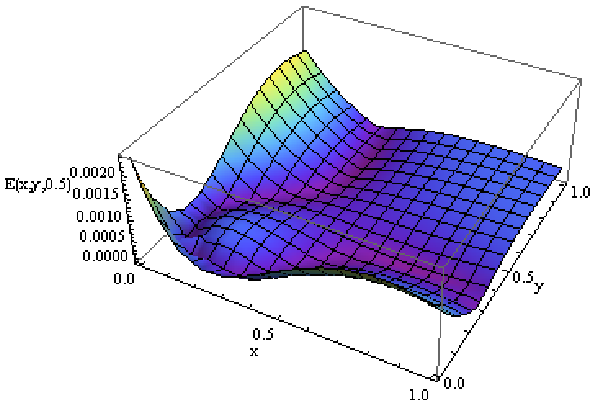

Fig. 5 – Space graph of absolute error at $t = 50$ for problem (39) with $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $N = M = K = 8$.

Using three different values of nodes, the maximum absolute errors and the infinity norm for the two-dimensional problem using the GLGRC method are presented in Table 4. The space graph of absolute error at $t = 50$ of problem (39) is plotted in Fig. 5 with values of parameters listed in its caption. In the case of $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $N = M = K = 8$, the t-direction absolute error curve for problem (39) is shown in Fig. 6. The obtained numerical results are accurate and compare favorably with the exact solution. The numerical results presented for this example are highlighting the applicability and validity of the proposed algorithm.

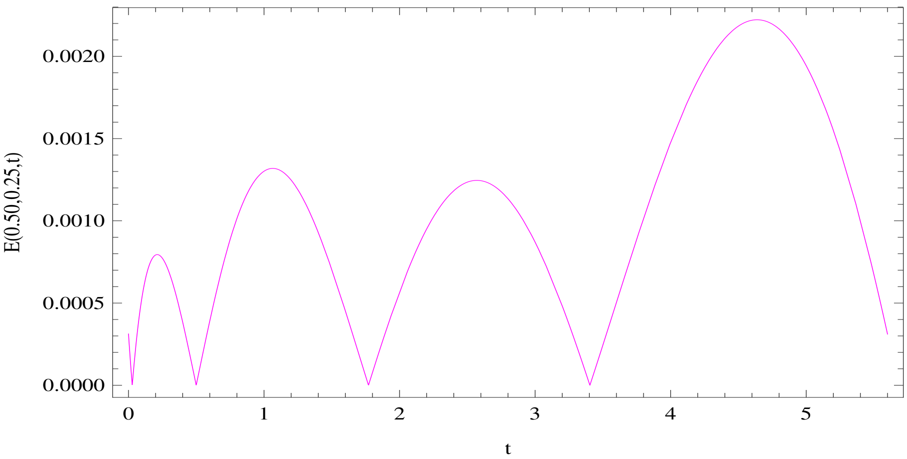

Fig. 6 – The t-direction curve of absolute error for problem (39) with $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $N = M = K = 8$.

5. CONCLUSION

We have developed a numerical approach to solve the hyperbolic PDEs of first-order in one- and two-spatial dimensions. In this approach the solution is approximated using the generalized Laguerre polynomials with applying the Gauss-Radau collocation scheme. The numerical simulations given in this article demonstrated the good accuracy of this approach. It was also confirmed that the generalized Laguerre collocation method is accurately approximating the exact solution. Moreover, the numerical approach proposed in this work can be well suited for handling general linear and nonlinear partial differential equations on a half line.

Acknowledgements. This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. (14-135-35-RG). The authors, therefore, acknowledge with thanks DSR technical and financial support.

REFERENCES

1. H. Fan, S. Jin, J.R. Miller, J. Diff. Equat. **189**, 267 (2003).
2. T. Li, J. Diff. Equat. **190**, 131 (2003).
3. T. Zang *et al.*, Comput. Chem. Eng. **32**, 2403 (2008).
4. S. Qamar *et al.*, Comput. Chem. Eng. **32**, 2403 (2010).
5. M. Oberguggenberger, J. Math. Anal. Appl. **142**, 452 (1989).
6. S.T.J. Yu *et al.*, Wave Motion **47**, 168 (2010).
7. S. Bonazzola, E. Gourgoulhon, J.-A. Marck, J. Comput. Appl. Math. **109**, 433 (1999).
8. E.H. Doha *et al.*, Appl Math. Infor. Sci. **8**, 535 (2014).
9. E.H. Doha *et al.*, Rom. J. Phys. **59**, 247 (2014).

10. M.A. Akinlar *et al.*, Appl. Math. Infor. Sci. **8**, 1633 (2014).
11. E.H. Doha *et al.*, Centr. Eur. Journ. Phys. **12**, 111 (2014)
12. K. Tabatabaei, E. Gunerhan, Appl. Math. Infor. Sci. Lett. **2**, 1 (2014).
13. A. Biswas *et al.*, Rom. J. Phys. **59**, 433 (2014).
14. A.H. Bhrawy, Appl. Math. Comput. **247**, 30 (2014).
15. A.H. Bhrawy *et al.*, Proc. Romanian Acad. A **15**, 235 (2014).
16. S.J. Sadati *et al.*, Rom. Rep. Phys. **65**, 94 (2013).
17. G.W. Wang, T.Z. Xu, Rom. Rep. Phys. **66**, 595 (2014).
18. H. Triki, Rom. J. Phys. **59**, 421 (2014).
19. M. Savescu *et al.*, Rom. J. Phys. **59**, 582 (2014).
20. A. Jafarian *et al.*, Rom. Rep. Phys. **66**, 296 (2014).
21. H. Triki *et al.*, Rom. Rep. Phys. **66**, 251 (2014).
22. A. Biswas *et al.*, Proc. Romanian Acad. A **15**, 123 (2014).
23. Lina Zhang, Aiyong Chen, Proc. Romanian Acad. A **15**, 11 (2014).
24. M. Alquran, J. B. Al-Shrida, Appl. Math. Infor. Sci. **9**, 825 (2015).
25. A.H. Bhrawy *et al.*, Rom. J. Phys. **59**, 646 (2014).
26. E.H. Doha *et al.*, Proc. Romanian Acad. A **15**, 130 (2014).
27. X. Ma, C. Huang, Xin Niu, Appl. Math. Comput. **250**, 131 (2015).
28. A.H. Bhrawy, M.A. Zaky, D. Baleanu, Rom. Rep. Phys. **67**, 340 (2015).
29. M.A. El-Khateb, M.E. Al-Hohaly, H.S. Hussien, Math. Sci. Lett. **1**, 33 (2012).
30. T. Sun, L.j. Yi, Math. Meth. Appl. Sci. **37**, 2257 (2014).
31. J.A. Rad *et al.*, Math. Meth. Appl. Sci. **37**, 329 (2014).
32. E.H. Doha *et al.*, Appl. Numer. Math. **77**, 43 (2014).
33. M.A. Abdelkawy, T.M. Taha, Walailak J. Sci. Tech. **11**, 1041 (2014).
34. E.H. Doha *et al.*, Nonlinear Anal. Modell. Contr. **19**, 537 (2014).
35. A. W. Lang, D. M. Sloan, J. Comput. Appl. Math. **140**, 499 (2002).
36. J.P. Boyd, J. Comput. Phys. **70**, 63 (1987).
37. B.Y. Guo, J. Comput. Math. **18**, 95 (2000).
38. J.P. Boyd, *Chebyshev and Fourier spectral methods* (2nd edn., New York: Dover, 2000).
39. Z-Q. Wang, B-Y. Guo, J. Math. Anal. Appl. **274**, 374 (2002).
40. D. Funaro, O. Kavian, Math. Comp. **57**, 597 (1990).
41. Y. Maday, B. Pernaud-Thomas, H. Vanderen, La Recherche Aerospatiale **6**, 13 (1985).
42. Z-Q. Zhang, H-P. Ma, J. Comput. Appl. Math. **230**, 614 (2009).
43. K. Parand *et al.*, Comput. Phys. Commun. **181**, 1096 (2010).
44. K. Parand, M. Shahini, M. Dehghan, J. Comput. Phys. **228**, 8830 (2009).
45. C. Canuto *et al.*, *Spectral methods: fundamentals in single domains* (Springer-Verlag, New York, 2006).
46. E.H. Doha *et al.*, Rom. J. Phys. **59**, 408 (2014).
47. A.H. Bhrawy, M.A. Zaky, J. Comput. Phys. **281**, 876 (2015).
48. E.H. Doha *et al.*, J. Comput. Phys. **261**, 244 (2014).
49. W. Kang *et al.*, Commun. Nonlinear Sci. **22**, 943 (2015)
50. X. Ma, C. Huang, Appl. Mathem. Model. **38**, 1434 (2014).
51. M.R. Eslahchi, M. Dehghan, Comput. Math. Appl. **61**, 2596 (2011).
52. M.R. Eslahchi, M. Dehghan, M. Parvizi, J. Comput. Appl. Math. **257**, 105 (2014).
53. S.R. Lau, H. Price, J. Comput. Phys. **231**, 7695 (2012).

54. A.H. Bhrawy, R.A. Van Gorder, Appl Math. Infor. Sci. **9**, 75 (2015).
55. G. Szegö, *Orthogonal Polynomials* (Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, RI, 4th edn., 1975).
56. D. Funaro, *Polynomial Approximations of Differential Equations* (Springer-Verlag, 1992).
57. D. Baleanu *et al.*, Abstr. Appl. Anal. **2013**, Article ID 546502 (2013).
58. D. Baleanu *et al.*, Abstr. Appl. Anal., **2013**, Article ID 413529 (2013).
59. B.-Y. Guo, X.-Y. Zhang, J. Comput. Appl. Math. **181**, 342 (2005).
60. D.K. Dimitrov, F. Marcellán, F.R. Rafaeli, J. Math. Anal. Appl. **368**, 80 (2010).