Special Issue Article

Advances in Mechanical Engineering

Advances in Mechanical Engineering
2015, Vol. 7(12) 1–11

DOI: 10.1177/1687814015619138
aime.sagepub.com
SAGE

A New fractional derivative for differential equation of fractional order under interval uncertainty

Soheil Salahshour[1], Ali Ahmadian[2], Fudzial Ismail[2], Dumitru Baleanu[3,4] and Norazak Senu[2]

Abstract

In this article, we develop a new definition of fractional derivative under interval uncertainty. This fractional derivative, which is called *conformable fractional derivative*, inherits some interesting properties from the integer differentiability which is more convenient to work with the mathematical models of the real-world phenomena. The interest for this new approach was born from the notion that makes a dependency just on the basic limit definition of the derivative. We will introduce and prove the main features of this well-behaved simple fractional derivative under interval arithmetic uncertainty. The actualization and usefulness of this approach are validated by solving two practical models.

Keywords

Interval arithmetic, fractional derivative, interval-valued function, generalized Hukuhara differentiability, viscoelastic models

Date received: 19 September 2015; accepted: 3 November 2015

Academic Editor: Hamid Reza Shaker

Introduction

Fractional calculus has been used widely to deal with some engineering problems in recent years.[1–11] The main advantage of fractional derivatives lies in that it is more suitable for describing memory and hereditary properties of various materials and processes in comparison with classical integer-order derivative. Moreover, the application of noninteger derivative has also been disseminated into the governing engineering models, and the curiosity of the scientists will achieve more in this area.[12–17] However, some objection has been revealed for the slightly burdensome mathematical formula of its definition and the resultant complexities in the solutions of the differential equations of fractional order. Khalil et al.[18] proposed a new fractional derivative that has some basic characteristics of the first-order derivative such as the product rule and the chain rule which seems more appropriate to describe the behavior of classical viscoelastic models under interval uncertainty. This new concept was followed up by Abdeljawad[19] to introduce conformable Gronwall's inequality, conformable exponential function, and conformable Laplace transform. Afterward, Batarfi et al.[20] employed this new fractional derivative for three-point boundary value problems, and then the

[1]Young Researchers and Elite Club, Mobarakeh Branch, Islamic Azad University, Mobarakeh, Iran
[2]Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, Serdang, Malaysia
[3]Department of Mathematics and Computer Science, Çankaya University, Ankara, Turkey
[4]Institute of Space Science, Magurele, Romania

Corresponding author:
Soheil Salahshour, Young Researchers and Elite Club, Islamic Azad University, Mobarakeh Branch, Mobarakeh 0098315, Iran.
Email: soheilsalahshour@yahoo.com

concept of the conformable timescale fractional calculus was proposed by Benkhettou et al.[21]

In many mathematical models of several crisp common world incidents in nature, a number of dynamical systems take control by ordinary differential equations (ODEs) in the presence of uncertainty. In practical dynamical systems, a collection of uncertainties is fundamental in the current state of the system, parameters, material properties, fraction tolerance, trajectories, and geometric dimensions, due to the complicatedness of physical-world problems. Uncertainties may lead to noticeable alterations of system dynamic feedback, specifically for mechanical dynamic systems. A considerable number of research papers have already been published incorporating various uncertainties in mechanical dynamic systems.[22–24] However, there is no report for applying the interval arithmetic with uncertainty in any physical model.

Probabilistic methods[25,26] have been extensively employed to a class of uncertain models, in which the uncertain variables are usually described as stochastic parameters with accurate probability distributions, under the hypothesis of perceiving total facts. However, it is not continually feasible to acquire the complete statistical data to explain probability distribution functions in physical models.

The interval-valued arithmetic and interval differential equations (IDEs) are the particular cases of the set-valued analysis and set differential equations, respectively. Interval arithmetic provides a possibility to measure uncertainties for uncertain variables regarding the lack of the knowledge of the complete information of the system. In recent times, the interval techniques have been attained much scrutiny and are encountering a huge demand in the field of uncertain analysis.[27–30]

The concept of Hukuhara derivative of set-valued mapping, presented by Hukuhara,[31] is rigorously combined with the theoretical foundation of IDEs and fuzzy differential equations (FDEs). The notion based on Hukuhara derivative has the drawback that any solution of a set differential equation has increasing length of its support. To overcome this situation, Stefanini and Bede[32] suggested a conception of generalized Hukuhara differentiability of interval-valued mapping based on the definition presented in Bede and Gal,[33] which permits them to achieve the solutions of IDEs with diminishing diameter of solution values. Thereafter, several researchers focused their investigations on the IDEs and FDEs with this new type of differentiability.[34–40]

In contrast to the crisp fractional calculus, the research works in the theory of the uncertain noninteger calculus are in the stage of infancy. Limited works have been done which are still traceless in the literature.[5,41–50] This partially motivated us to confine our focus to develop the new discussed fractional derivative for solving fractional IDEs. We introduce, for the first time in the literature, this new concept under interval uncertainty to cope with the solutions for the problems exhibiting high-level uncertainties. The new features which are proposed and proved in this study can be applied easily to express the behavior of the uncertain fractional dynamic systems based on the interval arithmetic.

This article is arranged as follows: In section "Preliminaries," we recall the most essential definitions from the interval arithmetic and fractional derivatives that will be used in the sequel. In section "Main results," we develop the new fractional derivative under generalized Hukuhara derivative for interval-valued functions. Some distinguishing features of these derivatives are introduced and proved in this section. To show the well-posedness of the considered fractional derivative, some concrete cases are solved in section "Applications." In the final section, the outcomes are surveyed and some points for the future research are recommended.

Preliminaries

In what follows, we draw some recent and necessary notions about differentiation of interval-valued and fuzzy functions (see, for example, Stefanini and Bede[32] and Stefanini.[38] Also, we review definition of the fractional conformable derivative with its features. A comprehensive study of the fractional derivatives can be found in Baleanu et al.[1] and Diethelm.[2]

Interval arithmetic

Let $\mathbb{F}$ indicate the family of all nonempty, compact, and convex intervals of the real line $\mathbb{R}$. The addition and scalar multiplication in $\mathbb{F}$, we define as usual, that is, for $M, N \in \mathbb{F}, M = [\underline{m}, \overline{m}], N = [\underline{n}, \overline{n}], \underline{m} \leq \overline{m}, \underline{n} \leq \overline{n}$, and $\lambda \geq 0$, we have

$$M + N = [\underline{m} + \underline{n}, \overline{m} + \overline{n}], \quad \lambda M = [\lambda \underline{m}, \lambda \overline{m}],$$
$$(-\lambda)M = [-\lambda \overline{m}, -\lambda \underline{m}]$$

Note that for $M \in \mathbb{F}, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}, \lambda_1, \lambda_2 \geq 0$, it holds

$$\lambda_1(\lambda_2 M) = (\lambda_1 \lambda_2)M \text{ and } (\lambda_3 + \lambda_4)M = \lambda_3 M + \lambda_4 M$$

The Hausdorff metric $\mathcal{H}$ in $\mathbb{F}$ is defined as follows

$$\mathcal{H}(M, N) = \max\{|\underline{m} - \underline{n}|, |\bar{m} - \bar{n}|\}$$

for $M = [\underline{m}, \overline{m}]$ and $N = [\underline{n}, \overline{n}]$. It is known (see, for example, Stefanini and Bede[32]) that $(\mathbb{F}, \mathcal{H})$ is a complete, separable, and locally compact metric space. For the metric $\mathcal{H}$, the following properties hold

$$\mathcal{H}(\mathrm{M}+\mathrm{N},\mathrm{S}+\mathrm{T}) \leq \mathcal{H}(\mathrm{M},\mathrm{S}) + \mathcal{H}(\mathrm{N},\mathrm{T})$$
$$\mathcal{H}(\xi\mathrm{M},\xi\mathrm{N}) = |\xi|\mathcal{H}(\mathrm{M},\mathrm{N})$$

for $\mathrm{M},\mathrm{N},\mathrm{S},\mathrm{T} \in \mathbb{F}, \xi \in \mathbb{R}$

Let $\mathrm{M},\mathrm{N} \in \mathbb{F}$. If there exists an interval $\mathrm{S} \in \mathbb{F}$ such that $\mathrm{M} = \mathrm{N} + \mathrm{S}$, then we call S the *Hukuhara difference* (H difference for short) of M and N. We denote the interval S by $\mathrm{M} \ominus \mathrm{N}$. Note that $\mathrm{M} \ominus \mathrm{N} \neq \mathrm{M} + (-1)\mathrm{N}$

For $\mathrm{M} = [\underline{\mathrm{m}}, \overline{\mathrm{m}}] \in \mathbb{F}$, denote the length and magnitude of M by

$$\mathrm{len}(\mathrm{M}) := \overline{\mathrm{m}} - \underline{\mathrm{m}} \quad \text{and}$$
$$\| \mathrm{M} \| := \mathcal{H}(\mathrm{M}, \{0\}) = \max_{\{|\overline{\mathrm{m}}|, |\underline{\mathrm{m}}|\}}$$

respectively. It is known that $\mathrm{M} \ominus \mathrm{N}$ exists in the case $\mathrm{len}(\mathrm{M}) \geq \mathrm{len}(\mathrm{N})$. Also, the following properties hold for $\mathrm{M},\mathrm{N},\mathrm{S},\mathrm{T} \in \mathbb{F}$:

1. If $\mathrm{M} \ominus \mathrm{N}, \mathrm{M} \ominus \mathrm{S}$ exist, then $\mathcal{H}(\mathrm{M} \ominus \mathrm{N}, \mathrm{M} \ominus \mathrm{S}) = \mathcal{H}(\mathrm{N},\mathrm{S})$;
2. If $\mathrm{M} \ominus \mathrm{N}, \mathrm{S} \ominus \mathrm{T}$ exist, $\mathcal{H}(\mathrm{M} \ominus \mathrm{N}, \mathrm{S} \ominus \mathrm{T}) = \mathcal{H}(\mathrm{M}+\mathrm{T}, \mathrm{N}+\mathrm{S})$;
3. If $\mathrm{M} \ominus \mathrm{N}, \mathrm{M} \ominus (\mathrm{N}+\mathrm{S})$ exist, then there exists $(\mathrm{M} \ominus \mathrm{N}) \ominus \mathrm{S}$ and $(\mathrm{M} \ominus \mathrm{N}) \ominus \mathrm{S} = \mathrm{M} \ominus (\mathrm{N}+\mathrm{S})$;
4. If $\mathrm{M} \ominus \mathrm{N}, \mathrm{M} \ominus \mathrm{S}, \mathrm{S} \ominus \mathrm{N}$ exist, then there exists $(\mathrm{M} \ominus \mathrm{N}) \ominus (\mathrm{M} \ominus \mathrm{S})$ and $(\mathrm{M} \ominus \mathrm{N}) \ominus (\mathrm{M} \ominus \mathrm{S}) = \mathrm{S} \ominus \mathrm{N}$.

As it was stated above, the H difference is unique, but it does not always exist. A generalization of the Hukuhara difference is proposed in Bede and Gal[33] to overcome this shortcoming.

Definition 1. The generalized Hukuhara difference of two fuzzy numbers $u_1, u_2 \in \mathbb{F}$ (gH difference for short) is defined as follows

$$u_1 \ominus_g u_2 = u_3 \Leftrightarrow \begin{cases} (1)\, u_1 = u_2 + u_3 \\ \quad\quad \text{or} \\ (2)\, u_2 = u_1 + (-1)u_3 \end{cases} \tag{1}$$

in which $u_3 \in \mathbb{F}$.

Several alternative definitions exist for the differentiability of an interval-valued function. The most important concept is a particularization of the fuzzy concepts proposed in Lupulescu[51] to the interval case, based on the generalized fuzzy differentiability.

Definition 2. Let $\mathcal{T} : (a,b) \to \mathbb{F}$ and $\omega \in (a,b)$. We say that f is strongly generalized (Hukuhara) differentiable at ω, if there exists an element $\mathcal{T}'(\omega) \in \mathbb{F}$, such that $\mathcal{T}'(\omega)$ satisfies one of the following Definitions 1–4:

1. For all $h>0$ sufficiently small, $\exists \mathcal{T}(\omega+h) \ominus \mathcal{T}(\omega)$, $\exists \mathcal{T}(\omega) \ominus \mathcal{T}(\omega-h)$, and
$$\lim_{h \searrow 0} \frac{\mathcal{T}(\omega+h) \ominus \mathcal{T}(\omega)}{h} = \lim_{h \searrow 0} \frac{\mathcal{T}(\omega) \ominus \mathcal{T}(\omega-h)}{h} = \mathcal{T}'(\omega)$$
2. For all $h>0$ sufficiently small, $\exists \mathcal{T}(\omega) \ominus \mathcal{T}(\omega+h)$, $\exists \mathcal{T}(\omega-h) \ominus \mathcal{T}(\omega)$, and
$$\lim_{h \searrow 0} \frac{\mathcal{T}(\omega) \ominus \mathcal{T}(\omega+h)}{-h} = \lim_{h \searrow 0} \frac{\mathcal{T}(\omega-h) \ominus \mathcal{T}(\omega)}{-h} = \mathcal{T}'(\omega)$$
3. For all $h>0$ sufficiently small, $\exists \mathcal{T}(\omega+h) \ominus \mathcal{T}(\omega)$, $\exists \mathcal{T}(\omega-h) \ominus \mathcal{T}(\omega)$, and
$$\lim_{h \searrow 0} \frac{\mathcal{T}(\omega+h) \ominus \mathcal{T}(\omega)}{h} = \lim_{h \searrow 0} \frac{\mathcal{T}(\omega-h) \ominus \mathcal{T}(\omega)}{-h} = \mathcal{T}'(\omega)$$
4. For all $h>0$ sufficiently small, $\exists \mathcal{T}(\omega) \ominus \mathcal{T}(\omega+h)$, $\exists \mathcal{T}(\omega) \ominus \mathcal{T}(\omega-h)$, and
$$\lim_{h \searrow 0} \frac{\mathcal{T}(\omega) \ominus \mathcal{T}(\omega+h)}{-h} = \lim_{h \searrow 0} \frac{\mathcal{T}(\omega) \ominus \mathcal{T}(\omega-h)}{h} = \mathcal{T}'(\omega)$$

Based on the gH difference, the generalized Hukuhara differentiability was introduced in Stefanini and Bede.[32]

Definition 3. Let $t \in (a,b)$ and h be such that $t+h \in (a,b)$, then the generalized Hukuhara derivative of a fuzzy-valued function $x : (a,b) \to \mathbb{F}$ is defined as

$$x'_{gH}(t) = \lim_{h \to 0} \frac{x(t+h) \ominus_g x(t)}{h} \tag{2}$$

If $x'_{gH}(t) \in \mathbb{F}$ satisfying equation (2) exists, we say that x is generalized Hukuhara differentiable (gH differentiable for short) at t. Also, we say that x is [1-gH] differentiable at t, if x satisfies in Definition 2-1, then we have $x'_{gH}(t) = \underline{x}'(t), \overline{x'}(t)]$; similarly, x is [2-gH] differentiable at t, if x satisfies in Definition 2-2, then we have $x'_{gH}(t) = [\overline{x'}(t), \underline{x}'(t)]$.

We say that an interval-valued function $F : [a,b] \to \mathbb{F}$ is *w-increasing* (w-decreasing) on $[a,b]$ if the real function $t \to w_F(t) := w(F(t))$ is increasing (decreasing) on $[a,b]$. If F is w-increasing or w-decreasing on $[a,b]$, then we say that F is w-monotone on $[a,b]$ (see Lupulescu[51]).

Proposition 1. Let $F : [a,b] \to \mathbb{F}$ be such that $F(t) = [f^-(t), f^+(t)], t \in [a,b]$. If F is w-monotone and gH differentiable on $[a,b]$, then $df^-(t)/dt$ and $df^+(t)/dt$

exist for all $t \in [a,b]$ (see Markov[52]). Moreover, we have that

1. $F'(t) = [df^{-}(t)/dt, df^{+}(t)/dt]$ for all $t \in [a,b]$, if F is w-increasing;
2. $F'(t) = [df^{+}(t)/dt, df^{-}(t)/dt]$ for all $t \in [a,b]$, if F is w-decreasing.

Proposition 2. Let $F : [a,b] \to \mathbb{F}$ be w-monotone and gH differentiable on $[a,b]$ (see Markov[52]), the following properties are then true:

1. For all $C \in \mathbb{F}$ and for all $\lambda \in \mathbb{R}$, the interval-valued functions $F + C$, $F \ominus_g C$, and λF are gH differentiable on $[a,b]$, and $(F + C)' = F'$, $(F \ominus_g C)' = F'$, and $(\lambda F)' = \lambda F'$, respectively;
2. If F and G are equally w-monotonic (i.e. both are w-increasing or both are w-decreasing), then $(F + G)' = F' + G'$ and $(F \ominus_g G)' = F' \ominus_g G'$;
3. If F and G are differently w-monotonic (i.e. one is w-increasing and the other is w-decreasing), then $(F + G)' = F' \ominus_g (-G')$ and $(F \ominus_g G)' = F' + (-G')$

Proposition 3. If $F : [a,b] \to \mathbb{F}$ is Lebesgue integrable on $[a,b]$, then the interval-valued function $G : [a,b] \to \mathbb{F}$, defined by $G(t) := \int_a^t F(s)ds$ for all $t \in [a,b]$, is absolutely continuous and $G'(t) = F(t)$ for a.e. $t \in [a,b]$ (see Markov[52]).

Proposition 4. If $F \in ([a,b], \mathbb{F})$, then F is gH differentiable a.e. on $[a,b]$ and $F' \in L^1([a,b], \mathbb{F})$. Moreover, if F is w-monotone on $[a,b]$, then (see Lupulescu[51])

$$F(t) \ominus_g F(a) = \int_a^t F'(s)ds \text{ for all } t \in [a,b].$$

Fractional conformable derivative

We recall the *conformable fractional derivative* presented by Khalil et al.[18] Also, the paramount features of this derivative are reviewed.

Definition 4. Let $\mathcal{T} : [0,\infty) \to \mathbb{F}$. Then, conformable fractional derivative of $\mathcal{T}$ of order α is defined by

$$T^{(\alpha)}\mathcal{T}(t) = \lim_{\varepsilon \to 0} \frac{\mathcal{T}(t + \varepsilon t^{1-\alpha}) - \mathcal{T}(x)}{\varepsilon} \quad (3)$$

in which $T^{(\alpha)}$ indicates the fractional derivative operator of order α for all $t > 0, \alpha \in (0,1)$. If $\mathcal{T}$ is α differentiable in some $(0,a), a > 0$, and $\lim_{t \to 0^+} \mathcal{T}^{(\alpha)}(t)$ exists, then define

$$\mathcal{T}^{(\alpha)}(0) = \lim_{t \to 0^+} \mathcal{T}^{(\alpha)}(t)$$

Henceforth, for more simplicity, if the conformable fractional derivative of $\mathcal{T}(t)$ of order α exists, then we simply say $\mathcal{T}$ is α differentiable. Also, the following properties are satisfied for the conformable fractional derivative.[18]

Let $\alpha \in (0,1]$ and $\mathcal{T}, \mathcal{S}$ be α differentiable at a point $x > 0$. Then,

1. $T^{(\alpha)}(\lambda\mathcal{T} + \eta\mathcal{S}) = \lambda T^{(\alpha)}\mathcal{T} + \eta T^{(\alpha)}\mathcal{S}$, for all $\lambda, \eta \in \mathbb{R}$;
2. $T^{(\alpha)}(t^{\lambda}) = \lambda t^{\lambda-\alpha}$, for all $\lambda \in \mathbb{R}$;
3. $T^{(\alpha)}(\zeta) = 0$, for all constant functions $\mathcal{T}(t) = \zeta$;
4. $T^{(\alpha)}(\mathcal{T}\mathcal{S}) = \mathcal{T}T^{(\alpha)}\mathcal{S} + \mathcal{S}T^{(\alpha)}\mathcal{T}$;
5. $T^{(\alpha)}(\mathcal{T}/\mathcal{S}) = (\mathcal{S}T^{(\alpha)}\mathcal{T} - \mathcal{T}T^{(\alpha)}\mathcal{S})/\mathcal{S}^2$; Now, it is the time to review the definition of the fractional conformable integral proposed by Khalil et al.[18]

Let $\alpha \in (0,\infty)$. Define $J_\alpha(t^p) = (t^{p+\alpha})/(p+\alpha)$ for any $p \in \mathbb{R}$, and $\alpha \neq -p$. If $\mathcal{T}(t) = \sum_{k=0}^{n} b_k t^k$, then we define $J_\alpha(\mathcal{T}) = \sum_{k=0}^{n} b_k J_\alpha(t^k) = \sum_{k=0}^{n} b_k((t^{k+\alpha})/(k+\alpha))$. Obviously, J_α is linear on its domain. Furthermore, if $\alpha = 1$, then J_α is the usual integral.

Regarding the above circumstances, the following definition was suggested by Khalil et al.[18] for the α fractional integral of a function $\mathcal{T}$ starting from $a \geq 0$.

Definition 5. $I_\alpha^a(\mathcal{T})(t) = I_1^a(t^{\alpha-1}\mathcal{T}) = \int_a^t (\mathcal{T}(x))/(x^{1-\alpha})dx$, where the integral is the usual Riemann improper integral and $\alpha \in (0,1)$.

Theorem 1. $T^{(\alpha)}I_\alpha^a(\mathcal{T})(t)$ for $t \geq a$, where $\mathcal{T}$ is any continuous function in domain of I_α (see Khalil et al.[18]). Also, we prove the following theorem which is needed for the main part of our results which is employed in the next sections.

Theorem 2. Let $\mathcal{T}$ be α differentiable, then

$$I_\alpha T^{(\alpha)}\mathcal{T}(t) = \mathcal{T}(t) - \mathcal{T}(a)$$

Proof. In fact, using the definition of I_α and $T^{(\alpha)}$, we obtain

$$\begin{aligned} I_\alpha T^{(\alpha)}\mathcal{T}(t) &= \int_a^t \frac{T^{(\alpha)}\mathcal{T}(s)}{s^{1-\alpha}} ds \\ &= \int_a^t \frac{s^{1-\alpha}\mathcal{T}'(s)}{s^{1-\alpha}} ds = \mathcal{T}(t) - \mathcal{T}(a) \end{aligned}$$

Main results

In this section, the conformable fractional derivative of order $\alpha \in (0,1]$ is developed under interval arithmetic with uncertainty. We introduce and study the continuity and the differentiability for the interval-valued functions based on the generalized differentiability described in Stefanini and Bede.[32]

Definition 6. Let $\mathcal{T}:(a,b) \to \mathbb{F}$ and $x_0 \in (a,b)$. We say that $\mathcal{T}$ is generalized conformable fractional differentiable at ω, if there exists $\mathcal{T}^{(\alpha)}(\omega) \in \mathbb{F}$ such that

$$\mathcal{T}_{gH}^{(\alpha)}(\omega) = \lim_{\varepsilon \to 0} \frac{\mathcal{T}(\omega + \varepsilon\omega^{1-\alpha}) \ominus_g \mathcal{T}(\omega)}{\varepsilon} \tag{4}$$

Now, we investigate the continuity of $\mathcal{T}$ at $\omega \in (a,b)$, when $\mathcal{T}$ is differentiable. For simplicity, we call throughout this article that $\mathcal{T}$ is α differentiable at $\omega \in (a,b)$, if $\mathcal{T}$ generalized conformable fractional differentiable of order α as stated in Definition 6.

Theorem 3. Let $\mathcal{T}:(a,b) \to \mathbb{F}$ be α differentiable at $\omega \in (a,b)$ and $\alpha \in (0,1]$, then $\mathcal{T}$ is continuous at ω.

Proof. Due to the fact that

$$\mathcal{T}\left(t + \varepsilon t^{1-\alpha}\right) \ominus_g \mathcal{T}(t) = \frac{\mathcal{T}(t + \varepsilon t^{1-\alpha}) \ominus_g \mathcal{T}(t)}{\varepsilon} \times \varepsilon$$

then

$$\lim_{\varepsilon \to 0} \mathcal{T}\left(t + \varepsilon t^{1-\alpha}\right) \ominus_g \mathcal{T}(t) = \lim_{\varepsilon \to 0} \frac{\mathcal{T}(t + \varepsilon t^{1-\alpha}) \ominus_g \mathcal{T}(t)}{\varepsilon} \times \lim_{\varepsilon \to 0} \varepsilon$$

Set $P = \varepsilon\omega^{1-\alpha}$, then we obtain

$$\lim_{P \to 0} \mathcal{T}(\omega + P) \ominus_g \mathcal{T}(\omega) = 0$$

Consequently, we finally obtain

$$\lim_{P \to 0} \mathcal{T}(\omega + P) = \mathcal{T}(\omega)$$

which denotes that $\mathcal{T}$ is continuous.

Similar to Definition 3, we can state Definition 6 as follows:

1. We say that $\mathcal{T}$ is $(\alpha,1)$ differentiable, if there exists $\mathcal{T}^{(\alpha)}(\omega) \in \mathbb{F}$ such that for all $\varepsilon > 0$ sufficiently small, $\mathcal{T}(\omega + \varepsilon\omega^{1-\alpha}) \ominus \mathcal{T}(\omega)$ and $\mathcal{T}(\omega) \ominus \mathcal{T}(\omega - \varepsilon\omega^{1-\alpha})$ exist and the limits

$$\lim_{\varepsilon \to 0} \frac{\mathcal{T}(\omega + \varepsilon\omega^{1-\alpha}) \ominus \mathcal{T}(\omega)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\mathcal{T}(\omega) \ominus \mathcal{T}(\omega - \varepsilon\omega^{1-\alpha})}{\varepsilon} = \mathcal{T}^{(\alpha)}(\omega)$$

2. We say that $\mathcal{T}$ is $(\alpha,2)$ differentiable, if there exists $\mathcal{T}^{(\alpha)}(\omega) \in \mathbb{F}$ such that for all $\varepsilon > 0$ sufficiently small, $f(\omega) \ominus \mathcal{T}(\omega + \varepsilon\omega^{1-\alpha})$ and $\mathcal{T}(\omega - \varepsilon\omega^{1-\alpha}) \ominus \mathcal{T}(\omega)$ exist and the limits

$$\lim_{\varepsilon \to 0} \frac{\mathcal{T}(\omega) \ominus \mathcal{T}(\omega + \varepsilon\omega^{1-\alpha})}{-\varepsilon} = \lim_{\varepsilon \to 0} \frac{\mathcal{T}(\omega - \varepsilon\omega^{1-\alpha}) \ominus \mathcal{T}(\omega)}{-\varepsilon} = \mathcal{T}^{(\alpha)}(\omega)$$

Theorem 4. Let $\mathcal{T}(\omega) = [\mathcal{T}_1(\omega), \mathcal{T}_2(\omega)]$ be α differentiable and w-monotone on (a,b), then for every $\omega \in (a,b)$, the derivatives $\mathcal{T}_1^{(\alpha)}(\omega)$ and $\mathcal{T}_2^{(\alpha)}(\omega)$ exist and $\mathcal{T}^{(\alpha)}(\omega) = [\mathcal{T}_1^{(\alpha)}(\omega), \mathcal{T}_2^{(\alpha)}(\omega)]$, if $\mathcal{T}$ is w-increasing and $\mathcal{T}^{(\alpha)}(\omega) = [\mathcal{T}_2^{(\alpha)}(\omega), \mathcal{T}_1^{(\alpha)}(\omega)]$, if $\mathcal{T}$ is w-decreasing.

Proof. The idea was introduced by Markov[52] for first-order IDEs. In fact, using Definition 6, we have for the case of w-increasing

$$\begin{aligned}
\mathcal{T}^{(\alpha)} &= \lim_{\varepsilon \to 0} \frac{\mathcal{T}(\omega + \varepsilon\omega^{1-\alpha}) \ominus \mathcal{T}(\omega)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\left[\mathcal{T}_1^{(\alpha)}(\omega + \varepsilon\omega^{1-\alpha}), \mathcal{T}_2^{(\alpha)}(\omega + \varepsilon\omega^{1-\alpha})\right] \ominus \left[\mathcal{T}_1^{(\alpha)}(\omega), \mathcal{T}_2^{(\alpha)}(\omega)\right]}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\left[\mathcal{T}_1^{(\alpha)}(\omega + \varepsilon\omega^{1-\alpha}) - \mathcal{T}_1^{(\alpha)(\omega)}, \mathcal{T}_2^{(\alpha)}(\omega + \varepsilon\omega^{1-\alpha}) - \mathcal{T}_2^{(\alpha)}(\omega)\right]}{\varepsilon} \\
&= [\mathcal{T}_1(\omega), \mathcal{T}_2(\omega)]
\end{aligned}$$

Also, it is easy to verify that

$$\mathcal{T}(\omega) = \lim_{\varepsilon \to 0} \frac{\mathcal{T}(\omega) \ominus \mathcal{T}(\omega - \varepsilon\omega^{1-\alpha})}{\varepsilon} = [\mathcal{T}_1(\omega), \mathcal{T}_2(\omega)]$$

Indeed, for the case of w-decreasing, the proof is similar as above and we omit it.

Remark 1. It is worth noting here that the result of Theorem 4 coincides with Proposition 1 for $\alpha = 1$. Now, in order to use the proposed interval type of conformable fractional differentiability, we should state the relation between conformable fractional derivative and conformable fractional integral. For this purpose, we state the following theorem results.

Theorem 5. Let $\mathcal{T}$ be α differentiable and w-monotone, then

$$I_\alpha \mathcal{T}^{(\alpha)} \mathcal{T}(\omega) = \mathcal{T}(\omega) \ominus_g \mathcal{T}(a)$$

for a.e. $t \in [a,b]$.

Proof. Using the definition of integrability, differentiability, and Theorem 4, the proof is straightforward.

Remark 2. It should be mentioned that the result of Theorem 5 coincides with Proposition 4 for $\alpha = 1$.

Remark 3. We suppose that the following propositions for simplicity of the presentation:

S1. For $\varepsilon>0$, sufficiently small, there exist $\mathcal{T}(x+\varepsilon x^{1-\alpha}) \ominus \mathcal{T}(x)$ and $\mathcal{T}(x) \ominus \mathcal{T}(x-\varepsilon x^{1-\alpha})$ at $x \in (a,b)$;
S2. For $\varepsilon>0$, sufficiently small, there exist $\mathcal{T}(x) \ominus \mathcal{T}(x+\varepsilon x^{1-\alpha})$ and $\mathcal{T}(x-\varepsilon x^{1-\alpha}) \ominus \mathcal{T}(x)$ at $x \in (a,b)$.

Now, we extend Theorem 5 stated in Bede et al.[53] to the aforesaid fractional derivative case under interval settings theory.

Theorem 6. Let $\mathcal{T}:\mathbb{R}\to\mathbb{R}$ and $\mathcal{S}:\mathbb{R}\to\mathbb{F}$ are two differentiable functions in the crisp sense and generalized conformable differentiable, respectively. Then,

1. If $\mathcal{T}(x)\mathcal{T}^{(\alpha)}(x)>0$, and $\mathcal{S}$ is $(\alpha,1)$ differentiable, then $\mathcal{T}.\mathcal{S}$ is $(\alpha,1)$ differentiable and $(\mathcal{T}.\mathcal{S})^{(\alpha)}(x) = \mathcal{T}^{(\alpha)}(x).\mathcal{S}(x) + \mathcal{T}(x).\mathcal{S}^{(\alpha)}(x)$;
2. If $\mathcal{T}(x)\mathcal{T}^{(\alpha)}(x)<0$, and $\mathcal{S}$ is $(\alpha,2)$ differentiable, then $\mathcal{T}.\mathcal{S}$ is $(\alpha,2)$ differentiable and $(\mathcal{T}.\mathcal{S})^{(\alpha)}(x) = \mathcal{T}^{(\alpha)}(x).\mathcal{S}(x) + \mathcal{T}(x).\mathcal{S}^{(\alpha)}(x)$;
3. If $\mathcal{T}(x)\mathcal{T}^{(\alpha)}(x)>0$, and $\mathcal{S}$ is $(\alpha,2)$ differentiable, then $\mathcal{T}.\mathcal{S}$ satisfies S1 at x, then $\mathcal{T}.\mathcal{S}$ is $(\alpha,1)$ differentiable and $(\mathcal{T}.\mathcal{S})^{(\alpha)}(x) = \mathcal{T}^{(\alpha)}(x).\mathcal{S}(x)\ominus(-1)\mathcal{T}(x).\mathcal{S}^{(\alpha)}(x)$;
4. If $\mathcal{T}(x)\mathcal{T}^{(\alpha)}(x)>0$, and $\mathcal{S}$ is $(\alpha,2)$ differentiable, then $\mathcal{T}.\mathcal{S}$ satisfies S2 at x, then $\mathcal{T}.\mathcal{S}$ is $(\alpha,2)$ differentiable and $(\mathcal{T}.\mathcal{S})^{(\alpha)}(x) = \mathcal{T}(x).\mathcal{S}^{(\alpha)}(x)\ominus(-1)\mathcal{T}(x)^{(\alpha)}.\mathcal{S}(x)$;
5. If $\mathcal{T}(x)\mathcal{T}^{(\alpha)}(x)<0$, and $\mathcal{S}$ is $(\alpha,1)$ differentiable, then $\mathcal{T}.\mathcal{S}$ satisfies S1 at x, then $\mathcal{T}.\mathcal{S}$ is $(\alpha,1)$ differentiable and $(\mathcal{T}.\mathcal{S})^{(\alpha)}(x) = \mathcal{T}(x).\mathcal{S}^{(\alpha)}(x)\ominus(-1)\mathcal{T}(x)^{(\alpha)}.\mathcal{S}(x)$;
6. If $\mathcal{T}(x)\mathcal{T}^{(\alpha)}(x)<0$, and $\mathcal{S}$ is $(\alpha,1)$ differentiable, then $\mathcal{T}.\mathcal{S}$ satisfies S2 at x, then $\mathcal{T}.\mathcal{S}$ is $(\alpha,2)$ differentiable and $(\mathcal{T}.\mathcal{S})^{(\alpha)}(x) = \mathcal{T}^{(\alpha)}(x).\mathcal{S}(x)\ominus(-\mathcal{T}(x)).\mathcal{S}^{(\alpha)}(x)$.

Proof. We prove Cases 2 and 3. The proofs for the other cases are obtained analogously to the demonstration of the proof of those two cases.

Case 2. Due to the $(\alpha,2)$ differentiability of $\mathcal{S}$ at point x, we have the following result for sufficiently small $\varepsilon>0$

$$\mathcal{S}(x) = \mathcal{S}\left(x+\varepsilon x^{1-\alpha}\right) + p(x,\varepsilon,\alpha)$$

Moreover, $\mathcal{T}(x) - \mathcal{T}(x+\varepsilon x^{1-\alpha}) := q(x,\varepsilon,\alpha)$ has the same sign as $\mathcal{T}(x)$ and $\mathcal{T}(x+\varepsilon x^{1-\alpha})$ for $\varepsilon>0$. Then, we obtain

$$\begin{aligned}\mathcal{T}(x).\mathcal{S}(x) = {} & \mathcal{T}\left(x+\varepsilon x^{1-\alpha}\right).\mathcal{S}\left(x+\varepsilon x^{1-\alpha}\right)\\ & + \mathcal{T}\left(x+\varepsilon x^{1-\alpha}\right)p(x,\varepsilon,\alpha)\\ & + q(x,\varepsilon,\alpha)\mathcal{S}\left(x+\varepsilon x^{1-\alpha}\right)\\ & + q(x,\varepsilon,\alpha)p(x,\varepsilon,\alpha)\end{aligned} \tag{5}$$

In fact, equation (5) is the Hukuhara difference of $\mathcal{T}(x).\mathcal{S}(x)$ and $(x+\varepsilon x^{1-\alpha}).\mathcal{S}(x+\varepsilon x^{1-\alpha})$. So, we obtain

$$\begin{aligned}&\mathcal{T}(x).\mathcal{S}(x) \ominus \mathcal{T}\left(x+\varepsilon x^{1-\alpha}\right).\mathcal{S}\left(x+\varepsilon x^{1-\alpha}\right)\\ &= \mathcal{T}\left(x+\varepsilon x^{1-\alpha}\right)p(x,\varepsilon,\alpha) + q(x,\varepsilon,\alpha)\mathcal{S}\left(x+\varepsilon x^{1-\alpha}\right)\\ &+ q(x,\varepsilon,\alpha)p(x,\varepsilon,\alpha)\end{aligned}$$

Then, we deduce that

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{\mathcal{T}(x).\mathcal{S}(x)\ominus\mathcal{T}(x+\varepsilon x^{1-\alpha}).\mathcal{S}(x+\varepsilon x^{1-\alpha})}{-\varepsilon}\\ &= \lim_{\varepsilon\to0}\mathcal{T}\left(x+\varepsilon x^{1-\alpha}\right).\frac{p(x,\varepsilon,\alpha)}{-\varepsilon}\\ &+ \lim_{\varepsilon\to0}\frac{q(x,\varepsilon,\alpha)}{-\varepsilon}\mathcal{S}\left(x+\varepsilon x^{1-\alpha}\right) + \lim_{\varepsilon\to0}\frac{q(x,\varepsilon,\alpha)p(x,\varepsilon,\alpha)}{-\varepsilon}\end{aligned}$$

Using the fact that $\mathcal{T},\mathcal{S}$ are continuous, $\lim_{\varepsilon\to0}p(x,\varepsilon,\alpha) = 0$, and we have

$$\begin{aligned}&\lim_{\varepsilon\to0}\mathcal{T}\left(x+\varepsilon x^{1-\alpha}\right).\frac{p(x,\varepsilon,\alpha)}{-\varepsilon} = \mathcal{T}(x).\mathcal{S}^{(\alpha)}(x),\\ &\lim_{\varepsilon\to0}\frac{q(x,\varepsilon,\alpha)}{-\varepsilon}\mathcal{S}\left(x+\varepsilon x^{1-\alpha}\right) = \mathcal{T}^{(\alpha)}(x).\mathcal{S}(x)\end{aligned}$$

Therefore

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{\mathcal{T}(x).\mathcal{S}(x)\ominus\mathcal{T}(x+\varepsilon x^{1-\alpha}).\mathcal{S}(x+\varepsilon x^{1-\alpha})}{-\varepsilon}\\ &= \mathcal{T}(x).\mathcal{S}^{(\alpha)}(x) + \mathcal{T}^{(\alpha)}(x).\mathcal{S}(x)\end{aligned}$$

In a closed discussion, one can easily obtain the following result

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{\mathcal{T}(x-\varepsilon x^{1-\alpha}).\mathcal{S}(x-\varepsilon x^{1-\alpha})\ominus\mathcal{T}(x).\mathcal{S}(x)}{-\varepsilon}\\ &= \mathcal{T}(x).\mathcal{S}^{(\alpha)}(x) + \mathcal{T}^{(\alpha)}(x).\mathcal{S}(x)\end{aligned}$$

So, $\mathcal{T}.\mathcal{S}$ is $(\alpha,2)$ differentiable and the proof is completed.

Case 3. Based on the $(\alpha,2)$ differentiability of $\mathcal{S}(x)$, we have

$$\mathcal{S}(x) = \mathcal{S}\left(x+\varepsilon x^{1-\alpha}\right) + p(x,\varepsilon,\alpha)$$

Since $\mathcal{T}$ is α differentiable, we have

$$\mathcal{T}\left(x + \varepsilon x^{1-\alpha}\right) = \mathcal{T}(x) + q(x, \varepsilon, \alpha)$$

Indeed, $p(x, \varepsilon, \alpha)$ is the Hukuhara difference between $\mathcal{S}(x)$ and $\mathcal{S}(x + \varepsilon x^{1-\alpha})$. Also, $q(x, \varepsilon, \alpha)$ has the same sign with $\mathcal{T}(x)$ and $\mathcal{T}(x + \varepsilon x^{1-\alpha})$ for small ε. So, we deduce that

$$\begin{aligned} &\mathcal{T}(x).\mathcal{S}(x) + q(x, \varepsilon, \alpha)\mathcal{S}(x) \\ &= \mathcal{T}\left(x + \varepsilon x^{1-\alpha}\right).\mathcal{S}\left(x + \varepsilon x^{1-\alpha}\right) + \mathcal{T}\left(x + \varepsilon x^{1-\alpha}\right).p(x, \varepsilon, \alpha) \end{aligned}$$

Using Proposition S1 and, consequently, the existence of $q(x, \varepsilon, \alpha).\mathcal{S}(x) \ominus \mathcal{T}(x + \varepsilon x^{1-\alpha}).p(x, \varepsilon, \alpha)$, we obtain

$$\begin{aligned} &\mathcal{T}\left(x + \varepsilon x^{1-\alpha}\right).\mathcal{S}\left(x + \varepsilon x^{1-\alpha}\right) \ominus \mathcal{T}(x).\mathcal{S}(x) \\ &= q(x, \varepsilon, \alpha).\mathcal{S}(x) \ominus \mathcal{T}\left(x + \varepsilon x^{1-\alpha}\right).p(x, \varepsilon, \alpha) \end{aligned}$$

and similar to the previous case, taking the limit for both sides

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{\mathcal{T}(x + \varepsilon x^{1-\alpha}).\mathcal{S}(x + \varepsilon x^{1-\alpha}) \ominus \mathcal{T}(x).\mathcal{S}(x)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{q(x, \varepsilon, \alpha)}{\varepsilon} \mathcal{S}(x) \ominus \lim_{\varepsilon \to 0} \mathcal{T}\left(x + \varepsilon x^{1-\alpha}\right) \lim_{\varepsilon \to 0} \frac{p(x, \varepsilon, \alpha)}{\varepsilon} \\ &= \mathcal{T}(x)^{(\alpha)}.\mathcal{S}(x) \ominus (-\mathcal{T}(x)).\mathcal{S}^{(\alpha)}(x) \end{aligned}$$

Similarly, one can easily obtain

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{\mathcal{T}(x).\mathcal{S}(x) \ominus \mathcal{T}(x - \varepsilon x^{1-\alpha}).\mathcal{S}(x - \varepsilon x^{1-\alpha})}{\varepsilon} \\ &= \mathcal{T}(x)^{(\alpha)}.\mathcal{S}(x) \ominus (-\mathcal{T}(x)).\mathcal{S}^{(\alpha)}(x) \end{aligned}$$

In fact, we establish that $(\mathcal{T}.\mathcal{S})(x)$ is $(\alpha, 1)$ differentiable and satisfies in the mentioned product rule. Now, some useful results are investigated and developed.

Theorem 7. Suppose that $\alpha \in (0, 1]$ and $\mathcal{T}, \mathcal{S} : (a, b) \to \mathbb{F}$ are two α differentiable interval-valued functions at point $x>0$. Then

1. $(\lambda\mathcal{T}(x) + \eta\mathcal{S}(x))^{(\alpha)} = \lambda\mathcal{T}^{(\alpha)}(x) + \eta\mathcal{S}^{(\alpha)}(x)$, for all $\lambda, \eta \in \mathbb{R}$;
2. $(\overline{A}t^{\lambda})^{(\alpha)} = \lambda\overline{A}t^{\lambda-\alpha}$, where $\overline{A} = [A_L, A_U] \in \mathbb{F}$ and $A_L, A_U \in \mathbb{R}$;
3. $(\overline{A}\zeta)^{(\alpha)} = \overline{0}$, for all crisp constant functions $\mathcal{T}(x) = \zeta$;
4. If $\mathcal{T}$ is differentiable, then $\mathcal{T}^{(\alpha)}(x) = x^{(1-\alpha)}\mathcal{T}'(x)$.

Proof. All the proofs are straightforward. So, we just prove Case 4 and the others are similar to this case.

Let $\mathcal{T}$ be $(\alpha, 2)$ differentiable, then

$$\begin{aligned} \mathcal{T}^{(\alpha)}(x) &= \lim_{\varepsilon \to 0} \frac{\mathcal{T}(x) \ominus \mathcal{T}(x + \varepsilon x^{1-\alpha})}{-\varepsilon} \\ &\overset{h = \varepsilon x^{1-\alpha}}{=} x^{1-\alpha} \lim_{h \to 0} \frac{\mathcal{T}(x) \ominus \mathcal{T}(x + h)}{-h} = x^{1-\alpha}\mathcal{T}'(x) \end{aligned}$$

Using the same substitution as above, we have

$$\begin{aligned} \mathcal{T}^{(\alpha)}(x) &= \lim_{\varepsilon \to 0} \frac{\mathcal{T}(x - \varepsilon x^{1-\alpha}) \ominus \mathcal{T}(x)}{-\varepsilon} \\ &= x^{1-\alpha} \lim_{h \to 0} \frac{\mathcal{T}(x) \ominus \mathcal{T}(x + h)}{-h} = x^{1-\alpha}\mathcal{T}'(x) \end{aligned}$$

Applications

We begin this section with the practical examples which reflect a part of motivations of our studying of the fractional interval initial value problem with generalized Hukuhara differentiability.

To obtain linear viscoelastic behavior, it is useful to consider the simpler behavior of analog mechanical models. They are constructed from linear springs and dashpots, disposed singly and in branches of two (in series or in parallel) as shown in Figure 1.[54,55] Now, it is advantageous to consider the simple models of Figure 1 by providing their governing stress–strain relations along with the related material functions.

The Hooke model

The spring in Figure 1(a) is the elastic (or storage) element, as for it the force is proportional to the extension; it represents a perfect elastic body obeying the Hooke law. This model is thus referred to as the Hooke model which is given by

$$\sigma(x) = E\varepsilon(x) \qquad (6)$$

The Newton model

The dashpot in Figure 1(b) is the viscous (or dissipative) element, the force being proportional to rate of extension; it represents a perfectly viscous body obeying the Newton law. This model is thus referred to as

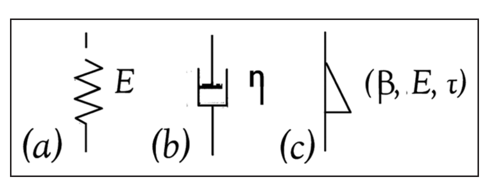

Figure 1. Single elements: (a) elastic, (b) viscous, and (c) fractional elements.
$w'(x)$).

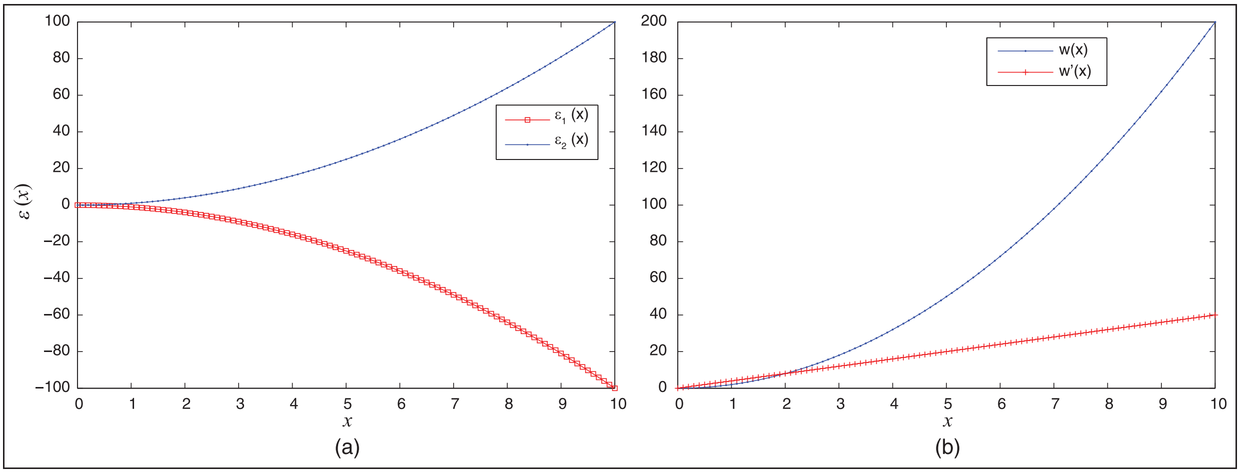

Figure 2. Trend of the solution of Example 1: (a) exact solution of Example 1 and (b) $w(x)$ with its derivative ($w'(x)$).

the Newton model. Denoting by b_1 the pertinent viscosity coefficient, we have

$$\sigma(x) = b_1 \frac{d}{dx}\varepsilon(x) \tag{7}$$

In equations (6) and (7), E and b_1 stand for the spring constant and the viscosity.

The Voigt model

A branch constituted by a spring in parallel with a dashpot is known as the Voigt model, as shown in Figure 1(c). We have

$$\sigma(x) = E\varepsilon(x) + b_1\varepsilon^{(\alpha)}(x) \tag{8}$$

in which b_1 is a constant and E can be a constant or set-valued function parameter.

The definite large part of viscoelastic variables in the actuality are specified just by ambitious notions which are dealt with by means of a common language. In the current section, we propose a new model using IDEs under strongly generalized differentiability concept. In this regard, the interval arithmetic exploited corresponds to frequency-dependent factors in the Kelvin–Voigt equations to depict the reality far better.

Example 1. Let us consider the following Kelvin–Voigt equation[55,56] model based on the interval fractional conformable derivative (Figure 2)

$$\begin{cases} \varepsilon^{(1/2)}(x) + \varepsilon(x) = \sigma(x) \\ \varepsilon(0) = 0, \quad \alpha \in (0,1] \end{cases} \tag{9}$$

in which $\sigma(x) = \overline{c}(x^2 + 2x^{3/2})$. It is easy to verify that, under $(\alpha, 1)$ differentiability, the solution is $\varepsilon(x) = \overline{c}x^2$, where $\overline{c}$ is an interval number such that $\overline{c} = [-1, 1]$.

In fact, Khalil et al.[18] stated that in order to obtain the solution of equation (9), under Riemann–Liouville differentiability, the problem must be as follows

$$\varepsilon^{(1/2)}(x) + \varepsilon(x) = x^2 + \frac{2}{\Gamma(2.5)}x^{3/2}$$

Indeed, obtaining the solution under conformable fractional differentiability is much more simpler than Riemann–Liouville derivative. This is also valid under interval uncertainty. Now, we solve another case of the Kelvin–Voigt model in the sense of the interval arithmetic.

Example 2

$$\begin{cases} \varepsilon^{(1/2)}(x) + \sqrt{x}\varepsilon(x) = \overline{c}xe^{-x} \\ \varepsilon(0) = 0, \quad \alpha \in (0,1] \end{cases} \tag{10}$$

in which $\overline{c} = [-1, 1]$ is an interval number and $x \in (0, 1.5)$.

Under $(\alpha, 1)$ differentiability, we have

$$\left[\varepsilon_1^{(1/2)}(x), \varepsilon_2^{(1/2)}(x)\right] + \left[\sqrt{x}\varepsilon_1(x), \sqrt{x}\varepsilon_2(x)\right] = \left[-xe^{-x}, xe^{-x}\right]$$

Thus, we obtain

$$\begin{cases} \varepsilon_1^{(1/2)}(x) + \sqrt{x}\varepsilon_1(x) = -xe^{-x} \\ \varepsilon_2^{(1/2)}(x) + \sqrt{x}\varepsilon_2(x) = xe^{-x} \end{cases}$$
$$\rightarrow \begin{cases} \varepsilon_1^{(1/2)}(x) + \sqrt{x}\varepsilon_1(x) = -xe^{-x} \\ \varepsilon_2^{(1/2)}(x) + \sqrt{x}\varepsilon_2(x) = xe^{-x} \end{cases}$$
$$\times e^x \rightarrow \begin{cases} \varepsilon_1^{(1/2)}(x) + \sqrt{x}e^x\varepsilon_1(x) = -x \\ \varepsilon_2^{(1/2)}(x) + \sqrt{x}e^x\varepsilon_2(x) = x \end{cases}$$
$$\begin{cases} (e^x\varepsilon_1(x))^{(1/2)} = x \\ (e^x\varepsilon_2(x))^{(1/2)} = -x \end{cases}$$

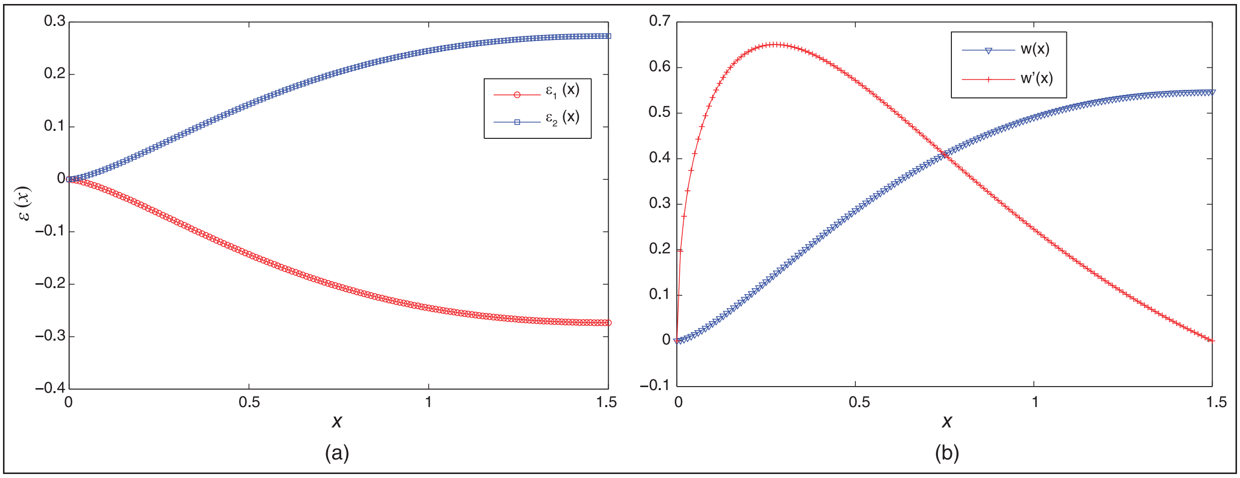

Figure 3. Trend of the solution of Example 2: (a) exact solution of Example 2 and (b) $w(x)$ with its derivative ($w'(x)$).

In fact, using Theorem 6, Case 1, we have

$$(e^x\varepsilon(x))^{1/2} = \bar{c}x$$

Then, taking the conformable fractional integral for both sides and using Theorem 5, we have

$$e^x\varepsilon(x) = \bar{c}I_{1/2}x + A \overset{\varepsilon(0)=0}{\longrightarrow} \varepsilon(x) = \frac{2}{3}\bar{c}e^{-x}x^{3/2}$$

Remark 4. It is easy to verify that $\varepsilon(x)$ is w-increasing on $(0, 1.5)$ (see Figure 3 which includes w and w').

Conclusion

A new fractional derivative with some simplifications in the formula and computations was proposed under interval uncertainty. We have developed the highlights of the conformable fractional derivatives which are more influential for the solution of fractional IDEs under generalized Hukuhara differentiability. In practice, a simple fractional derivative satisfying the main rules such as the product rule and the chain rule for the uncertain fractional differential equations simplifies considerably the cumbersome mathematical expressions of the mathematical modeling in the engineering sciences which was achieved in this article. In addition, to validate our claim and demonstrate the effectiveness of the proposed fractional derivative, two viscoelastic models were solved under interval uncertainty. Let us remark here that it is possible to extend the proposed definition for the fuzzy fractional differential equations under gH differentiability which will be considered by the authors as a future work.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: The authors received financial support from Ministry of Higher Education (MOHE), Malaysia under FRGS for publication of this article.

References

1. Baleanu D, Diethelm K, Scalas E, et al. *Fractional calculus: models and numerical methods*. Singapore: World Scientific, 2012.
2. Diethelm K. *The analysis of fractional differential equations* (Lecture notes in mathematics). Berlin: Springer-Verlag, 2010.
3. Hilfer R. *Applications of fractional calculus in physics*. Singapore: World Scientific, 2000.
4. Malinowska AB and Torres DFM. Towards a combined fractional mechanics and quantization. *Fract Calc Appl Anal* 2012; 15: 407–417.
5. Ahmadian A, Salahshour S, Baleanu D, et al. An efficient Tau method for numerical solution of a fuzzy fractional kinetic model and its application to oil palm frond as a promising source of xylose. *J Comput Phys* 2015; 264: 562–564.
6. Bhrawy AH, Doha EH, Baleanu D, et al. A spectral Tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations. *J Comput Phys* 2015; 293: 142–156.
7. Shivanian E, Abbasbandy S, Alhuthali MS, et al. Local integration of 2-D fractional telegraph equation via moving least squares approximation. *Eng Anal Bound Elem* 2015; 56: 98–105.

8. Tenreiro Machado J, Lopes AM, Duarte FB, et al. Rhapsody in fractional. *Fract Calc Appl Anal* 2014; 17: 1188–1214.
9. Lopes AM and Tenreiro Machado J. Modeling vegetable fractals by means of fractional-order equations. *J Vib Control*. Epub ahead of print 20 April 2015. DOI: 10.1177/1077546315581228.
10. Yang X-J, Baleanu D, Lazarević MP, et al. Fractal boundary value problems for integral and differential equations with local fractional operators. *Therm Sci* 2015; 19: 959–966.
11. Cattani C and Guariglia E. Fractional derivative of the Hurwitz ζ-function and chaotic decay to zero. *J King Saud Univ: Sci*. Epub ahead of print 2015. DOI: 10.1016/j.jksus.2015.04.003.
12. Baleanu D, Golmankhaneh AK and Golmankhaneh AK. The dual action and fractional multi time Hamilton equations. *Int J Theor Phys* 2009; 48: 2558–2569.
13. Caputo M and Fabrizio M. Damage and fatigue described by a fractional derivative model. *J Comput Phys* 2015; 293: 400–408.
14. Caputo M, Cametti C and Ruggiero V. Time and spatial concentration profile inside a membrane by means of a memory formalism. *Physica A* 2007; 387: 2010–2018.
15. Zhang Y, Cattani C and Yang X-J. Local fractional homotopy perturbation method for solving non-homogeneous heat conduction equations in fractal domains. *Entropy* 2015; 17: 6753–6764.
16. Zhang Y and Yang X-J. An efficient analytical method for solving local fractional nonlinear PDEs arising in mathematical physics. *Appl Math Model*. Epub ahead of print 30 September 2015. DOI: 10.1016/j.apm.2015.08.017.
17. Yang X-J, Tenreiro Machado A and Hristov J. Nonlinear dynamics for local fractional Burgers' equation arising in fractal flow. *Nonlinear Dynam*. Epub ahead of print 17 April 2015. DOI: 10.1007/s11071-015-2085-2.
18. Khalil R, Al Horani M, Yousef A, et al. A new definition of fractional derivative. *J Comput Appl Math* 2014; 264: 65–70.
19. Abdeljawad T. On conformable fractional calculus. *J Comput Appl Math* 2015; 279: 57–66.
20. Batarfi H, Losada J, Nieto JJ, et al. Three-point boundary value problems for conformable fractional differential equations. *J Funct Space* 2015; 2015: 706383 (1–6 pp.).
21. Benkhettou N, Hassani S and Delfim FM Torres. A conformable fractional calculus on arbitrary time scales. *J King Saud Univ: Sci*. Epub ahead of print 21 May 2015. DOI: 10.1016/j.jksus.2015.05.003.
22. Schukller GI and Jensen HA. Computational methods in optimization considering uncertainties—an overview. *Comput Method Appl M* 2008; 198: 2–13.
23. Wu J, Zhang Y, Chen L, et al. A Chebyshev interval method for nonlinear dynamic systems under uncertainty. *Appl Math Model* 2013; 37: 4578–4591.
24. Ben-Haim Y and Elishakoff I. *Convex models of uncertainty in applied mechanics*. Amsterdam: Elsevier Science, 1990.
25. Chen W, Jin R and Sudjianto A. Analytical variance-based global sensitivity analysis in simulation-based design under uncertainty. *J Mech Des: T ASME* 2005; 127: 875–886.
26. Yin X, Lee S, Chen W, et al. Efficient random field uncertainty propagation in design using multiscale analysis. *J Mech Des: T ASME* 2009; 131: 021006.
27. Alefeld G and Mayer G. Interval analysis: theory and applications. *J Comput Appl Math* 2000; 121: 421–464.
28. Alefeld G and Herzberger J. *Introduction to interval computations*. New York: Academic Press, 1983.
29. Moore RE. *Methods and applications of interval analysis*. Philadelphia, PA: SIAM Studies in Applied Mathematics, 1979.
30. Hu J and Qiu Z. Non-probabilistic convex models and interval analysis method for dynamic response of a beam with bounded uncertainty. *Appl Math Model* 2010; 34: 725–734.
31. Hukuhara M. Integration des applications mesurables dont la valeur est un compact convex. *Funkc Ekvacio* 1967; 10: 205–229.
32. Stefanini L and Bede B. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Anal: Theor* 2009; 71: 1311–1328.
33. Bede B and Gal SG. Generalizations of the differentiability of fuzzy number value functions with applications to fuzzy differential equations. *Fuzzy Set Syst* 2005; 151: 581–599.
34. Malinowski MT. Interval differential equations with a second type Hukuhara derivative. *Appl Math Lett* 2011; 24: 2118–2123.
35. Malinowski MT. Interval Cauchy problem with a second type Hukuhara derivative. *Inform Sciences* 2012; 213: 94–105.
36. Lupulescu V. Hukuhara differentiability of interval-valued functions and interval differential equations on time scales. *Inform Sciences* 2013; 248: 50–67.
37. Chalco-Cano Y, Lodwick WA and Bede B. Single level constraint interval arithmetic. *Fuzzy Set Syst* 2014; 257: 146–168.
38. Stefanini L. A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. *Fuzzy Set Syst* 2010; 161: 1564–1584.
39. Khastan A and Rodrmguez-Lspez R. On the solutions to first order linear fuzzy differential equations. *Fuzzy Set Syst*. Epub ahead of print 16 June 2015. DOI: 10.1016/j.fss.2015.06.005.
40. Khastan A, Nieto JJ and Rodrmguez-Lspez R. Variation of constant formula for first order fuzzy differential equations. *Fuzzy Set Syst* 2011; 177: 20–33.
41. Agarwal RP, Lakshmikantham V and Nieto JJ. On the concept of solution for fractional differential equations with uncertainty. *Nonlinear Anal: Theor* 2010; 72: 2859–2862.
42. Allahviranloo T, Salahshour S and Abbasbandy S. Explicit solutions of fractional differential equations with uncertainty. *Soft Comput* 2012; 16: 1297–1302.

43. Salahshour S, Allahviranloo T, Abbasbandy S, et al. Existence and uniqueness results for fractional differential equations with uncertainty. *Adv Differ Equ: NY* 2012; 2012: 112.
44. Salahshour S, Allahviranloo T and Abbasbandy S. Solving fuzzy fractional differential equations by fuzzy Laplace transforms. *Commun Nonlinear Sci* 2012; 17: 1372–1381.
45. Alikhani R and Bahrami F. Global solutions for nonlinear fuzzy fractional integral and integrodifferential equations. *Commun Nonlinear Sci* 2013; 18: 2007–2017.
46. Hoa NV. Fuzzy fractional functional differential equations under Caputo gH-differentiability. *Commun Nonlinear Sci* 2015; 22: 1134–1157.
47. Malinowski MT. Random fuzzy fractional integral equations—theoretical foundations. *Fuzzy Set Syst* 2015; 265: 39–62.
48. Mazandarani M and Vahidian Kamyad A. Modified fractional Euler method for solving Fuzzy Fractional Initial Value Problem. *Commun Nonlinear Sci* 2013; 18: 12–21.
49. Ahmadian A, Suleiman M, Salahshour S, et al. A Jacobi operational matrix for solving fuzzy linear fractional differential equation. *Adv Difference Equ: NY* 2013; 2013: 104.
50. Salahshour S, Ahmadian A, Senu N, et al. On analytical solutions of the fractional differential equation with uncertainty: application to the Basset Problem. *Entropy* 2015; 17: 885–902.
51. Lupulescu V. Fractional calculus for interval-valued functions. *Fuzzy Set Syst* 2015; 265: 63–85.
52. Markov S. Calculus for interval functions of a real variables. *Computing* 1979; 22: 325–337.
53. Bede B, Rudas IJ and Bencsik AL. First order linear fuzzy differential equations under generalized differentiability. *Inform Sciences* 2007; 177: 1648–1662.
54. Tschoegl NW. *The phenomenological theory of linear viscoelastic behavior: an introductio*n. Berlin: Springer-Verlag, 1989, 143–145.
55. Mainardi F. *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*. Singapore: World Scientific, 2010.
56. Podlubny I. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, vol. 198. San Diego, CA: Academic Press, 1998.