NEW SOLUTIONS OF THE TRANSPORT EQUATIONS
IN POROUS MEDIA WITHIN LOCAL FRACTIONAL DERIVATIVE

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Abstract. In this manuscript we use the series expansion method within local fractional derivative to obtain the solutions of both homogeneous and non-homogeneous transport equations. The new reported solutions are able to describe more efficiently the behavior of solutions of the transport phenomena in porous media.

Key words: transport equation, approximate solution, fractals, local fractional derivative, porous media.

1. INTRODUCTION

The linear transport equations [1–4] are crucial for modelling of the nonlinear transport equations arising in conservation laws [5–7], biology [8], and porous media [9]. We recall that the fractional calculus [10, 11] was successfully applied to find better mathematical models for real world problems arising in these fields. For example, Tarasov considered the fractional transport equations within Liouville equations [12], Uchaikin and Sibatov presented the fractional-order transport in disordered semiconductors [13] and Kadem and Baleanu [14] investigated the solution of the fractional transport equation.

It is well known that the fractal geometry is important for several processes in nature [15, 16]. However on fractal sets the fractional calculus is not efficient, therefore recently, the local fractional derivative [17] was used to debate the non-differentiable problems of local fractional partial differential equations (PDEs) arising in several branches of science and engineering [18–28]. Thus, we investigate the homogeneous transport equation with local fractional derivative in the fractal porous media, which was suggested as [29]

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = 0 \,, \tag{1}$$

and the non-homogeneous transport equation with local fractional derivative in fractal porous media, which is given by

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} = u(x,t), \tag{2}$$

in the presence of the initial condition

$$u(x,0) = f(x), \tag{3}$$

where $f(x)$ and $u(x,t)$ are non-differentiable functions, and the expression of the local fractional partial derivative is [17]

$$\frac{\partial^\alpha f(x_0, y)}{\partial x^\alpha} = \frac{\Delta^\alpha \left(f(x, y) - f(x_0, y) \right)}{(x - x_0)^\alpha},$$ (4)

where

$$\Delta^\alpha \left(f(x, y) - f(x_0, y) \right) \cong \Gamma(1+\alpha) \left[f(x, y) - f(x_0, y) \right].$$ (5)

The advantage of *series expansion method* (SEM) is related to the easy processing of the solutions in series form as well as in finding the exact solutions based upon the known series. Thus, we apply this method within the local fractional derivative [30] to investigate the solutions of the homogeneous and non-homogeneous transport equations in fractal porous media. In the following section, the local fractional SEM is given. In Section 3, the approximate solutions with non-differentiable terms are discussed. Finally, the conclusions are provided in Section 4.

2. THE LOCAL FRACTIONAL SEM

In order to introduce the local fractional SEM [30], we start with the following local fractional differentiable operator equation, namely

$$u_t^\alpha = L_\alpha u,$$ (6)

where $u(x,t)$ denotes a non-differentiable function, L_α represents a linear local fractional differential operator with respect to x, and u_t^α is the local fractional partial derivative of $u(x,t)$ with respect to t. By making use of (6) we express the non-differentiable function $u(x,t)$ as

$$u(x,t) = \sum_{i=0}^{\infty} T_i(t) X_i(x),$$ (7)

where $T_i(t)$ and $X_i(x)$ are non-differentiable functions too.

Let the term $T_i(t)$ written as

$$T_i(t) = \frac{t^{i\alpha}}{\Gamma(1+i\alpha)},$$ (8)

thus, by making use of (7) we get

$$u(x,t) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} X_i(x),$$ (9)

where $X_i(x)$ is a local fractional continuous function.

So, in view of (9), we obtain

$$u_t^\alpha = \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\alpha)} t^{i\alpha} X_{i+1}(x)$$ (10)

and

$$L_\alpha u = L_\alpha \left[\sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} X_i(x) \right] = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} (L_\alpha X_i)(x).$$ (11)

Now, following (10) and (11), we transfer (6) into

$$\sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} X_{i+1}(x) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} (L_\alpha X_i)(x). \tag{12}$$

For any $i > 0$ we have the iterative formula as

$$X_{i+1}(x) = (L_\alpha X_i)(x). \tag{13}$$

In view of (13), we express the non-differentiable solution of (6) as

$$u(x,t) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} X_i(x), \tag{14}$$

where the convergence condition is given by

$$\lim_{n\to\infty} \left[\frac{t^{i\alpha}}{\Gamma(1+i\alpha)} X_i(x) \right] = 0. \tag{15}$$

The operators defined in this manuscript are defined on the Cantor sets and their meanings are different with respect to the classical and the fractional calculus ones.

3. APPROXIMATE SOLUTIONS WITH NON-DIFFERENTIABLE TERMS

In this section, the approximate solutions for several homogeneous and non-homogeneous transport equations of fractal porous media are presented. Let us rewrite (1) as a local fractional differential operator equation, namely

$$u_t^\alpha = L_\alpha u, \tag{16}$$

where the linear local fractional differential operator with respect to x is given by

$$L_\alpha u = -\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}. \tag{17}$$

In the following we present some illustrative examples in order to show how the method works.

Example 1. We consider the following initial value of (16) as

$$u(x,0) = E_\alpha(x^\alpha). \tag{18}$$

In view of (13), (17), and (18), we have the iterative formula

$$X_{i+1}(x) = -\frac{\partial^\alpha X_i(x)}{\partial x^\alpha}, \tag{19}$$

where the initial value is presented as

$$X_0(x) = E_\alpha(x^\alpha). \tag{20}$$

Hence, we get the following expression

$$X_1(x) = -E_\alpha(x^\alpha), X_2(x) = E_\alpha(x^\alpha), \cdots, X_n(x) = (-1)^i E_\alpha(x^\alpha). \tag{21}$$

As a result, we obtain the non-differentiable solution of (1) with initial value (18), which is given by

$$u(x,t) = E_\alpha(x^\alpha)\left(\sum_{i=0}^\infty \frac{(-1)^i t^{2i\alpha}}{\Gamma(1+2i\alpha)} - \sum_{i=0}^\infty \frac{(-1)^i t^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)}\right) =$$
$$= E_\alpha(x^\alpha)\left(\cos_\alpha(t^\alpha) - \sin_\alpha(t^\alpha)\right). \tag{22}$$

The result presented in Eq. (22) is in full agreement with the result obtained in Ref. [29].

Example 2. Next, we consider the initial value of (16) as

$$u(x,0) = \sinh_\alpha(x^\alpha). \tag{23}$$

As a result, from (13), (17), and (23), we conclude that

$$X_{i+1}(x) = -\frac{\partial^\alpha X_i(x)}{\partial x^\alpha} \tag{24}$$

and

$$X_0(x) = \sinh_\alpha(x^\alpha). \tag{25}$$

Therefore, by using the method developed in this manuscript together with (24) we arrive at the following solution

$$u(x,t) = \sinh_\alpha(x^\alpha)\sum_{i=0}^\infty \frac{(-1)^i t^{2i\alpha}}{\Gamma(1+2i\alpha)} - \cosh_\alpha(x^\alpha)\sum_{i=0}^\infty \frac{(-1)^i t^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)} =$$
$$= \sinh_\alpha(x^\alpha)\cos_\alpha(t^\alpha) - \cosh_\alpha(x^\alpha)\sin_\alpha(t^\alpha). \tag{26}$$
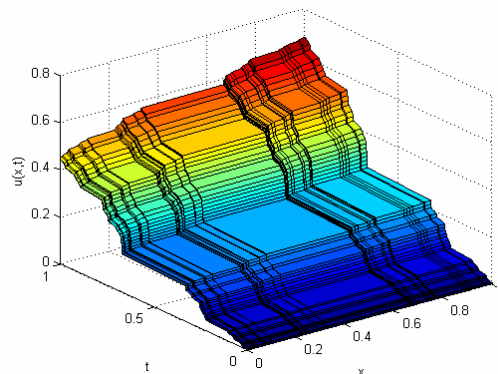
The plot of (26) is shown in Fig. 1.

Fig. 1 – The non-differentiable solution of (1) with initial value (23) when $\alpha = \ln 2 / \ln 3$.

Example 3. In this example the initial value of (16) reads as

$$u(x,0) = \cosh_\alpha(x^\alpha). \tag{27}$$

Making use of (13), (17), and (28), we obtain

$$X_{i+1}(x) = -\frac{\partial^\alpha X_i(x)}{\partial x^\alpha} \tag{28}$$

and

$$X_0(x) = \cosh_\alpha(x^\alpha).$$ (29)

The solution in this case it is written as

$$u(x,t) = \cosh_\alpha(x^\alpha)\sum_{i=0}^{\infty}\frac{(-1)^i t^{2i\alpha}}{\Gamma(1+2i\alpha)} - \sinh_\alpha(x^\alpha)\sum_{i=0}^{\infty}\frac{(-1)^i t^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)} =$$

$$= \cosh_\alpha(x^\alpha)\cos_\alpha(t^\alpha) - \sinh_\alpha(x^\alpha)\sin_\alpha(t^\alpha).$$ (30)

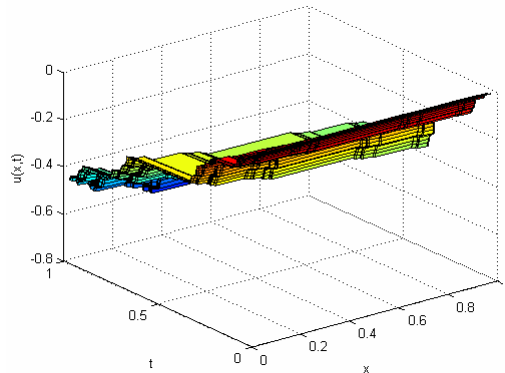The corresponding graph of (30) is depicted in Figure 2.

Fig. 2 – The non-differentiable solution of (1) with initial value (28) when $\alpha = \ln 2/\ln 3$.

In the following let us rewrite (2) as the local fractional differential operator equation, namely

$$u_t^\alpha = L_\alpha u,$$ (31)

where the linear local fractional differential operator with respect to x is given as

$$L_\alpha u = -\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + u(x,t).$$ (32)

Example 4. Here we consider the initial value of (31) as

$$u(x,0) = 2.$$ (33)

In view of (13), (31), and (32), the following iterative formula reads as

$$X_{i+1}(x) = -\frac{\partial^\alpha X_i(x)}{\partial x^\alpha} + X_i(x),$$ (34)

where the initial value is given by

$$X_0(x) = \mu.$$ (35)

With the help of (34) and (35), we conclude that

$$X_i(x) = \mu, i \in N.$$ (36)

Thus, the non-differentiable solution of (2) together with the initial value (33), is represented as

$$u(x,t) = \mu\sum_{i=0}^{1}\frac{t^{i\alpha}}{\Gamma(1+i\alpha)} = \mu E_\alpha(t^\alpha),$$ (37)
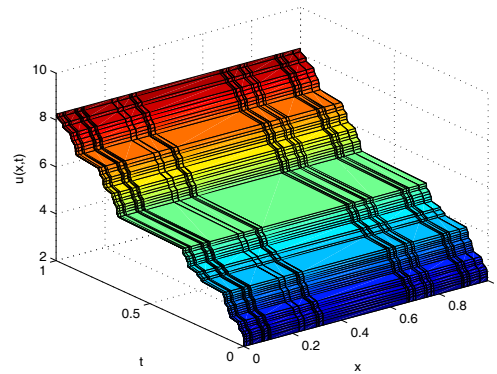
and its plot is depicted in Fig. 3.

Fig. 3 – The non-differentiable solution of (1) with initial value (37) when $\alpha = \ln 2 / \ln 3$.

Example 5. Let us choose the initial value of (31) given by

$$u(x,0) = 1 + E_\alpha\left(x^\alpha\right). \tag{38}$$

Employing (13), (31), and (38), the following iterative formula is given by

$$X_{i+1}(x) = -\frac{\partial^\alpha X_i(x)}{\partial x^\alpha} + X_i(x), \tag{39}$$

where the initial value has the form

$$X_0(x) = 1 + E_\alpha\left(x^\alpha\right). \tag{40}$$

Due to the formulas (39) and (40), the following iterative formula becomes

$$X_i(x) = 1, \ i \in N . \tag{41}$$

Consequently, the non-differentiable solution of (2) with initial value (40) is represented as

$$u(x,t) = E_\alpha\left(x^\alpha\right) + \sum_{i=0}^{1} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} =$$
$$= E_\alpha\left(x^\alpha\right) + E_\alpha\left(t^\alpha\right), \tag{42}$$
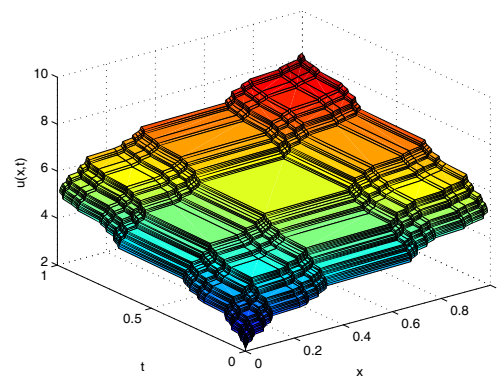
and it has the plot illustrated in Fig. 4.

Fig. 4 – The non-differentiable solution of (1) with initial value (42) when $\alpha = \ln 2 / \ln 3$.

4. CONCLUSIONS

In this work we applied the local fractional SEM to investigate both homogeneous and non-homogeneous transport equations in fractal porous media. The SEM is very effective for solving the local fractional PDEs because it leads rapidly to the desired solution. The reported graphs of the investigated non-differentiable solutions are shown to illustrate the studied examples. The values of the parameter α in all examples given in this work can be changed and the solutions of the investigated problems reflect the complexity and the diversity of the fractal structure.

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