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On existence results for impulsive fractional neutral stochastic integro-differential equations with nonlocal and state-dependent delay conditions

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Abstract
This manuscript deals with a new set of sufficient conditions for the existence of solutions for a class of impulsive fractional neutral stochastic integro-differential systems (IFNSIDS) with nonlocal conditions (NLCs) and state-dependent delay (SDD) in Hilbert spaces. An example is provided to illustrate the obtained theory.

MSC: 65C30; 26A33; 34A08

Keywords: fractional differential equations; stochastic differential system; non-local condition; state-dependent delay; fixed point theorem; semigroup theory

1 Introduction

In this paper, we establish the existence of mild solutions for IFNSIDS with SDD in Hilbert spaces through the utilization of the fixed point theorem of Krasnoselskii [1], Lemma 3.3. We discuss the impulsive neutral stochastic integro-differential equations of fractional order with NLCs and SDD of the model

$$
\begin{aligned}
&{}^C D_t^{\alpha}\big[u(t) - \mathscr{G}(t, u_{\varrho(t,u_t)})\big] \\
&\quad = \mathscr{A} u(t) + \mathscr{F}\left(t, u_{\varrho(t,u_t)}, \int_0^t e_1(t,s,u_{\varrho(s,u_s)})\,ds\right) \\
&\qquad + \Sigma\left(t, u_{\varrho(t,u_t)}, \int_0^t e_2(t,s,u_{\varrho(s,u_s)})\,ds\right)\frac{dw(t)}{dt}, \quad t \neq t_k, k = 1,2,\dots,n,
\end{aligned} \tag{1.1}
$$

$$
\Delta u(t_k) = \mathcal{I}_k\big(u(t_k^-)\big), \quad k = 1,2,\dots,n, \tag{1.2}
$$

$$
u(0) + h(u) = \varphi \in \mathcal{B}, \tag{1.3}
$$

where ${}^C D_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, $t \in \mathscr{I} = [0,T]$ is an operational interval, the state variable u takes values in a Hilbert space $\mathcal{H}$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $k = 1,2,\dots,n$, are jumps of the solution at impulsive points t_k $(0 < t_1 < t_2 < \cdots < t_n < T)$, $\mathscr{A} : D(\mathscr{A}) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{\mathbb{T}(t) : t \geq 0\}$, that is, $\|\mathbb{T}(t)\| \leq \mathcal{M}$ for some constant $\mathcal{M} \geq 1$ and all $t \geq 0$. The time history $u_t : (-\infty, 0] \to \mathcal{H}$, $u_t(\theta) = u(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ described axiomatically in Section 2, and $\varrho : \mathscr{I} \times \mathcal{B} \to (-\infty, T]$ is a continuous function. Let $\mathcal{K}$ be another Hilbert space and suppose that $\{W(t)\}_{t\geq 0}$ is a $\mathcal{K}$-valued Brownian motion (or Wiener process) with finite trace nuclear covariance operator $Q \geq 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We denote by $\mathcal{PC}(\mathscr{I}, \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{H}))$ the Banach space of piecewise continuous functions from $\mathscr{I}$ into $\mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{H})$ with norm $\|u\|_{\mathcal{PC}} = \sup_{t\in\mathscr{I}} |u(t)| < \infty$ (by piecewise continuous functions we mean the functions that are continuous everywhere except for some t_k at which $u(t_k^-)$ and $u(t_k^+)$ exist and $u(t_k^-) = u(t_k)$); $\mathcal{PC}(\mathscr{I}, \mathcal{L}^2)$ is the closed subspace of $\mathcal{PC}(\mathscr{I}, \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{H}))$ consisting of measurable $\mathcal{F}_t$-adapted $\mathcal{H}$-valued processes u with norm $\|u\|^2 = \sup\{\mathbb{E}\|u(t)\|^2, t \in \mathscr{I}\}$. The functions $\mathscr{G}$, $\mathscr{F}$, Σ, e_i, $i = 1, 2$, $\mathcal{I}_k$, and h are apposite functions to be specified later.

The concept of semigroups of bounded linear operators is precisely relevant to dealing with differential and integro-differential equations in Banach spaces. Lately, this strategy has been utilized to a substantial type of nonlinear differential equations in Banach spaces. For more points of interest on this concept, we refer to Pazy [2]. In the midst of the past two decades, fractional differential equations (FDEs) have picked up extensive vitality because of their use in numerous sciences, including physical science, mechanics, and engineering [3, 4]. There has been a lot of enthusiasm toward the solutions of fractional differential equations in systematic and mathematical thoughts. For fundamental certainties about fractional systems, we refer to the books [5–7], papers [8–12], and the references therein. FDEs with delay features happen in several areas such as medical and physical with SDD or nonconstant delay. Nowadays, the existence and controllability of mild solutions for such problems became very attractive. As of late, a few papers have been published on the fractional-order problems with SDD (see [13–20] and references therein). Particularly, in [13], the authors studied the existence of solutions for fractional integro-differential equations, whereas Benchohra et al. [14, 15] established the existence of mild solutions for fractional integro-differential equations in Banach spaces. In [17–20], the authors investigate the existence and approximate controllability results for neutral fractional differential (or integro-differential) equations (or inclusions) in Banach spaces.

On the one hand, various evolutionary processes from fields in physics, population dynamics, aeronautics, economics, and engineering are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. Because the duration of these changes is often negligible compared to the total duration of the process, such changes can be reasonably well approximated as being instantaneous changes of state or in the form of impulses. These processes tend to be more suitably modeled by impulsive differential equations, which allow for discontinuities in the evolution of the state. For more details on this theory and its applications, we refer to the monographs of Lakshmikantham et al. [21], Stamova [22], Graef et al. [23], Bainov et al. [24], Benchohra et al. [25], the papers [4, 26–32], and the references therein.

On the other hand, the study of differential equations with NLCs is of significance due to its applications in problems in physics and other areas of applied mathematics. Byszewski [33] proved the existence of mild, strong, and classical solutions for the nonlocal Cauchy problem. As commended by Byszewski and Lakshmikantham [34], the NLCs can be more helpful than the standard initial condition to depict some physical marvels. There are numerous papers involved with the NLCs; see [10, 35] and the references therein for illustrations.

A natural extension of a deterministic differential equation model is a system of stochastic differential equations, where relevant parameters are modeled as suitable stochastic processes. This is due to the fact that most problems in real-life situations are basically modeled by stochastic equations rather than deterministic. Furthermore, it should be mentioned that noise or stochastic perturbation is unavoidable in nature and man-made systems. Thus, stochastic differential equations have attracted great interest due to their extensive applications in describing many sophisticated dynamical systems in physical, biological, medical, and social sciences; see [1, 35–42] and the references therein for details.

The existence, controllability, and other qualitative and quantitative attributes of stochastic FDEs are the most advancing area of interest; for instance, see [1, 32, 35–40, 43–48]. In particular, Sakthivel et al. [1, 46] analyzed the existence and approximate controllability of fractional stochastic integro-differential equations with infinite delay by utilizing the Krasnoselskii fixed point theorem. Zang and Li [35] discussed a new set of sufficient conditions for approximate controllability of fractional impulsive neutral stochastic differential equations with nonlocal conditions in Hilbert spaces under the Krasnoselskii-Schaefer fixed point theorem and stochastic analysis concept. Recently, Guendouzi et al. [37, 39, 43] studied the existence and approximate controllability of different types of fractional stochastic differential and integro-differential systems with state-dependent delay in Hilbert spaces under different suitable fixed point theorems, whereas Yan et al. [38, 40, 45, 49] examined the existence and approximate controllability of fractional stochastic differential systems with infinite or state-dependent delay in Hilbert spaces with the help of suitable fixed point theorems. Very recently, Balasubramaniam and Tamilalagan [36] analyzed a new set of sufficient conditions for the approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay in Hilbert spaces by utilizing the Bohnenblust-Karlin fixed point theorem, Mainardi's function, operator semigroups, and fractional calculus. Lately, Zhang et al. [44] investigate the approximate controllability of impulsive fractional stochastic differential equations with state-dependent delay in Hilbert spaces with the help of fractional calculus and stochastic analysis.

Rather than the present results, this paper has some positive aspects, namely: We include the integral term in $\mathscr{F}$ and Σ and present an appropriate notion of mild solution of model (1.1)-(1.3) with the help of the Laplace transform and probability density functions. Further, in Lemma 2.5, we prove that our definition of mild solution satisfies the given system. Then, we analyze the existence of mild solutions for IFNSIDS with NLCs and SDD of problem (1.1)-(1.3) under the Krasnoselskii fixed point theorem in $\mathcal{B}$ phase spaces, and the results in [43, 46] might be observed as particular circumstances. By using the concept introduced in this paper, we can analyze the approximate controllability, stability, and so on with necessary modifications. The further developments are clearly given in the conclusion section.

Since impulsive effects also widely exist in fractional stochastic differential systems, it is important and necessary to discuss the qualitative properties for impulsive stochastic fractional integro-differential equations with nonlocal conditions and state-dependent delay. However, to the authors' knowledge, little is concerned with the existence results for IFNSIDS with NLCs and SDD in Hilbert spaces. The purpose of this paper is to analyze this fascinating model (1.1)-(1.3).

The rest of the paper is organized as follows. In Section 2, we give some preliminaries, basic definitions, lemmas, and results. In Section 3, we present and prove existence results for problem (1.1)-(1.3) under the Krasnoselskii fixed point theorem. In Section 4, as a last point, an example is given to illustrate our theoretical results.

2 Preliminaries

Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ and $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$ be two real separable Hilbert spaces. We utilize the same notations $\|\cdot\|$ and $(\cdot,\cdot)$ to represent the norms and inner products in $\mathcal{H}$ and $\mathcal{K}$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space furnished with a normal filtration $\mathcal{F}_t$, $t \in \mathscr{I}$, satisfying the usual conditions, and $\mathbb{E}(\cdot)$ means the expectation with respect to the measure $\mathcal{P}$. An $\mathcal{H}$-valued random variable is an $\mathcal{F}$-measurable function $u(t) : \Omega \to \mathcal{H}$, and a collection of random variables $\mathcal{W} = \{u(t,\omega) : \Omega \to \mathcal{H}|_{t\in T}\}$ is called a stochastic process. We suppress the dependence on $\omega \in \Omega$ and write $u(t)$ instead of $u(t,\omega)$ and $u(t) : \mathscr{I} \to \mathcal{H}$ instead of $\mathcal{W}$. Let $\{\beta_n\}_{n\geq 1}$ be a sequence of real-valued independent Brownian motions. We define $W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)\chi_n$, $t \geq 0$, where $\{\chi_n\}_{n\geq 1}$ is a complete orthonormal system in $\mathcal{K}$, and $\lambda_n \geq 0$ $(n = 1, 2, \dots)$ are nonnegative real numbers. Let $Q \in \mathcal{L}(\mathcal{K}, \mathcal{K})$ be an operator satisfying $Q\chi_n = \lambda_n\chi_n$ with $tr(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$. Then, the above $\mathcal{K}$-valued stochastic process $W(t)$ becomes a Q-Wiener process. We assume that $\mathcal{F}_t = \sigma(W(s) : 0 \leq s \leq t)$ is the σ-algebra generated by W and $\mathcal{F}_T = \mathcal{F}$.

Let $\mathcal{L}(\mathcal{K}, \mathcal{H})$ denote the space of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$ possessing the operator norm $\|\cdot\|$. For $\varphi \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, we define

$$\|\varphi\|_Q^2 = tr\bigl(\varphi Q\varphi^*\bigr) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\varphi\chi_n\|^2.$$

If $\|\varphi\|_Q^2 < \infty$, then φ is called a Q-Hilbert-Schmidt operator. Let $\mathcal{L}_Q(\mathcal{K}, \mathcal{H})$ be the space of all Q-Hilbert-Schmidt operators φ. The completion $\mathcal{L}_Q(\mathcal{K}, \mathcal{H})$ of $\mathcal{L}(\mathcal{K}, \mathcal{H})$ with respect to the topology induced by the norm $\|\cdot\|_Q$ such that $\|\varphi\|_Q^2 = \langle\varphi, \varphi\rangle$ is a Hilbert space with this norm topology.

Without loss of generality, we assume that $0 \in \wp(\mathscr{A})$, the resolvent set of $\mathscr{A}$. Then for $0 < \eta \leq 1$, it is possible to define the fractional power $\mathscr{A}^\eta$ as a closed linear operator on its domain $D(\mathscr{A}^\eta)$, which is dense in $\mathcal{H}$, and we denote by $\mathcal{H}_\eta$ the Banach space of $D(\mathscr{A}^\eta)$ endowed with the norm $\|u\|_\eta = \|\mathscr{A}^\eta u\|$, which is equivalent to the graph norm of $\mathscr{A}^\eta$.

Lemma 2.1 [2] *Suppose that the preceding conditions are satisfied.*

(i) *If $0 < \eta \leq 1$, then $\mathcal{H}_\eta$ is a Banach space.*

(ii) *If $0 < \nu \leq \eta$, then the embedding $\mathcal{H}_\nu \subset \mathcal{H}_\eta$ is compact whenever the resolvent operator of $\mathscr{A}$ is compact.*

(iii) *For every $\eta \in (0, 1]$, there exists a positive constant $\mathcal{C}_\eta$ such that*

$$\bigl\|\mathscr{A}^\eta\mathbb{T}(t)\bigr\| \leq \frac{\mathcal{C}_\eta}{t^\eta}, \quad t > 0.$$

It needs to be outlined that, once the delay is infinite, we should talk about the theoretical phase space $\mathcal{B}$ in a beneficial way.

We assume that the phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of $\mathcal{F}_0$-measurable functions mapping $(-\infty, 0]$ into $\mathcal{H}$ and fulfilling the subsequent elementary axioms as a result of Hale and Kato (see [4, 50, 51]).

If $u:(-\infty,T]\to\mathcal{H}$, $T>0$, is continuous on $\mathscr{I}$ and $u_0\in\mathcal{B}$, then, for every $t\in\mathscr{I}$, the accompanying conditions hold:

(P_1) u_t is in $\mathcal{B}$;
(P_2) $\|u(t)\|\leq H\|u_t\|_{\mathcal{B}}$;
(P_3) $\|u_t\|_{\mathcal{B}}\leq\mathscr{E}_1(t)\sup\{\|u(s)\|:0\leq s\leq t\}+\mathscr{E}_2(t)\|u_0\|_{\mathcal{B}}$, where $H>0$ is a constant, $\mathscr{E}_1(\cdot):[0,+\infty)\to[0,+\infty)$ is continuous, $\mathscr{E}_2(\cdot):[0,+\infty)\to[0,+\infty)$ is locally bounded, and $\mathscr{E}_1$, $\mathscr{E}_2$ are independent of $u(\cdot)$.
(P_4) The function $t\to\varphi_t$ is well described and continuous from the set

$$\mathcal{R}(\varrho^-)=\{\varrho(s,\psi):(s,\psi)\in\mathscr{I}\times\mathcal{B}\},$$

into $\mathcal{B}$, and there is a continuous and bounded function $J^\varphi:\mathcal{R}(\varrho^-)\to(0,\infty)$ such that $\mathbb{E}\|\varphi_t\|^2_{\mathcal{B}}\leq J^\varphi(t)\mathbb{E}\|\varphi\|^2_{\mathcal{B}}$ for every $t\in\mathcal{R}(\varrho^-)$.
(P_5) For the function $u(\cdot)$ in (P_1), u_t is a $\mathcal{B}$-valued continuous function on $[0,T)$.
(P_6) The space $\mathcal{B}$ is complete.

Let $u:(-\infty,T]\to\mathcal{H}$ be an $\mathcal{F}_t$-adapted measurable process such that $u_0=\varphi(t)\in\mathcal{L}^2(\Omega,\mathcal{B})$ is an $\mathscr{F}_0$-adapted process. Then

$$\mathbb{E}\|u_t\|^2_{\mathcal{B}}\leq\mathscr{E}_1^{*2}\sup_{0\leq s\leq T}\{\mathbb{E}\|u(s)\|^2\}+\mathscr{E}_2^{*2}\mathbb{E}\|\varphi\|^2_{\mathcal{B}},$$

where $\mathscr{E}_1^*=\sup_{s\in\mathscr{I}}\mathscr{E}_1(s)$ and $\mathscr{E}_2^*=\sup_{s\in\mathscr{I}}\mathscr{E}_2(s)$.

Lemma 2.2 [52] *Let $u:(-\infty,T]\to\mathcal{H}$ be a function such that $u_0=\varphi$, $u\in\mathcal{PC}(\mathscr{I},\mathcal{L}^2)$, and* ($P_4$) *holds. Then*

$$\begin{aligned}\mathbb{E}\|u_s\|^2_{\mathcal{B}}\leq{}&\mathscr{E}_1^{*2}\sup\{\mathbb{E}\|u(\theta)\|^2_{\mathcal{H}}:\theta\in[0,\max\{0,s\}]\}\\&+(\mathscr{E}_2^*+J^\varphi)^2\mathbb{E}\|u_0\|^2_{\mathcal{B}},\quad s\in\mathcal{R}(\varrho^-)\cup\mathscr{I},\end{aligned}$$

where $J^\varphi=\sup_{t\in\mathcal{R}(\varrho^-)}J^\varphi(t)$.

Consider the space

$$\mathscr{B}_T=\{u:(-\infty,T]\to\mathcal{H}\text{ such that }u_0\in\mathcal{B}\text{ and }u|_{\mathscr{I}}\in\mathcal{PC}(\mathscr{I},\mathcal{L}^2)\}.$$

The function $\|\cdot\|_{\mathscr{B}_T}$ defined as

$$\|u\|_{\mathscr{B}_T}=\|\varphi\|_{\mathcal{B}}+\sup\{(\mathbb{E}\|u(s)\|^2)^{\frac{1}{2}}:s\in[0,T]\},\quad u\in\mathscr{B}_T,$$

is a seminorm in $\mathscr{B}_T$.

Now, we provide some additional fundamental definitions and outcomes of the fractional calculus theory.

Definition 2.1 [53] The fractional integral of order γ with the lower limit zero for a function f is defined by

$$I_t^\gamma f(t)=\frac{1}{\Gamma(\gamma)}\int_0^t\frac{f(s)}{(t-s)^{1-\gamma}}\,ds,\quad t>0,\gamma>0,$$

provided that the right part is pointwise defined on $[0,+\infty)$, where Γ is the gamma function.

Definition 2.2 [53] The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f \in \mathcal{L}^1(\mathscr{I},\mathcal{H})$ is defined by

$$D_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{1-n+\gamma}}\,ds, \quad t>0, n-1<\gamma<n.$$

Definition 2.3 [53, 54] The Caputo derivative of order γ for a function $f \in \mathcal{L}^1(\mathscr{I},\mathcal{H})$ is defined by

$${}^C D_t^\gamma f(t) = D_t^\gamma \bigl(f(t)-f(0)\bigr), \quad t>0, 0<\gamma<1.$$

Definition 2.4 ([55], Definition 4.59) The generalized Mittag-Leffler special function $E_{\alpha,\beta}$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)} = \frac{1}{2\pi i}\int_{\Upsilon_0}\frac{\lambda^{\alpha-\beta}e^\lambda}{\lambda^\alpha - z}\,d\lambda, \quad \alpha,\beta>0, z\in\widetilde{C},$$

where Υ_0 is a contour which starts and ends at $-\infty$ and encircles the disc $|\lambda| \le |z|^{\frac{1}{\alpha}}$ counter-clockwise, and $\widetilde{C}$ denotes the complex plane. For short, set $E_\alpha(z) = E_{\alpha,1}(z)$.

Definition 2.5 [55] The Wright-type function is defined by

$$\phi_\alpha(z) = \sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(-\alpha n-\alpha+1)} = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-z)^n}{(n-1)!}\Gamma(n\alpha)\sin(n\pi\alpha), \quad 0<\alpha<1, z\in\widetilde{C}.$$

Presently, we are in a position to characterize the mild solution for system (1.1)-(1.3). For this, we first consider the system

$${}^C D_t^\alpha x(t) = \mathscr{A}x(t) + \mathscr{F}(t)\frac{dw(t)}{dt}, \tag{2.1}$$

$$x(0) = x_0, \tag{2.2}$$

where ${}^C D_t^\alpha$ and $\mathscr{A}$ are defined in (1.1)-(1.3). Now, we first consider the classical solutions to problem (2.1)-(2.2). Then, based on the expression of such solutions, we define the mild solutions of problem (2.1)-(2.2). Finally, we obtain relations between the analytic semigroup $\{\mathbb{T}(t)\}_{t\ge 0}$ and some solution operators.

Lemma 2.3 ([56], Lemma 6) *Let $\mathscr{A}$ be the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t\ge 0}$. Then, if $\mathscr{F}$ satisfies the uniform Hölder condition with exponent $\beta \in (0,1]$, then the solutions of the Cauchy system* (2.1)-(2.2) *are fixed points of the operator equation*

$$\Psi x(t) = \mathcal{T}_\alpha(t)x_0 + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s)\,dw(s), \tag{2.3}$$

where

$$\mathcal{T}_\alpha(t) = \frac{1}{2\pi i}\int_C e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,\mathscr{A})\,d\lambda \quad \textit{and} \quad \mathcal{S}_\alpha(t) = \frac{1}{2\pi i}\int_C e^{\lambda t}R(\lambda^\alpha,\mathscr{A})\,d\lambda.$$

Here C is a suitable path satisfying $\lambda^\alpha \notin \mu + S_\theta$ *for some* $\lambda \in C$.

Proof According to the Definitions 2.1 and 2.2, we rewrite the Cauchy system (2.1)-(2.2) as the equivalent integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\mathscr{A}x(s)}{(t-s)^{1-\alpha}}\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\mathscr{F}(s)}{(t-s)^{1-\alpha}}\,dw(s). \tag{2.4}$$

Let $\lambda > 0$. Using the Laplace transforms

$$(\mathscr{L}x)(\lambda) = \int_0^\infty e^{-\lambda s}x(s)\,ds \quad \text{and} \quad \big(\mathscr{L}\mathscr{F}(t)\big)(\lambda) = \int_0^\infty e^{-\lambda s}\mathscr{F}(s)\,dw(s)$$

to (2.4), we get

$$\begin{aligned}
&(\mathscr{L}x)(\lambda) \\
&\quad = \int_0^\infty e^{-\lambda s}\left[x_0 + \frac{1}{\Gamma(\alpha)}\int_0^s \frac{\mathscr{A}x(\theta)}{(s-\theta)^{1-\alpha}}\,d\theta + \frac{1}{\Gamma(\alpha)}\int_0^s \frac{\mathscr{F}(\theta)}{(s-\theta)^{1-\alpha}}\,dw(\theta)\right]ds \\
&\quad = \int_0^\infty e^{-\lambda s}x_0\,ds + \int_0^\infty e^{-\lambda s}\left[\frac{1}{\Gamma(\alpha)}\int_0^s \frac{\mathscr{A}x(\theta)}{(s-\theta)^{1-\alpha}}\,d\theta\right]ds \\
&\qquad + \int_0^\infty e^{-\lambda s}\left[\frac{1}{\Gamma(\alpha)}\int_0^s \frac{\mathscr{F}(\theta)}{(s-\theta)^{1-\alpha}}\,dw(\theta)\right]ds \\
&\quad = \frac{1}{\lambda}\big[e^{-\lambda s}\big]_\infty^0 x_0 + \frac{1}{\lambda^\alpha}\mathscr{A}(\mathscr{L}x)(\lambda) + \frac{1}{\lambda^\alpha}\big(\mathscr{L}\mathscr{F}(t)\big)(\lambda), \\
&(\mathscr{L}x)(\lambda) - \frac{1}{\lambda^\alpha}\mathscr{A}(\mathscr{L}x)(\lambda) = \frac{1}{\lambda}x_0 + \frac{1}{\lambda^\alpha}\big(\mathscr{L}\mathscr{F}(t)\big)(\lambda), \\
&\big(\lambda^\alpha I - \mathscr{A}\big)(\mathscr{L}x)(\lambda) = \frac{\lambda^\alpha}{\lambda}x_0 + \big(\mathscr{L}\mathscr{F}(t)\big)(\lambda), \\
&(\mathscr{L}x)(\lambda) = \lambda^{\alpha-1}\big(\lambda^\alpha I - \mathscr{A}\big)^{-1}x_0 + \big(\lambda^\alpha I - \mathscr{A}\big)^{-1}\big(\mathscr{L}\mathscr{F}(t)\big)(\lambda).
\end{aligned}$$

Since $\lambda^\alpha(\lambda^\alpha - \mathscr{A})^{-1} = I + \mathscr{A}(\lambda^\alpha - \mathscr{A})^{-1}$, taking the inverse Laplace transform of the above equation, we obtain

$$x(t) = \mathcal{T}_\alpha(t)x_0 + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s)\,dw(s).$$

Note that $\mathscr{F}$ satisfies the uniform Hölder condition with exponent $\beta \in (0,1)$. Hence, the classical solutions of Cauchy system (2.1)-(2.2) are fixed points of the operator equation (2.3). □

In view of Lemma 2.3, we determine the mild solutions of system (2.1)-(2.2).

Definition 2.6 A function $x : \mathscr{I} \to \mathcal{H}$ is said to be a mild solution of problem (2.1)-(2.2) if $x \in C(\mathscr{I}, \mathcal{H})$ fulfills the accompanying integral equation

$$x(t) = \mathcal{T}_\alpha(t)x_0 + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s)\,dw(s), \quad t \in \mathscr{I}.$$

Remark 2.1 It is straightforward to confirm that the classical solution of system (2.1)-(2.2) is a mild solution of the same system. Thus, Definition 2.6 is correct (see [2, 57]).

Lemma 2.4 ([56], Lemma 9) *Let $\mathscr{A}$ be the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$ and suppose that $0 \in \wp(\mathscr{A})$. Then we have*

$$\mathcal{S}_\alpha(t) = \alpha \int_0^\infty r\phi_\alpha(r)t^{\alpha-1}\mathbb{T}(t^\alpha r)\,dr \quad \text{and} \quad \mathcal{T}_\alpha(t) = \int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)\,dr. \tag{2.5}$$

Here $\phi_\alpha(r)$ is the probability density function on $(0,\infty)$ with the Laplace transform

$$\int_0^\infty e^{-rx}\phi_\alpha(r)\,dr = \sum_{j=0}^\infty \frac{(-x)^j}{\Gamma(1+\alpha j)}, \quad x > 0,$$

which fulfills

$$\int_0^\infty \phi_\alpha(r)\,dr = 1 \quad \text{and} \quad \int_0^\infty r^\eta \phi_\alpha(r)\,dr \leq 1, \quad 0 \leq \eta \leq 1.$$

Proof For all $x \in D(\mathscr{A}) \subset \mathcal{H}$, we have

$$(\lambda - \mathscr{A})^{-1}x = \int_0^\infty e^{-\lambda s}\mathbb{T}(s)x\,ds.$$

Let

$$\int_0^\infty e^{-\lambda r}\psi_\alpha(r)\,dr = e^{-\lambda^\alpha},$$

where $\alpha \in (0,1)$, $\psi_\alpha(r) = \frac{1}{\pi}\sum_{1\leq n<\infty}(-1)^n r^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha)$, and $r \in (0,\infty)$ (see [58]). Thus, we get

$$\begin{aligned}
(\lambda^\alpha - \mathscr{A})^{-1}x &= \int_0^\infty e^{-\lambda^\alpha s}\mathbb{T}(s)x\,ds \\
&= \int_0^\infty \alpha t^{\alpha-1}e^{-(\lambda t)^\alpha}\mathbb{T}(t^\alpha)x\,dt \\
&= \int_0^\infty \alpha t^{\alpha-1}\left[\int_0^\infty e^{-\lambda t r}\psi_\alpha(r)\,dr\right]\mathbb{T}(t^\alpha)x\,dt \\
&= \int_0^\infty \alpha\left[\int_0^\infty e^{-\lambda t}\psi_\alpha(r)\,dr\right]\mathbb{T}\left(\frac{t^\alpha}{r^\alpha}\right)x\frac{t^{\alpha-1}}{r^\alpha}\,dt \\
&= \int_0^\infty e^{-\lambda t}\left(\alpha\int_0^\infty r\phi_\alpha(r)t^{\alpha-1}\mathbb{T}(t^\alpha r)x\,dr\right)dt,
\end{aligned} \tag{2.6}$$

where $\phi_\alpha(r) = (\frac{1}{\alpha})r^{-1-\frac{1}{\alpha}}\psi_\alpha(r^{\frac{-1}{\alpha}})$ is the probability density function on $(0,\infty)$ such that

$$\int_0^\infty \phi_\alpha(r)\,dr = 1 \quad \text{and} \quad \int_0^\infty r^\eta \phi_\alpha(r)\,dr \le 1, \quad 0 \le \eta \le 1.$$

By Lemma 2.3 and Eq. (2.6), we have

$$\begin{aligned}
\mathcal{S}_\alpha(t) &= \frac{1}{2\pi i}\int_C e^{\lambda t} R(\lambda^\alpha, \mathscr{A})\,d\lambda \\
&= \int_0^\infty e^{\lambda t}(\lambda^\alpha - \mathscr{A})^{-1}\,dt \\
&= \alpha \int_0^\infty r\phi_\alpha(r) t^{\alpha-1}\mathbb{T}(t^\alpha r)\,dr.
\end{aligned}$$

Further, we estimate $\mathcal{S}_\alpha(t)$:

$$\begin{aligned}
\|\mathcal{S}_\alpha(t)\| &= \left\| \alpha \int_0^\infty r\phi_\alpha(r) t^{\alpha-1}\mathbb{T}(t^\alpha r)\,dr \right\| \\
&\le \alpha \left[\int_0^\infty r\phi_\alpha(r)\,dr\right] t^{\alpha-1} \|\mathbb{T}(t^\alpha r)\| \\
&\le \alpha \frac{\Gamma(2)}{\Gamma(1+\alpha)} t^{\alpha-1}\mathcal{M} \\
&\le \frac{\mathcal{M}}{\Gamma(\alpha)} t^{\alpha-1},
\end{aligned}$$

where $\int_0^\infty r^\beta \phi_\alpha(r)\,dr = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}$.

Then again, for all $x \in D(\mathscr{A}) \subset \mathcal{H}$, we notice that

$$\begin{aligned}
\lambda^{\alpha-1}(\lambda^\alpha - \mathscr{A})^{-1}x &= \int_0^\infty \lambda^{\alpha-1} e^{-\lambda^\alpha s}\mathbb{T}(s)x\,ds \\
&= \int_0^\infty \alpha(\lambda t)^{\alpha-1} e^{-(\lambda t)^\alpha}\mathbb{T}(t^\alpha)x\,dt \\
&= \int_0^\infty \frac{-1}{\lambda}\frac{d}{dt}\left[e^{-(\lambda t)^\alpha}\right]\mathbb{T}(t^\alpha)x\,dt \\
&= \int_0^\infty \frac{-1}{\lambda}\frac{d}{dt}\left[\int_0^\infty e^{-\lambda t r}\psi_\alpha(r)\,dr\right]\mathbb{T}(t^\alpha)x\,dt \\
&= \int_0^\infty \left[\int_0^\infty \frac{-1}{\lambda}\left[-\lambda r e^{-\lambda t r}\right]\psi_\alpha(r)\,dr\right]\mathbb{T}(t^\alpha)x\,dt \\
&= \int_0^\infty \int_0^\infty r e^{-\lambda t r}\psi_\alpha(r)\,dr\,\mathbb{T}(t^\alpha)x\,dt \\
&= \int_0^\infty e^{-\lambda t}\left[\int_0^\infty \psi_\alpha(r)\mathbb{T}\left(\frac{t^\alpha}{r^\alpha}\right)x\,dr\right]dt \\
&= \int_0^\infty e^{-\lambda t}\left[\int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)x\,dr\right]dt.
\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{T}_\alpha(t) &= \frac{1}{2\pi i}\int_C e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha,\mathscr{A})\,d\lambda\\ &= \int_0^\infty e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha-\mathscr{A})^{-1}\,dt\\ &= \int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)\,dr.\end{aligned}$$

Moreover, we have an estimate of $\mathcal{T}_\alpha(t)$,

$$\begin{aligned}\|\mathcal{T}_\alpha(t)\| &= \left\|\int_0^\infty \phi_\alpha(r)\mathbb{T}(t^\alpha r)\,dr\right\|\\ &\le \left(\int_0^\infty \phi_\alpha(r)\,dr\right)\|\mathbb{T}(t^\alpha r)\|\\ &\le \mathcal{M},\end{aligned}$$

where $\int_0^\infty \phi_\alpha(r)\,dr = 1$. □

Before we characterize the mild solution for system (1.1)-(1.3), we finally treat the following system:

$$\begin{aligned}&{}^C D_t^\alpha\big[x(t)-\mathscr{G}(t,x(t))\big]\\ &\quad= \mathscr{A}x(t)+\mathscr{F}(t,x(t))+\Sigma(t,x(t))\frac{dw(t)}{dt},\quad t\neq t_k, k=1,2,\ldots,n, &(2.7)\\ &\Delta x(t_k)=\mathcal{I}_k(x(t_k^-)),\quad k=1,2,\ldots,n, &(2.8)\\ &x(0)+h(x)=\varphi(0), &(2.9)\end{aligned}$$

where ${}^C D_t^\alpha$ and $\mathscr{A}$ are as defined in (1.1)-(1.3), and $\mathscr{F}$, Σ, $\mathscr{G}$ are appropriate functions.

By Definitions 2.1 and 2.2, the general integral equation of system (2.7)-(2.9) can be expressed as

$$\begin{aligned}x(t) &= \varphi(0)-\mathscr{G}(0,\varphi)-h(x)+\mathscr{G}(t,x(t))+\sum_{k=1}^n \mathcal{I}_k(x(t_k^-))+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathscr{A}x(s)\,ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathscr{F}(s,x(s))\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Sigma(s,x(s))\,dw(s). \qquad (2.10)\end{aligned}$$

Now, arguing as in [59] and applying the Laplace transform to (2.10), we get

$$\begin{aligned}u(\lambda) &= \lambda^{\alpha-1}(\lambda^\alpha I-\mathscr{A})^{-1}\big[\varphi(0)-\mathscr{G}(0,\varphi)-h(x)\big]+\lambda^\alpha(\lambda^\alpha I-\mathscr{A})^{-1}w(\lambda)\\ &\quad+(\lambda^\alpha I-\mathscr{A})^{-1}v(\lambda)+(\lambda^\alpha I-\mathscr{A})^{-1}z(\lambda)+\lambda^{\alpha-1}(\lambda^\alpha I-\mathscr{A})^{-1}\sum_{k=1}^n \mathcal{I}_k(x(t_k^-)),\end{aligned}$$

where

$$u(\lambda) = \int_0^\infty e^{-\lambda s} x(s)\, ds, \qquad v(\lambda) = \int_0^\infty e^{-\lambda s} \mathscr{F}\big(s, x(s)\big)\, ds,$$
$$w(\lambda) = \int_0^\infty e^{-\lambda s} \mathscr{G}\big(s, x(s)\big)\, ds, \qquad z(\lambda) = \int_0^\infty e^{-\lambda s} \Sigma\big(s, x(s)\big)\, dw(s).$$

By the same calculations as in [59] and by the properties of the Laplace transform we obtain a mild solution of system (2.7)-(2.9),

$$\begin{aligned} x(t) &= \mathcal{T}_\alpha(t)\big[\varphi(0) - \mathscr{G}(0,\varphi) - h(x)\big] + \mathscr{G}\big(t, x(t)\big) + \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\big(s, x(s)\big)\, ds \\ &\quad + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}\big(s, x(s)\big)\, ds + \int_0^t \mathcal{S}_\alpha(t-s)\Sigma\big(s, x(s)\big)\, dw(s) \\ &\quad + \sum_{0<t_k<t} \mathcal{T}_\alpha(t-t_k)\mathcal{I}_k\big(x(t_k^-)\big), \end{aligned} \tag{2.11}$$

where $\mathcal{T}_\alpha$ and $\mathcal{S}_\alpha$ are defined in (2.5).

Next, we shall show that, indeed, this mild solution satisfies system (2.7)-(2.9). To prove this, we first prove the following crucial lemma.

Lemma 2.5 ([60], Lemma 3.3) *Let $\mathscr{A}$ be the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$. If $0 < \alpha < 1$, then*

$${}^C D_t^\alpha\big[\mathcal{T}_\alpha(t)x_0\big] = \mathscr{A}\big[\mathcal{T}_\alpha(t)x_0\big]$$

and

$$\begin{aligned} &{}^C D_t^\alpha\left(\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}\big(s, x(s)\big) + \mathscr{F}\big(s, x(s)\big) + \Sigma\big(s, x(s)\big)\frac{dw(s)}{ds}\right] ds\right) \\ &\quad = \mathscr{A}\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}\big(s, x(s)\big) + \mathscr{F}\big(s, x(s)\big) + \Sigma\big(s, x(s)\big)\frac{dw(s)}{ds}\right] ds \\ &\qquad + \mathscr{A}\mathscr{G}\big(t, x(t)\big) + \mathscr{F}\big(t, x(t)\big) + \Sigma\big(t, x(t)\big)\frac{dw(t)}{dt}, \end{aligned}$$

where $\mathcal{T}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are defined in (2.5).

Proof By the well-known result from [60], Lemma 3.3, we have

$${}^C D_t^\alpha\big[\mathcal{T}_\alpha(t)x_0\big] = \mathscr{A}\big[\mathcal{T}_\alpha(t)x_0\big].$$

Furthermore,

$$\begin{aligned} &\mathscr{L}\left(\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}\big(s, x(s)\big) + \mathscr{F}\big(s, x(s)\big) + \Sigma\big(s, x(s)\big)\frac{dw(s)}{ds}\right] ds\right) \\ &\quad = \mathscr{L}\big(\mathcal{S}_\alpha(t)\big)\mathscr{L}\left(\mathscr{A}\mathscr{G}\big(t, x(t)\big) + \mathscr{F}\big(t, x(t)\big) + \Sigma\big(t, x(t)\big)\frac{dw(t)}{dt}\right) \\ &\quad = R\big(\lambda^\alpha, \mathscr{A}\big)\mathscr{L}\left(\mathscr{A}\mathscr{G}\big(t, x(t)\big) + \mathscr{F}\big(t, x(t)\big) + \Sigma\big(t, x(t)\big)\frac{dw(t)}{dt}\right) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&\mathscr{L}\left({}^{C}D_t^{\alpha}\left(\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}\big(s,x(s)\big)+\mathscr{F}\big(s,x(s)\big)+\Sigma\big(s,x(s)\big)\frac{dw(s)}{ds}\right]ds\right)\right)\\
&\quad=\lambda^{\alpha}\left[R\big(\lambda^{\alpha},\mathscr{A}\big)\mathscr{L}\left(\mathscr{A}\mathscr{G}\big(t,x(t)\big)+\mathscr{F}\big(t,x(t)\big)+\Sigma\big(t,x(t)\big)\frac{dw(t)}{dt}\right)\right]-\lambda^{\alpha-1}.0\\
&\quad=\big(\lambda^{\alpha}I-\mathscr{A}+\mathscr{A}\big)R\big(\lambda^{\alpha},\mathscr{A}\big)\mathscr{L}\left(\mathscr{A}\mathscr{G}\big(t,x(t)\big)+\mathscr{F}\big(t,x(t)\big)+\Sigma\big(t,x(t)\big)\frac{dw(t)}{dt}\right)\\
&\quad=\big(\lambda^{\alpha}I-\mathscr{A}\big)R\big(\lambda^{\alpha},\mathscr{A}\big)\mathscr{L}\left(\mathscr{A}\mathscr{G}\big(t,x(t)\big)+\mathscr{F}\big(t,x(t)\big)+\Sigma\big(t,x(t)\big)\frac{dw(t)}{dt}\right)\\
&\qquad+\mathscr{A}R\big(\lambda^{\alpha},\mathscr{A}\big)\mathscr{L}\left(\mathscr{A}\mathscr{G}\big(t,x(t)\big)+\mathscr{F}\big(t,x(t)\big)+\Sigma\big(t,x(t)\big)\frac{dw(t)}{dt}\right).
\end{aligned}\tag{2.13}$$

Thus, it follows from (2.12) and (2.13) that

$$\begin{aligned}
&{}^{C}D_t^{\alpha}\left(\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}\big(s,x(s)\big)+\mathscr{F}\big(s,x(s)\big)+\Sigma\big(s,x(s)\big)\frac{dw(s)}{ds}\right]ds\right)\\
&\quad=\mathscr{A}\int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}\big(s,x(s)\big)+\mathscr{F}\big(s,x(s)\big)+\Sigma\big(s,x(s)\big)\frac{dw(s)}{ds}\right]ds\\
&\qquad+\mathscr{A}\mathscr{G}\big(t,x(t)\big)+\mathscr{F}\big(t,x(t)\big)+\Sigma\big(t,x(t)\big)\frac{dw(t)}{dt}.
\end{aligned}$$

□

Now, it is time to show that the mild solution satisfies model (2.7)-(2.9). From Eq. (2.11) we have

$$\begin{aligned}
x(t)-\mathscr{G}\big(t,x(t)\big)&=\mathcal{T}_\alpha(t)\big[\varphi(0)-\mathscr{G}(0,\varphi)-h(x)\big]+\int_0^t \mathcal{S}_\alpha(t-s)\Big[\mathscr{A}\mathscr{G}\big(s,x(s)\big)\\
&\quad+\mathscr{F}\big(s,x(s)\big)+\Sigma\big(s,x(s)\big)\frac{dw(s)}{ds}\Big]ds+\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k\big(x\big(t_k^-\big)\big).
\end{aligned}$$

Taking the Caputo derivative of both sides, in view of Lemma 2.5, we have

$$\begin{aligned}
&{}^{C}D_t^{\alpha}\big(x(t)-\mathscr{G}\big(t,x(t)\big)\big)\\
&\quad={}^{C}D_t^{\alpha}\big(\mathcal{T}_\alpha(t)\big[\varphi(0)-\mathscr{G}(0,\varphi)-h(x)\big]\big)+{}^{C}D_t^{\alpha}\Big(\int_0^t \mathcal{S}_\alpha(t-s)\Big[\mathscr{A}\mathscr{G}\big(s,x(s)\big)\\
&\qquad+\mathscr{F}\big(s,x(s)\big)+\Sigma\big(s,x(s)\big)\frac{dw(s)}{ds}\Big]ds\Big)+{}^{C}D_t^{\alpha}\Big(\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k\big(x\big(t_k^-\big)\big)\Big)\\
&\quad=\mathscr{A}\mathcal{T}_\alpha(t)\big[\varphi(0)-\mathscr{G}(0,\varphi)-h(x)\big]+\mathscr{A}\Big(\int_0^t \mathcal{S}_\alpha(t-s)\Big[\mathscr{A}\mathscr{G}\big(s,x(s)\big)\\
&\qquad+\mathscr{F}\big(s,x(s)\big)+\Sigma\big(s,x(s)\big)\frac{dw(s)}{ds}\Big]ds\Big)+\mathscr{A}\mathscr{G}\big(t,x(t)\big)+\mathscr{F}\big(t,x(t)\big)\\
&\qquad+\Sigma\big(t,x(t)\big)\frac{dw(t)}{dt}+\mathscr{A}\Big(\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k\big(x\big(t_k^-\big)\big)\Big)\\
&\quad=\mathscr{A}\Big(\mathcal{T}_\alpha(t)\big[\varphi(0)-\mathscr{G}(0,\varphi)-h(x)\big]+\mathscr{G}\big(t,x(t)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t \mathcal{S}_\alpha(t-s)\left[\mathscr{A}\mathscr{G}\big(s,x(s)\big) + \mathscr{F}\big(s,x(s)\big) + \Sigma\big(s,x(s)\big)\frac{dw(s)}{ds}\right]ds \\
&+ \sum_{0<t_k<t} \mathcal{T}_\alpha(t-t_k)\mathcal{I}_k\big(x\big(t_k^-\big)\big)\Big) + \mathscr{F}\big(t,x(t)\big) + \Sigma\big(t,x(t)\big)\frac{dw(t)}{dt} \\
= {}& \mathscr{A}x(t) + \mathscr{F}\big(t,x(t)\big) + \Sigma\big(t,x(t)\big)\frac{dw(t)}{dt},
\end{aligned}
$$

that is,

$$
{}^C D_t^\alpha\big(x(t) - \mathscr{G}\big(t,x(t)\big)\big) = \mathscr{A}x(t) + \mathscr{F}\big(t,x(t)\big) + \Sigma\big(t,x(t)\big)\frac{dw(t)}{dt}.
$$

From the previous discussion we observe that our definition of a mild solution satisfies the given system (2.7)-(2.9). In accordance with discussion, we define a mild solution of the model (1.1)-(1.3).

Definition 2.7 ([44], Definition 2.1) A stochastic process $u:(-\infty,T]\to\mathcal{H}$ is called a mild solution of system (1.1)-(1.3) if

(i) $u(t)$ is measurable and $\mathcal{F}_t$-adapted for each $t\in\mathscr{I}$;

(ii) $\Delta u(t_k) = u(t_k^+) - u(t_k^-) = \mathcal{I}_k(x(t_k^-))$, $k=1,2,\ldots,n$;

(iii) $u(0)+h(u)=\varphi$;

(iv) $u(t)$ is continuous on $\mathscr{I}$, the function $\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s,u_{\varrho(s,u_s)})$ is integrable, and the following stochastic integral equation is satisfied:

$$
\begin{aligned}
u(t) = {}& \mathcal{T}_\alpha(t)\big[\varphi(0) - h(u) - \mathscr{G}(0,\varphi)\big] + \mathscr{G}(t,u_{\varrho(t,u_t)}) \\
&+ \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s,u_{\varrho(s,u_s)})\,ds \\
&+ \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}\left(s,u_{\varrho(s,u_s)},\int_0^s e_1(s,\tau,u_{\varrho(\tau,u_\tau)})\,d\tau\right)ds \\
&+ \int_0^t \mathcal{S}_\alpha(t-s)\Sigma\left(s,u_{\varrho(s,u_s)},\int_0^s e_2(s,\tau,u_{\varrho(\tau,u_\tau)})\,d\tau\right)dw(s) \\
&+ \sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k\big(u\big(t_k^-\big)\big);
\end{aligned}
\tag{2.14}
$$

(v) $u_0(\cdot)=\varphi\in\mathcal{B}$ on $(-\infty,0]$, and $\|\varphi\|_\mathcal{B}^2<\infty$.

3 Existence results

In this section, we show the existence of solutions for model (1.1)-(1.3) under the Krasnoselskii fixed point theorem [1] using the operator semigroup theory and fractional calculus.

We introduce the following hypotheses:

(H1) The function $\mathscr{G}:\mathscr{I}\times\mathcal{B}\to\mathcal{H}$ is continuous, and there exist constants $\beta\in(0,1)$ and $\mathcal{M}_\mathscr{G}>0$ such that $\mathscr{G}$ is $\mathcal{H}_\beta$-valued and satisfies the following conditions:

$$
\begin{aligned}
&\mathbb{E}\big\|\mathscr{A}^\beta\mathscr{G}(t,x) - \mathscr{A}^\beta\mathscr{G}(t,y)\big\|^2 \le \mathcal{M}_\mathscr{G}\|x-y\|_\mathcal{B}^2, \quad t\in\mathscr{I}, x,y\in\mathcal{B}, \\
&\mathbb{E}\big\|\mathscr{A}^\beta\mathscr{G}(t,x)\big\|^2 \le \mathcal{M}_\mathscr{G}\big(1+\|x\|_\mathcal{B}^2\big), \quad t\in\mathscr{I}, x\in\mathcal{B}.
\end{aligned}
$$

(H2) The function $\mathscr{F} : \mathscr{I} \times \mathcal{B} \times \mathcal{H} \to \mathcal{H}$ satisfies the following properties:
   (i) For each $t \in \mathscr{I}$, the function $\mathscr{F}(t,\cdot,\cdot) : \mathcal{B} \times \mathcal{H} \to \mathcal{H}$ is continuous.
   (ii) For each $(x,\phi) \in \mathcal{B} \times \mathcal{H}$, the function $\mathscr{F}(\cdot,x,\phi) : \mathscr{I} \to \mathcal{H}$ is strongly measurable.
   (iii) There exists a positive integrable function $m_{\mathscr{F}} \in \mathcal{L}^1(\mathscr{I})$ and a continuous nondecreasing function $\Psi_{\mathscr{F}} : [0,\infty) \to (0,\infty)$ such that for all $(t,x,\phi) \in \mathscr{I} \times \mathcal{B} \times \mathcal{H}$, we have

$$\mathbb{E}\big\|\mathscr{F}(t,x,\phi)\big\|^2 \le m_{\mathscr{F}}(s)\Psi_{\mathscr{F}}\big(\|x\|^2_{\mathcal{B}} + \mathbb{E}\|\phi\|^2_{\mathcal{H}}\big),$$
$$\liminf_{r\to\infty} \frac{\Psi_{\mathscr{F}}(r)}{r} = \Lambda < \infty.$$

(H3) The function $e_i : \mathcal{D} \times \mathcal{B} \to \mathcal{H}$, where $\mathcal{D} = \{(t,s) \in \mathscr{I} \times \mathscr{I}; 0 \le s \le t \le T\}$, satisfies:
   (i) For each $(t,s) \in \mathcal{D}$, the function $e_i(t,s,\cdot) : \mathcal{B} \to \mathcal{H}$ is continuous, and for each $x \in \mathcal{B}$, the function $e_i(\cdot,\cdot,x) : \mathcal{D} \to \mathcal{H}$ is strongly measurable.
   (ii) There exist constants $\widetilde{\mathcal{M}}_0, \widetilde{\mathcal{M}}_1 > 0$ such that, for all $t,s \in \mathscr{I}$ and $x \in \mathcal{B}$, we have

$$\mathbb{E}\big\|e_i(t,s,x)\big\|^2 \le \widetilde{\mathcal{M}}_j\big(1 + \|x\|^2_{\mathcal{B}}\big) \qquad \text{for } i = 1,2 \text{ and } j = 0,1.$$

(H4) The function $\Sigma : \mathscr{I} \times \mathcal{B} \times \mathcal{H} \to \mathcal{L}(\mathcal{K},\mathcal{H})$ satisfies the following properties:
   (i) For each $t \in \mathscr{I}$, the function $\Sigma(t,\cdot,\cdot) : \mathcal{B} \times \mathcal{H} \to \mathcal{L}(\mathcal{K},\mathcal{H})$ is continuous.
   (ii) For each $(x,\phi) \in \mathcal{B} \times \mathcal{H}$, the function $\Sigma(\cdot,x,\phi) : \mathscr{I} \to \mathcal{L}(\mathcal{K},\mathcal{H})$ is strongly measurable.
   (iii) There exists a positive integrable function $m_\Sigma \in \mathcal{L}^1(\mathscr{I})$ and a continuous non-decreasing function $\Psi_\Sigma : [0,\infty) \to (0,\infty)$ such that, for all $(t,x,\phi) \in \mathscr{I} \times \mathcal{B} \times \mathcal{H}$, we have

$$\mathbb{E}\big\|\Sigma(t,x,\phi)\big\|^2 \le m_\Sigma(s)\Psi_\Sigma\big(\|x\|^2_{\mathcal{B}} + \mathbb{E}\|\phi\|^2_{\mathcal{H}}\big),$$
$$\liminf_{r\to\infty} \frac{\Psi_\Sigma(r)}{r} = \widetilde{\Lambda} < \infty.$$

(H5) The functions $\mathcal{I}_k : \mathcal{B} \to \mathcal{H}, k = 1,2,\ldots,n$, are continuous, and there exist nondecreasing continuous functions $\mathcal{M}_{\mathcal{I}_k} : \mathcal{R}^+ \to \mathcal{R}^+$ such that, for all $x \in \mathcal{B}$, we have

$$\mathbb{E}\big\|\mathcal{I}_k(x)\big\|^2 \le \mathcal{M}_{\mathcal{I}_k}\big(\mathbb{E}\|x\|^2_{\mathcal{B}}\big), \qquad \liminf_{r\to\infty} \frac{\mathcal{M}_{\mathcal{I}_k}(r)}{r} = \gamma_k < \infty.$$

(H6) The function $h : \mathcal{B} \to \mathcal{H}$ is continuous, and there exists a constant $\mathcal{M}_h > 0$ such that, for all $x \in \mathcal{B}$, we have

$$\mathbb{E}\big\|h(x)\big\|^2 \le \mathcal{M}_h\|x\|^2_{\mathcal{B}}.$$

Now, we are in a position to derive the existence results for model (1.1)-(1.3).

Theorem 3.1 *Let assumptions* (H1)-(H6) *hold. Then system* (1.1)-(1.3) *has a mild solution on* $\mathscr{I}$, *provided that*

$$\begin{aligned} &28\mathcal{M}^2\mathscr{E}_1^{*2}\left[\mathcal{M}_h + H^2 n\sum_{k=1}^{n}\gamma_k\right] + 28\mathscr{E}_1^{*2}\left[\mathcal{M}_{\mathscr{G}}\left(\mathcal{N}_0{}^2 + \left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)T^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right)^2\right)\right. \\ &\quad + \left(\frac{\mathcal{M}T^{\alpha}}{\Gamma(1+\alpha)}\right)^2\left(\sup_{s\in\mathscr{I}} m_{\mathscr{F}}(s)(1+\widetilde{\mathcal{M}}_0 T)\Lambda\right. \\ &\quad \left.\left. + tr(Q)\sup_{s\in\mathscr{I}} m_{\Sigma}(s)(1+\widetilde{\mathcal{M}}_1 T)\widetilde{\Lambda}\right)\right] < 1, \end{aligned} \tag{3.1}$$

where $\mathcal{N}_0 = \|\mathscr{A}^{-\beta}\|$.

Proof Let us transform model (1.1)-(1.3) into a fixed-point problem. Consider the operator $\Upsilon : \mathscr{B}_T \to \mathscr{B}_T$ specified by

$$(\Upsilon u)(t) = \begin{cases} \varphi(t), \quad t \in (-\infty, 0], \\ \mathcal{T}_\alpha(t)[\varphi(0) - h(u) - \mathscr{G}(0,\varphi)] + \mathscr{G}(t, u_{\varrho(t,u_t)}) \\ \quad + \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, u_{\varrho(s,u_s)})\,ds \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)\mathscr{F}(s, u_{\varrho(s,u_s)}, \int_0^s e_1(s,\tau,u_{\varrho(\tau,u_\tau)})\,d\tau)\,ds \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)\Sigma(s, u_{\varrho(s,u_s)}, \int_0^s e_2(s,\tau,u_{\varrho(\tau,u_\tau)})\,d\tau)\,dw(s) \\ \quad + \sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(u(t_k^-)), \quad t \in \mathscr{I}. \end{cases}$$

By Lemma 2.1, for any $u \in \mathcal{H}$ and $\beta \in (0,1)$, we have

$$\begin{aligned} &\left\|\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, u_{\varrho(s,u_s)})\right\|_{\mathcal{H}}^2 \\ &\quad = \left\|\mathscr{A}^{1-\beta}\mathcal{S}_\alpha(t-s)\mathscr{A}^{\beta}\mathscr{G}(s, u_{\varrho(s,u_s)})\right\|_{\mathcal{H}}^2 \\ &\quad \le \left\|\left[\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathscr{A}^{1-\beta}\mathbb{T}\big((t-s)^\alpha r\big)\,dr\right]\mathscr{A}^{\beta}\mathscr{G}(s, u_{\varrho(s,u_s)})\right\|_{\mathcal{H}}^2 \\ &\quad \le \big(\alpha\mathcal{C}_{1-\beta}(t-s)^{\alpha\beta-1}\big)^2\left[\int_0^\infty r^\beta\phi_\alpha(r)\,dr\right]^2\left\|\mathscr{A}^{\beta}\mathscr{G}(s, u_{\varrho(s,u_s)})\right\|_{\mathcal{H}}^2. \end{aligned} \tag{3.2}$$

On the other hand, from the equality $\int_0^\infty r^{-q}\psi_\alpha(r)\,dr = \frac{\Gamma(1+\frac{q}{\alpha})}{\Gamma(1+q)}$ for all $q \in [0,1]$ (see [59], Lemma 3.2) we have

$$\int_0^\infty r^\beta\phi_\alpha(r)\,dr = \int_0^\infty \frac{1}{r^{\beta\alpha}}\psi_\alpha(r)\,dr = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}. \tag{3.3}$$

Then, by (3.2) and (3.3) it is easy to see that

$$\left\|\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, u_{\varrho(s,u_s)})\right\|_{\mathcal{H}}^2 \le \left(\frac{\alpha\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)(t-s)^{1-\alpha\beta}}\right)^2\left\|\mathscr{A}^{\beta}\mathscr{G}(s, u_{\varrho(s,u_s)})\right\|_{\mathcal{H}}^2. \tag{3.4}$$

It is obvious that the function $s \to \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s, u_{\varrho(s,u_s)})$ is integrable on $[0,t)$ for every $t > 0$.

It is evident that the fixed points of the operator Υ are mild solutions of model (1.1)-(1.3). We define the function $x(\cdot):(-\infty,T]\to\mathcal{H}$ by

$$x(t)=\begin{cases}\varphi(t), & t\leq 0,\\ \mathcal{T}_\alpha(t)\varphi(0), & t\in\mathscr{I}.\end{cases}$$

Then $x_0=\varphi$. For every function $z\in C(\mathscr{I},\mathcal{R}^+)$ with $z(0)=0$, we define the function $\tilde{z}$ by

$$\tilde{z}(t)=\begin{cases}0, & t\leq 0,\\ z(t), & t\in\mathscr{I}.\end{cases}$$

If u fulfills (2.14), then we are able to split it as $u(t)=z(t)+x(t)$, $t\in\mathscr{I}$, which suggests to take $u_t=z_t+x_t$ for $t\in\mathscr{I}$, and the function z satisfies

$$\begin{aligned}
z(t)={}&\mathcal{T}_\alpha(t)\big[-h(z_t+x_t)-\mathscr{G}(0,\varphi)\big]+\mathscr{G}(t,z_{\varrho(t,z_t+x_t)}+x_{\varrho(t,z_t+x_t)})\\
&+\int_0^t\mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)})\,ds\\
&+\int_0^t\mathcal{S}_\alpha(t-s)\mathscr{F}\left(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right)ds\\
&+\int_0^t\mathcal{S}_\alpha(t-s)\Sigma\Bigg(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\\
&\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\Bigg)dw(s)\\
&+\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(z(t_k^-)+x(t_k^-)),\quad t\in\mathscr{I}.
\end{aligned}$$

Let $\mathscr{B}_T^0=\{z\in\mathscr{B}_T:z_0=0\in\mathcal{B}\}$. Let $\|\cdot\|_{\mathscr{B}_T^0}$ be the seminorm in $\mathscr{B}_T^0$ defined by

$$\|z\|_{\mathscr{B}_T^0}=\sup_{s\in\mathscr{I}}\big(\mathbb{E}\big\|z(s)\big\|^2\big)^{\frac{1}{2}}+\|z_0\|_{\mathcal{B}}=\sup_{s\in\mathscr{I}}\big(\mathbb{E}\big\|z(s)\big\|^2\big)^{\frac{1}{2}},\quad z\in\mathscr{B}_T^0.$$

As a result, $(\mathscr{B}_T^0,\|\cdot\|_{\mathscr{B}_T^0})$ is a Banach space. Set $B_r=\{z\in\mathscr{B}_T^0:\|z\|^2\leq r\}$ for some $r\geq 0$; then, for each $r, B_r\subset\mathscr{B}_T^0$ clearly is a bounded closed convex set. For $z\in B_r$, by Lemma 2.2 and by the above discussion we get

$$\begin{aligned}
&\mathbb{E}\|z_{\varrho(t,z_t+x_t)}+x_{\varrho(t,z_t+x_t)}\|_{\mathcal{B}}^2\\
&\quad\leq 2\big(\mathbb{E}\|z_{\varrho(t,z_t+x_t)}\|_{\mathcal{B}}^2+\mathbb{E}\|x_{\varrho(t,z_t+x_t)}\|_{\mathcal{B}}^2\big)\\
&\quad\leq 4\Bigg(\mathscr{E}_1^{*2}\sup_{\substack{0\leq s\leq\max(0,t)\\ t\in\mathcal{R}(\varrho^-)\cup\mathscr{I}}}\mathbb{E}\big\|z(s)\big\|^2+\big(\mathscr{E}_2^*+J^\varphi\big)^2\mathbb{E}\|z_0\|_{\mathcal{B}}^2+\mathscr{E}_1^{*2}\sup_{\substack{0\leq s\leq\max(0,t)\\ t\in\mathcal{R}(\varrho^-)\cup\mathscr{I}}}\mathbb{E}\big\|x(s)\big\|^2\\
&\qquad+\big(\mathscr{E}_2^*+J^\varphi\big)^2\mathbb{E}\|x_0\|_{\mathcal{B}}^2\Bigg)\\
&\quad\leq 4\big(\mathscr{E}_1^{*2}r+\mathscr{E}_1^{*2}\mathbb{E}\big\|\mathcal{T}_\alpha(t)\varphi(0)\big\|^2+\big(\mathscr{E}_2^*+J^\varphi\big)^2\mathbb{E}\|x_0\|_{\mathcal{B}}^2\big)\\
&\quad\leq 4\left(\mathscr{E}_1^{*2}r+\mathscr{E}_1^{*2}\left\|\int_0^\infty\phi_\alpha(r)\mathbb{T}(t^\alpha r)\,dr\right\|^2\mathbb{E}\big\|\varphi(0)\big\|_{\mathcal{H}}^2+\big(\mathscr{E}_2^*+J^\varphi\big)^2\mathbb{E}\|x_0\|_{\mathcal{B}}^2\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq 4\mathscr{E}_1^{*2}\big(r + \mathcal{M}^2\mathbb{E}\big\|\varphi(0)\big\|_{\mathcal{H}}^2\big) + 4\big(\mathscr{E}_2^{*} + J^{\varphi}\big)^2\mathbb{E}\|\varphi\|_{\mathcal{B}}^2 \\
&\leq 4\mathscr{E}_1^{*2}r + c_n = r^*,
\end{aligned}
\tag{3.5}
$$

where $c_n = 4[\mathscr{E}_1^{*2}\mathcal{M}^2\mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2 + (\mathscr{E}_2^{*} + J^{\varphi})^2\mathbb{E}\|\varphi\|_{\mathcal{B}}^2]$, and

$$
\begin{aligned}
\mathbb{E}\|z_t + x_t\|_{\mathcal{B}}^2 &\leq 2\big(\mathbb{E}\|z_t\|_{\mathcal{B}}^2 + \mathbb{E}\|x_t\|_{\mathcal{B}}^2\big) \\
&\leq 4\Big(\mathscr{E}_2^{*2}\mathbb{E}\|z_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\sup_{s\in\mathscr{I}}\mathbb{E}\big\|z(s)\big\|^2 + \mathscr{E}_2^{*2}\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\sup_{s\in\mathscr{I}}\mathbb{E}\big\|x(s)\big\|^2\Big) \\
&\leq 4\big(\mathscr{E}_1^{*2}r + \mathscr{E}_2^{*2}\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\mathbb{E}\big\|\mathcal{T}_\alpha(t)\varphi(0)\big\|^2\big) \\
&\leq 4\left(\mathscr{E}_1^{*2}r + \mathscr{E}_2^{*2}\mathbb{E}\|x_0\|_{\mathcal{B}}^2 + \mathscr{E}_1^{*2}\left\|\int_0^\infty \phi_\alpha(r)\mathbb{T}\big(t^\alpha r\big)\,dr\right\|^2\mathbb{E}\big\|\varphi(0)\big\|_{\mathcal{H}}^2\right) \\
&\leq 4\big(\mathscr{E}_1^{*2}\big(r + \mathcal{M}^2\mathbb{E}\big\|\varphi(0)\big\|_{\mathcal{H}}^2\big) + \mathscr{E}_2^{*2}\mathbb{E}\|\varphi\|_{\mathcal{B}}^2\big) \\
&\leq 4\mathscr{E}_1^{*2}r + \widetilde{c}_n = \widetilde{r},
\end{aligned}
\tag{3.6}
$$

where $\widetilde{c}_n = 4[\mathscr{E}_1^{*2}\mathcal{M}^2\mathbb{E}\|\varphi(0)\|_{\mathcal{H}}^2 + \mathscr{E}_2^{*2}\mathbb{E}\|\varphi\|_{\mathcal{B}}^2]$. We define the operator $\overline{\Upsilon}:\mathscr{B}_T^0 \to \mathscr{B}_T^0$ by

$$
\begin{aligned}
(\overline{\Upsilon}z)(t) &= \mathcal{T}_\alpha(t)\big[-h(z_t + x_t) - \mathscr{G}(0,\varphi)\big] + \mathscr{G}\big(t, z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}\big) \\
&\quad + \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\big(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\big)\,ds \\
&\quad + \int_0^t \mathcal{S}_\alpha(t-s) \\
&\quad \times \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau\right)ds \\
&\quad + \int_0^t \mathcal{S}_\alpha(t-s) \\
&\quad \times \Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau\right)dw(s) \\
&\quad + \sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k\big(z(t_k^-) + x(t_k^-)\big), \quad t \in \mathscr{I}.
\end{aligned}
$$

We see that the operator Υ has a fixed point if and only if $\overline{\Upsilon}$ has a fixed point. Thus, let us demonstrate that $\overline{\Upsilon}$ has a fixed point.

Now, for $t \in \mathscr{I}$, we split $\overline{\Upsilon}$ as $\overline{\Upsilon}_1 + \overline{\Upsilon}_2$, where

$$
\begin{aligned}
\overline{\Upsilon}_1 z(t) &= \mathscr{G}\big(t, z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}\big) \\
&\quad + \int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\big(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}\big)\,ds, \quad t \in \mathscr{I},
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\Upsilon}_2 z(t) &= \mathcal{T}_\alpha(t)[-h(z_t + x_t) - \mathscr{G}(0,\varphi)] \\
&\quad + \int_0^t \mathcal{S}_\alpha(t-s)
\end{aligned}
$$

$$
\begin{aligned}
&\times \mathscr{F}(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau)\,ds \\
&+ \int_0^t \mathcal{S}_\alpha(t-s) \\
&\times \Sigma(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau)\,dw(s) \\
&+ \sum_{0<t_k<t} \mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(z(t_k^-) + x(t_k^-)), \quad t \in \mathscr{I}.
\end{aligned}
$$

The proof of the theorem is lengthy and technical. Therefore, it is practical to split it into several steps.

Step 1: $\overline{\Upsilon}(B_r) \subset B_r$ for some $r > 0$.

We assume that there exists a positive integer r such that $\overline{\Upsilon}(B_r) \subset B_r$. If this were not true, then for each positive number r, we could find a function $z^r(\cdot) \in B_r$ such that $\overline{\Upsilon}(z^r) \notin B_r$, that is, $\mathbb{E}\|\overline{\Upsilon}(z^r)(t)\|^2 > r$ for some $t \in \mathscr{I}$. Then

$$
\begin{aligned}
r &\le \mathbb{E}\big\|\overline{\Upsilon}\big(z^r\big)(t)\big\|^2 \\
&\le 7\mathbb{E}\big\|\mathcal{T}_\alpha(t)\big[-h\big(z_t^r + x_t\big) - \mathscr{G}(0,\varphi)\big]\big\|^2 + 7\mathbb{E}\big\|\mathscr{G}\big(t, z^r_{\varrho(t,z_t^r+x_t)} + x_{\varrho(t,z_t^r+x_t)}\big)\big\|^2 \\
&\quad + 7\mathbb{E}\bigg\|\int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\big(s, z^r_{\varrho(s,z_s^r+x_s)} + x_{\varrho(s,z_s^r+x_s)}\big)\,ds\bigg\|^2 \\
&\quad + 7\mathbb{E}\bigg\|\int_0^t \mathcal{S}_\alpha(t-s) \\
&\quad \times \mathscr{F}\bigg(s, z^r_{\varrho(s,z_s^r+x_s)} + x_{\varrho(s,z_s^r+x_s)}, \int_0^s e_1\big(s,\tau, z^r_{\varrho(\tau,z_\tau^r+x_\tau)} + x_{\varrho(\tau,z_\tau^r+x_\tau)}\big)\,d\tau\bigg)\,ds\bigg\|^2 \\
&\quad + 7\mathbb{E}\bigg\|\int_0^t \mathcal{S}_\alpha(t-s) \\
&\quad \times \Sigma\bigg(s, z^r_{\varrho(s,z_s^r+x_s)} + x_{\varrho(s,z_s^r+x_s)}, \int_0^s e_2\big(s,\tau, z^r_{\varrho(\tau,z_\tau^r+x_\tau)} + x_{\varrho(\tau,z_\tau^r+x_\tau)}\big)\,d\tau\bigg)\,dw(s)\bigg\|^2 \\
&\quad + 7\mathbb{E}\bigg\|\sum_{0<t_k<t} \mathcal{T}_\alpha(t-t_k)\mathcal{I}_k\big(z^r\big(t_k^-\big) + x\big(t_k^-\big)\big)\bigg\|^2 \\
&= \sum_{i=1}^6 J_i.
\end{aligned}
\tag{3.7}
$$

Using (3.4), (3.5), (3.6), (H1)-(H6), and Hölder's inequality, we get:

$$
\begin{aligned}
J_1 &= 7\mathbb{E}\big\|\mathcal{T}_\alpha(t)\big[-h\big(z_t^r + x_t\big) - \mathscr{G}(0,\varphi)\big]\big\|^2 \\
&\le 7\big\|\mathcal{T}_\alpha(t)\big\|^2\big[\mathbb{E}\big\|h\big(z_t^r + x_t\big)\big\|^2_{\mathcal{B}} + \mathbb{E}\big\|\mathscr{G}(0,\varphi)\big\|^2\big] \\
&\le \bigg\|\int_0^\infty \phi_\alpha(r)\mathbb{T}\big(t^\alpha r\big)\,dr\bigg\|^2\big[\mathbb{E}\big\|h\big(z_t^r + x_t\big)\big\|^2_{\mathcal{B}} + \mathbb{E}\big\|\mathscr{G}(0,\varphi)\big\|^2\big] \\
&\le 7\mathcal{M}^2\big[\mathcal{M}_h\big\|z_t^r + x_t\big\|^2_{\mathcal{B}} + \big\|\mathscr{A}^{-\beta}\big\|^2\mathbb{E}\big\|\mathscr{A}^\beta\mathscr{G}(0,\varphi)\big\|^2\big] \\
&\le 7\mathcal{M}^2\big[\mathcal{M}_h\big(4\mathscr{E}_1^{*2}r + \widetilde{c}_n\big) + \mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\big(1 + \|\varphi\|^2_{\mathcal{B}}\big)\big] \\
&\le 28\mathcal{M}^2\mathcal{M}_h\mathscr{E}_1^{*2}r + C_1,
\end{aligned}
$$

where $\mathcal{N}_0 = \|A^{-\beta}\|$ and $C_1 = 7\mathcal{M}^2\mathcal{M}_h\widetilde{c}_n + 7\mathcal{M}^2\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1 + \|\varphi\|_{\mathcal{B}}^2)$;

$$\begin{aligned}
J_2 &= 7\mathbb{E}\bigl\|\mathscr{G}\bigl(t, z^r_{\varrho(t,z^r_t+x_t)} + x_{\varrho(t,z^r_t+x_t)}\bigr)\bigr\|^2 \\
&\leq 7\bigl\|\mathscr{A}^{-\beta}\bigr\|^2\mathbb{E}\bigl\|\mathscr{A}^{\beta}\mathscr{G}\bigl(t, z^r_{\varrho(t,z^r_t+x_t)} + x_{\varrho(t,z^r_t+x_t)}\bigr)\bigr\|^2 \\
&\leq 7\bigl\|\mathscr{A}^{-\beta}\bigr\|^2\mathcal{M}_{\mathscr{G}}\bigl(1 + \bigl\|z^r_{\varrho(t,z^r_t+x_t)} + x_{\varrho(t,z^r_t+x_t)}\bigr\|^2_{\mathcal{B}}\bigr) \\
&\leq 28\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}r + C_2,
\end{aligned}$$

where $C_2 = 7\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1 + c_n)$;

$$\begin{aligned}
J_3 &= 7\mathbb{E}\biggl\|\int_0^t \mathscr{A}\mathcal{S}_\alpha(t-s)\mathscr{G}\bigl(s, z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}\bigr)\,ds\biggr\|^2 \\
&\leq 7\biggl\|\int_0^t \biggl\{\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathscr{A}^{1-\beta}\mathbb{T}\bigl((t-s)^\alpha r\bigr)\,dr\biggr\}\,ds\biggr\|^2 \\
&\quad\times \mathbb{E}\bigl\|\mathscr{A}^{\beta}\mathscr{G}\bigl(s, z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}\bigr)\bigr\|^2 \\
&\leq 7\mathcal{M}_{\mathscr{G}}\biggl(\frac{\alpha\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\biggr)^2\int_0^t (t-s)^{\alpha\beta-1}\,ds \\
&\quad\times \int_0^t (t-s)^{\alpha\beta-1}\bigl(1 + \bigl\|z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}\bigr\|^2_{\mathcal{B}}\bigr)\,ds \\
&\leq 7\mathcal{M}_{\mathscr{G}}\biggl(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\biggr)^2\bigl(1 + 4\mathscr{E}_1^{*2}r + c_n\bigr) \\
&\leq 28\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}r\biggl(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\biggr)^2 + C_3,
\end{aligned}$$

where $C_3 = 7\mathcal{M}_{\mathscr{G}}\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2(1 + c_n)$;

$$\begin{aligned}
J_4 &= 7\mathbb{E}\biggl\|\int_0^t \mathcal{S}_\alpha(t-s) \\
&\quad\times \mathscr{F}\biggl(s, z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}, \int_0^s e_1\bigl(s,\tau, z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)}\bigr)\,d\tau\biggr)\,ds\biggr\|^2 \\
&\leq 7\biggl\|\int_0^t \biggl\{\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathbb{T}\bigl((t-s)^\alpha r\bigr)\,dr\biggr\}\,ds\biggr\|^2 m_{\mathscr{F}}(s) \\
&\quad\times \Psi_{\mathscr{F}}\biggl(\bigl\|z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}\bigr\|^2_{\mathcal{B}} \\
&\quad+ \int_0^s \mathbb{E}\bigl\|e_1\bigl(s,\tau, z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)}\bigr)\bigr\|^2\,d\tau\biggr) \\
&\leq 7\alpha^2\bigl\|\mathbb{T}(t-s)^\alpha r\bigr\|^2\biggl\|\int_0^\infty r\phi_\alpha(r)\,dr\biggr\|^2\int_0^t (t-s)^{\alpha-1}\,ds\int_0^t (t-s)^{\alpha-1} \\
&\quad\times m_{\mathscr{F}}(s)\Psi_{\mathscr{F}}\biggl(\bigl\|z^r_{\varrho(s,z^r_s+x_s)} + x_{\varrho(s,z^r_s+x_s)}\bigr\|^2_{\mathcal{B}} \\
&\quad+ \int_0^s \mathbb{E}\bigl\|e_1\bigl(s,\tau, z^r_{\varrho(\tau,z^r_\tau+x_\tau)} + x_{\varrho(\tau,z^r_\tau+x_\tau)}\bigr)\bigr\|^2\,d\tau\biggr)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq 7\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\frac{T^\alpha}{\alpha}\int_0^t (t-s)^{\alpha-1}m_{\mathscr{F}}(s)\Psi_{\mathscr{F}}\left(\left\|z^r_{\varrho(s,z^r_s+x_s)}+x_{\varrho(s,z^r_s+x_s)}\right\|^2_{\mathcal{B}}\right.\\
&\quad\left.+\int_0^s \mathbb{E}\left\|e_1\left(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)}+x_{\varrho(\tau,z^r_\tau+x_\tau)}\right)\right\|^2 d\tau\right)ds\\
&\leq 7\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\Psi_{\mathscr{F}}\left[4\mathscr{E}_1^{*2}r(1+\widetilde{\mathcal{M}}_0T)+(1+\widetilde{\mathcal{M}}_0T)c_n+\widetilde{\mathcal{M}}_0T\right]\sup_{s\in\mathscr{I}}m_{\mathscr{F}}(s);
\end{aligned}
$$

$$
\begin{aligned}
J_5 &= 7\mathbb{E}\left\|\int_0^t \mathcal{S}_\alpha(t-s)\right.\\
&\quad\left.\times\Sigma\left(s,z^r_{\varrho(s,z^r_s+x_s)}+x_{\varrho(s,z^r_s+x_s)},\int_0^s e_2\left(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)}+x_{\varrho(\tau,z^r_\tau+x_\tau)}\right)d\tau\right)dw(s)\right\|^2\\
&\leq 7\left\|\int_0^t\left\{\alpha\int_0^\infty r\phi_\alpha(r)(t-s)^{\alpha-1}\mathbb{T}\left((t-s)^\alpha r\right)dr\right\}ds\right\|^2 tr(Q)\\
&\quad\times\mathbb{E}\left\|\Sigma\left(s,z^r_{\varrho(s,z^r_s+x_s)}+x_{\varrho(s,z^r_s+x_s)},\int_0^s e_2\left(s,\tau,z^r_{\varrho(\tau,z^r_\tau+x_\tau)}+x_{\varrho(\tau,z^r_\tau+x_\tau)}\right)d\tau\right)\right\|^2\\
&\leq 7\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2 tr(Q)\Psi_\Sigma\left[4\mathscr{E}_1^{*2}r(1+\widetilde{\mathcal{M}}_1T)+(1+\widetilde{\mathcal{M}}_1T)c_n+\widetilde{\mathcal{M}}_1T\right]\sup_{s\in\mathscr{I}}m_\Sigma(s);
\end{aligned}
$$

$$
\begin{aligned}
J_6 &= 7n\sum_{k=1}^n\left\|\mathcal{T}_\alpha(t-t_k)\right\|^2\mathbb{E}\left\|\mathcal{I}_k\left(z^r\left(t_k^-\right)+x\left(t_k^-\right)\right)\right\|^2\\
&\leq 7n\sum_{k=1}^n\left\|\int_0^\infty\mathbb{T}\left((t-t_k)^\alpha r\right)\phi_\alpha(r)\,dr\right\|^2\mathbb{E}\left\|\mathcal{I}_k\left(z^r\left(t_k^-\right)+x\left(t_k^-\right)\right)\right\|^2\\
&\leq 7\mathcal{M}^2n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}\mathbb{E}\left\|\left(z^r\left(t_k^-\right)+x\left(t_k^-\right)\right)\right\|^2\\
&\leq 7\mathcal{M}^2n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}\left(\sup_{t\in\mathscr{I}}\mathbb{E}\left\|z^r(t)+x(t)\right\|^2\right)\\
&\leq 7\mathcal{M}^2H^2n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}\mathbb{E}\left\|z^r_t+x_t\right\|^2_{\mathcal{B}}\\
&\leq 7\mathcal{M}^2H^2n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}\left[4\mathscr{E}_1^{*2}r+\widetilde{c}_n\right]\\
&\leq 28\mathcal{M}^2H^2\mathscr{E}_1^{*2}n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}r+C_4,
\end{aligned}
$$

where $C_4 = 7\mathcal{M}^2H^2\widetilde{c}_n n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}$. Combining the estimates (J_1)-(J_6) together with (3.7), we obtain

$$
\begin{aligned}
r &\leq \mathbb{E}\left\|\overline{\Upsilon}\left(z^r\right)(t)\right\|^2\\
&\leq 28\mathcal{M}^2\mathcal{M}_h\mathscr{E}_1^{*2}r+C_1+28\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}r+C_2\\
&\quad+28\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}r\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2+C_3\\
&\quad+7\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\Psi_{\mathscr{F}}\left[4\mathscr{E}_1^{*2}r(1+\widetilde{\mathcal{M}}_0T)+(1+\widetilde{\mathcal{M}}_0T)c_n+\widetilde{\mathcal{M}}_0T\right]\sup_{s\in\mathscr{I}}m_{\mathscr{F}}(s)
\end{aligned}
$$

$$+ 7\left(\frac{\mathcal{M}T^{\alpha}}{\Gamma(1+\alpha)}\right)^2 tr(Q)\Psi_{\Sigma}\big[4\mathscr{E}_1^{*2}r(1+\widetilde{\mathcal{M}}_1T) + (1+\widetilde{\mathcal{M}}_1T)c_n + \widetilde{\mathcal{M}}_1T\big]\sup_{s\in\mathscr{I}} m_{\Sigma}(s)$$
$$+ 28\mathcal{M}^2H^2\mathscr{E}_1^{*2}n\sum_{k=1}^{n}\mathcal{M}_{\mathcal{I}_k}r + C_4,$$

where C_1-C_4 are independent of r. Dividing both sides by r and taking the limit as $r \to \infty$, we get

$$28\mathcal{M}^2\mathscr{E}_1^{*2}\left[\mathcal{M}_h + H^2n\sum_{k=1}^{n}\gamma_k\right] + 28\mathscr{E}_1^{*2}\left[\mathcal{M}_{\mathscr{G}}\left({\mathcal{N}_0}^2 + \left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)T^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right)^2\right)\right.$$
$$\left. + \left(\frac{\mathcal{M}T^{\alpha}}{\Gamma(1+\alpha)}\right)^2\left(\sup_{s\in\mathscr{I}} m_{\mathscr{F}}(s)(1+\widetilde{\mathcal{M}}_0T)\Lambda + tr(Q)\sup_{s\in\mathscr{I}} m_{\Sigma}(s)(1+\widetilde{\mathcal{M}}_1T)\widetilde{\Lambda}\right)\right] \geq 1,$$

which is a contradiction to (3.1). For this reason, for some positive number r, we have $\overline{\Upsilon}(B_r) \subset B_r$.

Step 2: $\overline{\Upsilon}_1$ is contraction.

For $z, z^* \in B_r$, we have

$$\begin{aligned}
&\mathbb{E}\big\|(\overline{\Upsilon}_1z)(t) - (\overline{\Upsilon}_1z^*)(t)\big\|^2 \\
&\quad= \mathbb{E}\bigg\|\mathscr{G}(t, z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}) - \mathscr{G}\big(t, z^*_{\varrho(t,z_t^*+x_t)} + x_{\varrho(t,z_t^*+x_t)}\big) \\
&\qquad + \int_0^t \mathscr{A}\mathcal{S}_{\alpha}(t-s)\mathscr{G}(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)})\,ds \\
&\qquad - \int_0^t \mathscr{A}\mathcal{S}_{\alpha}(t-s)\mathscr{G}\big(s, z^*_{\varrho(s,z_s^*+x_s)} + x_{\varrho(s,z_s^*+x_s)}\big)\,ds\bigg\|^2 \\
&\quad\leq 2\mathbb{E}\big\|\mathscr{G}(t, z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}) - \mathscr{G}\big(t, z^*_{\varrho(t,z_t^*+x_t)} + x_{\varrho(t,z_t^*+x_t)}\big)\big\|^2 \\
&\qquad + 2\mathbb{E}\bigg\|\int_0^t \mathscr{A}\mathcal{S}_{\alpha}(t-s)\big[\mathscr{G}(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}) - \mathscr{G}\big(s, z^*_{\varrho(s,z_s^*+x_s)} + x_{\varrho(s,z_s^*+x_s)}\big)\big]\,ds\bigg\|^2 \\
&\quad= J_7 + J_8.
\end{aligned} \tag{3.8}$$

From (3.4), Lemma 2.2, and (H1) we get

$$\begin{aligned}
J_7 &= 2\mathbb{E}\big\|\mathscr{G}(t, z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}) - \mathscr{G}\big(t, z^*_{\varrho(t,z_t^*+x_t)} + x_{\varrho(t,z_t^*+x_t)}\big)\big\|^2 \\
&\leq 2\big\|\mathscr{A}^{-\beta}\big\|^2\mathbb{E}\big\|\mathscr{A}^{\beta}\mathscr{G}(t, z_{\varrho(t,z_t+x_t)} + x_{\varrho(t,z_t+x_t)}) - \mathscr{A}^{\beta}\mathscr{G}\big(t, z^*_{\varrho(t,z_t^*+x_t)} + x_{\varrho(t,z_t^*+x_t)}\big)\big\|^2 \\
&\leq 2\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\mathbb{E}\big\|z_{\varrho(t,z_t+x_t)} - z^*_{\varrho(t,z_t^*+x_t)}\big\|_{\mathcal{B}}^2 \leq 2\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}\sup_{s\in\mathscr{I}}\mathbb{E}\big\|z(s) - z^*(s)\big\|^2, \\
J_8 &= 2\mathbb{E}\bigg\|\int_0^t \mathscr{A}\mathcal{S}_{\alpha}(t-s)\big[\mathscr{G}(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}) - \mathscr{G}\big(s, z^*_{\varrho(s,z_s^*+x_s)} + x_{\varrho(s,z_s^*+x_s)}\big)\big]\,ds\bigg\|^2 \\
&\leq 2\left(\frac{\alpha\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\right)^2\int_0^t (t-s)^{\alpha\beta-1}\,ds\int_0^t (t-s)^{\alpha\beta-1}\mathcal{M}_{\mathscr{G}}\mathbb{E}\big\|z_{\varrho(t,z_t+x_t)} - z^*_{\varrho(t,z_t^*+x_t)}\big\|_{\mathcal{B}}^2\,ds \\
&\leq 2\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}\sup_{s\in\mathscr{I}}\mathbb{E}\big\|z(s) - z^*(s)\big\|^2.
\end{aligned}$$

By combining estimates (J_7)-(J_8) along with (3.8), we get

$$\begin{aligned}
&\mathbb{E}\|(\overline{\Upsilon}_1 z)(t)-(\overline{\Upsilon}_1 z^*)(t)\|^2 \\
&\quad\le 2\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}\left[\mathcal{N}_0^2+\left(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta}\right)^2\right]\sup_{s\in\mathscr{I}}\mathbb{E}\|z(s)-z^*(s)\|^2 \\
&\quad\le \mathcal{N}_1\sup_{s\in\mathscr{I}}\mathbb{E}\|z(s)-z^*(s)\|^2,
\end{aligned}$$

where $\mathcal{N}_1 = 2\mathcal{M}_{\mathscr{G}}\mathscr{E}_1^{*2}[\mathcal{N}_0^2+(\frac{\mathcal{C}_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\frac{T^{\alpha\beta}}{\beta})^2]<1$. Hence, $\overline{\Upsilon}_1$ is a contraction.

Step 3: $\overline{\Upsilon}_2$ maps bounded sets into bounded sets in B_r.

It is sufficient to show that there exists a positive constant $\overline{\Lambda}_1$ such that, for each $z\in B_r=\{z\in\mathscr{B}_T^0:\|z\|^2\le r\}$, we have $\|\overline{\Upsilon}_2 z\|^2\le\overline{\Lambda}_1$. Now, for $t\in\mathscr{I}$,

$$\begin{aligned}
&\mathbb{E}\|(\overline{\Upsilon}_2 z)(t)\|^2 \\
&\quad\le 5\mathbb{E}\|\mathcal{T}_\alpha(t)[-h(z_t+x_t)-\mathscr{G}(0,\varphi)]\|^2 \\
&\qquad+5\mathbb{E}\left\|\int_0^t \mathcal{S}_\alpha(t-s)\right. \\
&\qquad\left.\times\mathscr{F}\left(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right)ds\right\|^2 \\
&\qquad+5\mathbb{E}\left\|\int_0^t \mathcal{S}_\alpha(t-s)\right. \\
&\qquad\left.\times\Sigma\left(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right)dw(s)\right\|^2 \\
&\qquad+5\mathbb{E}\left\|\sum_{0<t_k<t}\mathcal{T}_\alpha(t-t_k)\mathcal{I}_k(z(t_k^-)+x(t_k^-))\right\|^2 \\
&\quad=\sum_{i=9}^{12}J_i.
\end{aligned}\tag{3.9}$$

From (3.4), (3.5), (3.6), (H1)-(H6), and Hölder's inequality we get:

$$\begin{aligned}
J_9 &= 5\mathbb{E}\|\mathcal{T}_\alpha(t)[-h(z_t+x_t)-\mathscr{G}(0,\varphi)]\|^2 \\
&\le\left\|\int_0^\infty\phi_\alpha(r)\mathbb{T}(t^\alpha r)\,dr\right\|^2\left[\mathbb{E}\|h(z_t+x_t)\|_{\mathcal{B}}^2+\|\mathscr{A}^{-\beta}\|^2\mathbb{E}\|\mathscr{A}^\beta\mathscr{G}(0,\varphi)\|^2\right] \\
&\le 5\mathcal{M}^2\left[\mathbb{E}\|h(z_t+x_t)\|_{\mathcal{B}}^2+\|\mathscr{A}^{-\beta}\|^2\mathbb{E}\|\mathscr{A}^\beta\mathscr{G}(0,\varphi)\|^2\right] \\
&\le 5\mathcal{M}^2\left[\mathcal{M}_h\widetilde{r}+\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}(1+\|\varphi\|_{\mathcal{B}}^2)\right], \\
J_{10} &= 5\mathbb{E}\left\|\int_0^t\mathcal{S}_\alpha(t-s)\right. \\
&\qquad\left.\times\mathscr{F}\left(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right)ds\right\|^2 \\
&\le 5\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\frac{T^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}m_{\mathscr{F}}(s)\Psi_{\mathscr{F}}
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\|z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\|_{\mathcal{B}}^2+\int_0^s\mathbb{E}\left\|e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})\right\|^2d\tau\right)ds\\
&\le 5\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\Psi_{\mathscr{F}}\left(r^*+\widetilde{\mathcal{M}}_0\left(1+r^*\right)T\right)\sup_{s\in\mathscr{I}}m_{\mathscr{F}}(s),\\
J_{11}&=5\mathbb{E}\bigg\|\int_0^t\mathcal{S}_\alpha(t-s)\\
&\quad\times\Sigma\left(s,z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)},\int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right)dw(s)\bigg\|^2\\
&\le 5\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\frac{T^\alpha}{\alpha}\int_0^t(t-s)^{\alpha-1}tr(Q)m_\Sigma(s)\Psi_\Sigma\bigg(\|z_{\varrho(s,z_s+x_s)}+x_{\varrho(s,z_s+x_s)}\|_{\mathcal{B}}^2\\
&\quad+\int_0^s\mathbb{E}\left\|e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)}+x_{\varrho(\tau,z_\tau+x_\tau)})\right\|^2d\tau\bigg)ds\\
&\le 5\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2tr(Q)\Psi_\Sigma\left(r^*+\widetilde{\mathcal{M}}_1\left(1+r^*\right)T\right)\sup_{s\in\mathscr{I}}m_\Sigma(s),\\
J_{12}&=5n\left\|\mathcal{T}_\alpha(t-t_k)\right\|^2\sum_{k=1}^n\left(\mathbb{E}\left\|\mathcal{I}_k\left(z\left(t_k^-\right)+x\left(t_k^-\right)\right)\right\|^2\right)\\
&\le 5\mathcal{M}^2n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}\left(\mathbb{E}\left\|\left(z\left(t_k^-\right)+x\left(t_k^-\right)\right)\right\|^2\right)\\
&\le 5\mathcal{M}^2H^2n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}\mathbb{E}\|z_t+x_t\|_{\mathcal{B}}^2\\
&\le 5\mathcal{M}^2H^2n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}\widetilde{r}.
\end{aligned}
$$

Substituting the estimates (J_9)-(J_{12}) into (3.9), we get

$$
\begin{aligned}
\mathbb{E}\left\|(\overline{\Upsilon}_2z)(t)\right\|^2&\le 5\mathcal{M}^2\left[\mathcal{M}_h\widetilde{r}+\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\left(1+\|\varphi\|_{\mathcal{B}}^2\right)+H^2n\sum_{k=1}^n\mathcal{M}_{\mathcal{I}_k}\widetilde{r}\right]\\
&\quad+5\left(\frac{\mathcal{M}T^\alpha}{\Gamma(1+\alpha)}\right)^2\Big[\sup_{s\in\mathscr{I}}m_{\mathscr{F}}(s)\Psi_{\mathscr{F}}\left(r^*+\widetilde{\mathcal{M}}_0\left(1+r^*\right)T\right)\\
&\quad+tr(Q)\sup_{s\in\mathscr{I}}m_\Sigma(s)\Psi_\Sigma\left(r^*+\widetilde{\mathcal{M}}_1\left(1+r^*\right)T\right)\Big]\\
&=\overline{\Lambda}_1.
\end{aligned}
$$

Then, for all $z\in B_r$, we have $\|\overline{\Upsilon}_2z\|^2\le\overline{\Lambda}_1$. Hence, $\overline{\Upsilon}_2$ maps bounded sets to bounded sets in B_r.

Step 4: $\overline{\Upsilon}_2$ maps bounded sets into equicontinuous sets of B_r. Let $0<\epsilon<t<T$ and $\delta>0$ be such that $\|\mathbb{T}(t_1^\alpha)-\mathbb{T}(t_2^\alpha)\|<\epsilon$ for any $z\in B_r$ and $0\le t_1\le t_2\le T$. Then we have

$$
\begin{aligned}
&\mathbb{E}\left\|(\overline{\Upsilon}_2z)(t_2)-(\overline{\Upsilon}_2z)(t_1)\right\|^2\\
&\quad\le 9\mathbb{E}\left\|\left[\mathbb{T}\left(t_2^\alpha r\right)-\mathbb{T}\left(t_1^\alpha r\right)\right]\left[-h(z_t+x_t)-\mathscr{G}(0,\varphi)\right]\right\|^2\\
&\quad\quad+9\mathbb{E}\bigg\|\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)\int_0^{t_1}(t_1-s)^{\alpha-1}\left[\mathbb{T}\left((t_2-s)^\alpha r\right)-\mathbb{T}\left((t_1-s)^\alpha r\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\times \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) ds\bigg\|^2 \\
&+ 9\mathbb{E}\bigg\|\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\mathbb{T}\big((t_2-s)^\alpha r\big) \\
&\times \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) ds\bigg\|^2 \\
&+ 9\mathbb{E}\bigg\|\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{T}\big((t_2-s)^\alpha r\big) \\
&\times \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) ds\bigg\|^2 \\
&+ 9\mathbb{E}\bigg\|\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)\int_0^{t_1}(t_1-s)^{\alpha-1}\big[\mathbb{T}\big((t_2-s)^\alpha r\big)-\mathbb{T}\big((t_1-s)^\alpha r\big)\big] \\
&\times \Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) dw(s)\bigg\|^2 \\
&+ 9\mathbb{E}\bigg\|\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\mathbb{T}\big((t_2-s)^\alpha r\big) \\
&\times \Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) dw(s)\bigg\|^2 \\
&+ 9\mathbb{E}\bigg\|\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\mathbb{T}\big((t_2-s)^\alpha r\big) \\
&\times \Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) dw(s)\bigg\|^2 \\
&+ 9\mathbb{E}\bigg\|\sum_{0<t_k<t_1}\big[\mathbb{T}\big((t_2-t_k)^\alpha r\big)-\mathbb{T}\big((t_1-t_k)^\alpha r\big)\big]\mathbb{E}\big[\mathcal{I}_k\big(z(t_k^-)+x(t_k^-)\big)\big]\bigg\|^2 \\
&+ 9\mathbb{E}\bigg\|\sum_{t_1<t_k<t_2}\mathbb{T}\big((t_2-t_k)^\alpha r\big)\mathbb{E}\big[\mathcal{I}_k\big(z(t_k^-)+x(t_k^-)\big)\big]\bigg\|^2 \\
=&\sum_{i=13}^{21} J_i.
\end{aligned}
\tag{3.10}
$$

From (3.4), (3.5), (3.6), (H1)-(H6) and Hölder's inequality we get:

$$
\begin{aligned}
J_{13} &= 9\mathbb{E}\big\|\big[\mathbb{T}(t_2^\alpha r)-\mathbb{T}(t_1^\alpha r)\big]\big[-h(z_t+x_t)-\mathscr{G}(0,\varphi)\big]\big\|^2 \\
&\le 18\big\|\mathbb{T}(t_2^\alpha r)-\mathbb{T}(t_1^\alpha r)\big\|^2\big[\mathbb{E}\big\|h(z_t+x_t)\big\|_{\mathcal{B}}^2+\mathbb{E}\big\|\mathscr{G}(0,\varphi)\big\|^2\big] \\
&\le 18\epsilon^2\big[\mathcal{M}_h\|z_t+x_t\|_{\mathcal{B}}^2+\big\|\mathscr{A}^{-\beta}\big\|^2\mathbb{E}\big\|\mathscr{A}^\beta\mathscr{G}(0,\varphi)\big\|^2\big] \\
&\le 18\epsilon^2\big[\mathcal{M}_h\widetilde{r}+\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\big(1+\|\varphi\|_{\mathcal{B}}^2\big)\big], \\
J_{14} &= 9\mathbb{E}\bigg\|\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)\int_0^{t_1}(t_1-s)^{\alpha-1}\big[\mathbb{T}\big((t_2-s)^\alpha r\big)-\mathbb{T}\big((t_1-s)^\alpha r\big)\big] \\
&\quad\times \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) ds\bigg\|^2 \\
&\le 9\left(\frac{\alpha}{\Gamma(1+\alpha)}\right)^2\big\|\mathbb{T}\big((t_2-s)^\alpha r\big)-\mathbb{T}\big((t_1-s)^\alpha r\big)\big\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \int_o^{t_1} (t_1 - s)^{\alpha-1}\,ds \int_0^{t_1} (t_1 - s)^{\alpha-1} \\
&\quad \times \mathbb{E}\left\| \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau \right) \right\|^2 ds \\
&\leq 9\left(\frac{\alpha\epsilon}{\Gamma(1+\alpha)} \right)^2 \frac{t_1^\alpha}{\alpha} \int_0^{t_1} (t_1 - s)^{\alpha-1} m_{\mathscr{F}}(s) \Psi_{\mathscr{F}}\big(r^* + \widetilde{\mathcal{M}}_0\big(1 + r^*\big)T\big)\,ds, \\
J_{15} &= 9\mathbb{E}\left\| \left(\frac{\alpha}{\Gamma(1+\alpha)} \right) \int_0^{t_1} \big[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\big] \mathbb{T}\big((t_2 - s)^\alpha r\big) \right. \\
&\quad \left. \times \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau \right) ds \right\|^2 \\
&\leq 9\left(\frac{\alpha}{\Gamma(1+\alpha)} \right)^2 \big\| \mathbb{T}\big((t_2 - s)^\alpha r\big) \big\|^2 \\
&\quad \times \int_0^{t_1} \big[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\big]\,ds \int_0^{t_1} \big[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\big] \\
&\quad \times \mathbb{E}\left\| \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau \right) \right\|^2 ds \\
&\leq 9\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)} \right)^2 \int_0^{t_1} \big[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\big]\,ds \int_0^{t_1} \big[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\big] \\
&\quad \times m_{\mathscr{F}}(s) \Psi_{\mathscr{F}}\big(r^* + \widetilde{\mathcal{M}}_0\big(1 + r^*\big)T\big)\,ds, \\
J_{16} &= 9\mathbb{E}\left\| \left(\frac{\alpha}{\Gamma(1+\alpha)} \right) \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathbb{T}\big((t_2 - s)^\alpha r\big) \right. \\
&\quad \left. \times \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau \right) ds \right\| \\
&\leq 9\left(\frac{\alpha}{\Gamma(1+\alpha)} \right)^2 \big\| \mathbb{T}\big((t_2 - s)^\alpha r\big) \big\|^2 \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\,ds \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \\
&\quad \times \mathbb{E}\left\| \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau \right) \right\|^2 ds \\
&\leq 9\left(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)} \right)^2 \frac{(t_2 - t_1)^\alpha}{\alpha} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} m_{\mathscr{F}}(s) \Psi_{\mathscr{F}}\big(r^* + \widetilde{\mathcal{M}}_0\big(1 + r^*\big)T\big)\,ds, \\
J_{17} &= 9\mathbb{E}\left\| \left(\frac{\alpha}{\Gamma(1+\alpha)} \right) \int_0^{t_1} (t_1 - s)^{\alpha-1} \big[\mathbb{T}\big((t_2 - s)^\alpha r\big) - \mathbb{T}\big((t_1 - s)^\alpha r\big)\big] \right. \\
&\quad \left. \times \Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau \right) dw(s) \right\|^2 \\
&\leq 9\left(\frac{\alpha}{\Gamma(1+\alpha)} \right)^2 \big\| \mathbb{T}\big((t_2 - s)^\alpha r\big) - \mathbb{T}\big((t_1 - s)^\alpha r\big) \big\|^2 \\
&\quad \times \int_o^{t_1} (t_1 - s)^{\alpha-1}\,ds \int_0^{t_1} (t_1 - s)^{\alpha-1} tr(Q) \\
&\quad \times \mathbb{E}\left\| \Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau \right) \right\|^2 ds \\
&\leq 9\left(\frac{\alpha\epsilon}{\Gamma(1+\alpha)} \right)^2 \frac{t_1^\alpha}{\alpha} \int_0^{t_1} (t_1 - s)^{\alpha-1} tr(Q) m_\Sigma(s) \Psi_\Sigma\big(r^* + \widetilde{\mathcal{M}}_1\big(1 + r^*\big)T\big)\,ds,
\end{aligned}
$$

$$
\begin{aligned}
J_{18} &= 9\mathbb{E}\bigg\| \bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg) \int_0^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big] \mathbb{T}\big((t_2-s)^{\alpha} r\big) \\
&\quad \times \Sigma\bigg(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\, d\tau\bigg)\, dw(s)\bigg\|^2 \\
&\le 9\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)^2 \big\|\mathbb{T}\big((t_2-s)^{\alpha} r\big)\big\|^2 \\
&\quad \times \int_0^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big]\, ds \int_0^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big] tr(Q) \\
&\quad \times \mathbb{E}\bigg\|\Sigma\bigg(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\, d\tau\bigg)\bigg\|^2\, ds \\
&\le 9\bigg(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\bigg)^2 \int_0^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big]\, ds \\
&\quad \times \int_0^{t_1} \big[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\big] tr(Q) m_\Sigma(s) \Psi_\Sigma\big(r^* + \widetilde{\mathcal{M}}_1\big(1+r^*\big)T\big)\, ds, \\
J_{19} &= 9\mathbb{E}\bigg\| \bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg) \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \mathbb{T}\big((t_2-s)^{\alpha} r\big) \\
&\quad \times \Sigma\bigg(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\, d\tau\bigg)\, dw(s)\bigg\|^2 \\
&\le 9\bigg(\frac{\alpha}{\Gamma(1+\alpha)}\bigg)^2 \big\|\mathbb{T}\big((t_2-s)^{\alpha} r\big)\big\|^2 \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\, ds \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} tr(Q) \\
&\quad \times \mathbb{E}\bigg\|\Sigma\bigg(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2\big(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\, d\tau\bigg)\bigg\|^2\, ds \\
&\le 9\bigg(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\bigg)^2 \frac{(t_2-t_1)^{\alpha}}{\alpha} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} tr(Q) m_\Sigma(s) \Psi_\Sigma\big(r^* + \widetilde{\mathcal{M}}_1\big(1+r^*\big)T\big)\, ds, \\
J_{20} &= 9\mathbb{E}\bigg\| \sum_{0<t_k<t_1} \big[\mathbb{T}\big((t_2-t_k)^{\alpha} r\big) - \mathbb{T}\big((t_1-t_k)^{\alpha} r\big)\big] \mathcal{I}_k\big(z\big(t_k^-\big) + x\big(t_k^-\big)\big)\bigg\|^2 \\
&\le 9\big\|\mathbb{T}\big((t_2-t_k)^{\alpha} r\big) - \mathbb{T}\big((t_1-t_k)^{\alpha} r\big)\big\|^2 \mathbb{E}\bigg\| \sum_{0<t_k<t_1} \mathcal{I}_k\big(z\big(t_k^-\big) + x\big(t_k^-\big)\big)\bigg\|^2 \\
&\le 9\epsilon^2 n \sum_{k=1}^n \mathbb{E}\big\|\mathcal{I}_k\big(z\big(t_k^-\big) + x\big(t_k^-\big)\big)\big\|^2 \\
&\le 9\epsilon^2 n \sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k} \mathbb{E}\big\|\big(z\big(t_k^-\big) + x\big(t_k^-\big)\big)\big\|^2 \\
&\le 9\epsilon^2 n \sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k} \Big(\sup_{t\in\mathscr{I}} \mathbb{E}\big\|z(t) + x(t)\big\|^2\Big) \\
&\le 9\epsilon^2 n \sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k} H^2 \|z_t + x_t\|_{\mathcal{B}}^2 \\
&\le 9\epsilon^2 H^2 n \sum_{k=1}^n \mathcal{M}_{\mathcal{I}_k} \widetilde{r},
\end{aligned}
$$

$$\begin{aligned}
J_{21} &= 9\mathbb{E}\Big\| \sum_{t_1<t_k<t_2} \mathbb{T}\big((t_2-t_k)^\alpha r\big)\mathcal{I}_k\big(z\big(t_k^-\big)+x\big(t_k^-\big)\big)\Big\|^2 \\
&\leq 9\big\|\mathbb{T}\big((t_2-t_k)^\alpha r\big)\big\|^2\mathbb{E}\Big\| \sum_{t_1<t_k<t_2} \mathcal{I}_k\big(z\big(t_k^-\big)+x\big(t_k^-\big)\big)\Big\|^2 \\
&\leq 9\mathcal{M}^2H^2 n\sum_{k=1}^{n}\mathcal{M}_{\mathcal{I}_k}\widetilde{r}.
\end{aligned}$$

Combining estimates (J_{13})-(J_{21}) together with (3.10), we get

$$\begin{aligned}
&\mathbb{E}\big\|(\overline{\Upsilon}_2 z)(t_2)-(\overline{\Upsilon}_2 z)(t_1)\big\|^2 \\
&\quad\leq 18\epsilon^2\big[\mathcal{M}_h\widetilde{r}+\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\big(1+\|\varphi\|_{\mathcal{B}}^2\big)\big]+9\bigg(\frac{\alpha\epsilon}{\Gamma(1+\alpha)}\bigg)^2\frac{t_1^\alpha}{\alpha} \\
&\qquad\times\int_0^{t_1}(t_1-s)^{\alpha-1}m_{\mathscr{F}}(s)\Psi_{\mathscr{F}}\big(r^*+\widetilde{\mathcal{M}}_0\big(1+r^*\big)T\big)\,ds \\
&\qquad+9\bigg(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\bigg)^2\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\,ds\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big] \\
&\qquad\times m_{\mathscr{F}}(s)\Psi_{\mathscr{F}}\big(r^*+\widetilde{\mathcal{M}}_0\big(1+r^*\big)T\big)\,ds \\
&\qquad+9\bigg(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\bigg)^2\frac{(t_2-t_1)^\alpha}{\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}m_{\mathscr{F}}(s)\Psi_{\mathscr{F}}\big(r^*+\widetilde{\mathcal{M}}_0\big(1+r^*\big)T\big)\,ds \\
&\qquad+9\bigg(\frac{\alpha\epsilon}{\Gamma(1+\alpha)}\bigg)^2\frac{t_1^\alpha}{\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}tr(Q)m_{\Sigma}(s)\Psi_{\Sigma}\big(r^*+\widetilde{\mathcal{M}}_1\big(1+r^*\big)T\big)\,ds \\
&\qquad+9\bigg(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\bigg)^2\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\,ds \\
&\qquad\times\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]tr(Q)m_{\Sigma}(s)\Psi_{\Sigma}\big(r^*+\widetilde{\mathcal{M}}_1\big(1+r^*\big)T\big)\,ds \\
&\qquad+9\bigg(\frac{\alpha\mathcal{M}}{\Gamma(1+\alpha)}\bigg)^2\frac{(t_2-t_1)^\alpha}{\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}tr(Q)m_{\Sigma}(s)\Psi_{\Sigma}\big(r^*+\widetilde{\mathcal{M}}_1\big(1+r^*\big)T\big)\,ds \\
&\qquad+9\epsilon^2H^2n\sum_{k=1}^{n}\mathcal{M}_{\mathcal{I}_k}\widetilde{r}+9\mathcal{M}^2H^2n\sum_{k=1}^{n}\mathcal{M}_{\mathcal{I}_k}\widetilde{r}.
\end{aligned}$$

Therefore, for ϵ sufficiently small, the right-hand side of the above inequality tends to zero as $t_1\to t_2$. By [1], Lemma 2.2, the compactness of $\mathcal{T}_\alpha(t)$ signifies the continuity in the uniform operator topology on $\mathscr{I}$. As a result, the set $\mathcal{V}(t)=\{\overline{\Upsilon}_2 z: z\in B_r\}$ is equicontinuous.

Step 5: $\overline{\Upsilon}_2$ maps B_r into a precompact set in $\mathcal{H}$.

Now, we shall prove that $\mathcal{V}(t)=\{(\overline{\Upsilon}_2 z)(t): z\in B_r\}$ is relatively compact in $\mathcal{H}$. Obviously, $\mathcal{V}(t)$ is relatively compact in $\mathscr{B}_T^0$ for $t=0$. Let $0<t\leq T$ be fixed, and let ϵ be a real number such that $0<\epsilon<t$. For $\delta>0$ and $z\in B_r$, define the operator $\overline{\Upsilon}_2^{\epsilon,\delta}$ on B_r by

$$\begin{aligned}
&\big(\overline{\Upsilon}_2^{\epsilon,\delta}z\big)(t) \\
&\quad=\int_\delta^\infty\phi_\alpha(r)\mathbb{T}\big(t^\alpha r\big)\big[-h(z_t+x_t)-\mathscr{G}(0,\varphi)\big]\,dr
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha \int_0^{t-\epsilon} \int_\delta^\infty r(t-s)^{\alpha-1} \phi_\alpha(r) \mathbb{T}\bigl((t-s)^\alpha r\bigr) \\
&\times \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) dr\,ds \\
&+ \alpha \int_0^{t-\epsilon} \int_\delta^\infty r(t-s)^{\alpha-1} \phi_\alpha(r) \mathbb{T}\bigl((t-s)^\alpha r\bigr) \\
&\times \Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) dr\,dw(s) \\
&+ \sum_{0<t_k<t} \int_\delta^\infty \phi_\alpha(r) \mathbb{T}\bigl((t-t_k)^\alpha r\bigr) \mathcal{I}_k\bigl(z\bigl(t_k^-\bigr) + x\bigl(t_k^-\bigr)\bigr)\,dr \\
\le{}& \mathbb{T}\bigl(\epsilon^\alpha \delta\bigr) \int_\delta^\infty \phi_\alpha(r) \mathbb{T}\bigl(t^\alpha r - \epsilon^\alpha \delta\bigr)\bigl[-h(z_t + x_t) - \mathscr{G}(0,\varphi)\bigr]\,dr \\
&+ \alpha \mathbb{T}\bigl(\epsilon^\alpha \delta\bigr) \int_0^{t-\epsilon} \int_\delta^\infty r(t-s)^{\alpha-1} \phi_\alpha(r) \mathbb{T}\bigl((t-s)^\alpha r - \bigl(\epsilon^\alpha \delta\bigr)\bigr) \\
&\times \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) dr\,ds \\
&+ \alpha \mathbb{T}\bigl(\epsilon^\alpha \delta\bigr) \int_0^{t-\epsilon} \int_\delta^\infty r(t-s)^{\alpha-1} \phi_\alpha(r) \mathbb{T}\bigl((t-s)^\alpha r - \bigl(\epsilon^\alpha \delta\bigr)\bigr) \\
&\times \Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) dr\,dw(s) \\
&+ \mathbb{T}\bigl(\epsilon^\alpha \delta\bigr) \sum_{0<t_k<t} \int_\delta^\infty \phi_\alpha(r) \mathbb{T}\bigl((t-t_k)^\alpha r - \bigl(\epsilon^\alpha \delta\bigr)\bigr) \mathcal{I}_k\bigl(z\bigl(t_k^-\bigr) + x\bigl(t_k^-\bigr)\bigr)\,dr.
\end{aligned}
$$

Then from the compactness of $\mathbb{T}(\epsilon^\alpha \delta)$ $(\epsilon^\alpha \delta > 0)$ it follows that the set $\mathcal{V}^{\epsilon,\delta}(t) = \{(\overline{\Upsilon}_2^{\epsilon,\delta} z)(t) : z \in B_r\}$ is relatively compact in $\mathcal{H}$ for all $\epsilon > 0$ and $\delta > 0$. Also, for every $z \in B_r$, we have

$$
\begin{aligned}
&\mathbb{E}\bigl\|(\overline{\Upsilon}_2 z)(t) - \bigl(\overline{\Upsilon}_2^{\epsilon,\delta} z\bigr)(t)\bigr\|^2 \\
&\quad\le 7\mathbb{E}\left\| \int_0^\delta \phi_\alpha(r) \mathbb{T}\bigl(t^\alpha r\bigr)\bigl[-h(z_t + x_t) - \mathscr{G}(0,\varphi)\bigr]\,dr \right\|^2 \\
&\qquad+ 7\alpha^2 \mathbb{E}\left\| \int_0^t \int_0^\delta r(t-s)^{\alpha-1} \phi_\alpha(r) \mathbb{T}\bigl((t-s)^\alpha r\bigr) \right. \\
&\qquad\times \left. \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) dr\,ds \right\|^2 \\
&\qquad+ 7\alpha^2 \mathbb{E}\left\| \int_{t-\epsilon}^t \int_\delta^\infty r(t-s)^{\alpha-1} \phi_\alpha(r) \mathbb{T}\bigl((t-s)^\alpha r\bigr) \right. \\
&\qquad\times \left. \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) dr\,ds \right\|^2 \\
&\qquad+ 7\alpha^2 \mathbb{E}\left\| \int_0^t \int_0^\delta r(t-s)^{\alpha-1} \phi_\alpha(r) \mathbb{T}\bigl((t-s)^\alpha r\bigr) \right. \\
&\qquad\times \left. \Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2(s,\tau, z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)})\,d\tau\right) dr\,dw(s) \right\|^2 \\
&\qquad+ 7\alpha^2 \mathbb{E}\left\| \int_{t-\epsilon}^t \int_\delta^\infty r(t-s)^{\alpha-1} \phi_\alpha(r) \mathbb{T}\bigl((t-s)^\alpha r\bigr) \right.
\end{aligned}
$$

$$\begin{aligned}
&\times \Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2\big(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau\right) dr\,dw(s)\Bigg\|^2 \\
&+ 7\mathbb{E}\Bigg\| \sum_{0<t_k<t} \int_0^\delta \phi_\alpha(r)\mathbb{T}\big((t-t_k)^\alpha r\big)\mathcal{I}_k\big(z(t_k^-) + x(t_k^-)\big)\,dr\Bigg\|^2 \\
= {} & \sum_{i=22}^{27} J_i.
\end{aligned} \tag{3.11}$$

From (3.4), (3.5), (3.6), (H1)-(H6), and Hölder's inequality we get:

$$\begin{aligned}
J_{22} &= 7\mathbb{E}\Bigg\| \int_0^\delta \phi_\alpha(r)\mathbb{T}\big(t^\alpha r\big)\big[-h(z_t+x_t) - \mathscr{G}(0,\varphi)\big]\,dr\Bigg\|^2 \\
&\leq 7\big\|\mathbb{T}\big(t^\alpha r\big)\big\|^2\big[\mathbb{E}\big\|h(z_t+x_t)\big\|^2 + \mathbb{E}\big\|\mathscr{G}(0,\varphi)\big\|^2\big]\left(\int_0^\delta \phi_\alpha(r)\,dr\right)^2 \\
&\leq 7\mathcal{M}^2\big[\mathcal{M}_h\widetilde{r} + \mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\big(1 + \|\varphi\|_{\mathcal{B}}^2\big)\big]\left(\int_0^\delta \phi_\alpha(r)\,dr\right)^2, \\
J_{23} &= 7\alpha^2\mathbb{E}\Bigg\| \int_0^t \int_0^\delta r(t-s)^{\alpha-1}\phi_\alpha(r)\mathbb{T}\big((t-s)^\alpha r\big) \\
&\quad \times \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1\big(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau\right) dr\,ds\Bigg\|^2 \\
&\leq 7\alpha T^\alpha\mathcal{M}^2 \int_0^t (t-s)^{\alpha-1} m_{\mathscr{F}}(s)\Psi_{\mathscr{F}}\big(r^* + \widetilde{\mathcal{M}}_0\big(1+r^*\big)T\big)\,ds\left(\int_0^\delta r\phi_\alpha(r)\,dr\right)^2, \\
J_{24} &= 7\alpha^2\mathbb{E}\Bigg\| \int_{t-\epsilon}^t \int_\delta^\infty r(t-s)^{\alpha-1}\phi_\alpha(r)\mathbb{T}\big((t-s)^\alpha r\big) \\
&\quad \times \mathscr{F}\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_1\big(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau\right) dr\,ds\Bigg\|^2 \\
&\leq 7\alpha\left(\frac{\mathcal{M}}{\Gamma(1+\alpha)}\right)^2 \epsilon^\alpha \int_{t-\epsilon}^t (t-s)^{\alpha-1} m_{\mathscr{F}}(s)\Psi_{\mathscr{F}}\big(r^* + \widetilde{\mathcal{M}}_0\big(1+r^*\big)T\big)\,ds, \\
J_{25} &= 7\alpha^2\mathbb{E}\Bigg\| \int_0^t \int_0^\delta r(t-s)^{\alpha-1}\phi_\alpha(r)\mathbb{T}\big((t-s)^\alpha r\big) \\
&\quad \times \Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2\big(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau\right) dr\,dw(s)\Bigg\|^2 \\
&\leq 7\alpha T^\alpha\mathcal{M}^2 \int_0^t (t-s)^{\alpha-1} tr(Q) m_\Sigma(s)\Psi_\Sigma\big(r^* + \widetilde{\mathcal{M}}_1\big(1+r^*\big)T\big)\,ds\left(\int_0^\delta r\phi_\alpha(r)\,dr\right)^2, \\
J_{26} &= 7\alpha^2\mathbb{E}\Bigg\| \int_{t-\epsilon}^t \int_\delta^\infty r(t-s)^{\alpha-1}\phi_\alpha(r)\mathbb{T}\big((t-s)^\alpha r\big) \\
&\quad \times \Sigma\left(s, z_{\varrho(s,z_s+x_s)} + x_{\varrho(s,z_s+x_s)}, \int_0^s e_2\big(s,\tau,z_{\varrho(\tau,z_\tau+x_\tau)} + x_{\varrho(\tau,z_\tau+x_\tau)}\big)\,d\tau\right) dr\,dw(s)\Bigg\|^2 \\
&\leq 7\alpha\left(\frac{\mathcal{M}}{\Gamma(1+\alpha)}\right)^2 \epsilon^\alpha \int_{t-\epsilon}^t (t-s)^{\alpha-1} tr(Q) m_\Sigma(s)\Psi_\Sigma\big(r^* + \widetilde{\mathcal{M}}_1\big(1+r^*\big)T\big)\,ds, \\
J_{27} &= 7\mathbb{E}\Bigg\| \sum_{0<t_k<t} \int_0^\delta \phi_\alpha(r)\mathbb{T}\big((t-t_k)^\alpha r\big)\mathcal{I}_k\big(z(t_k^-) + x(t_k^-)\big)\,dr\Bigg\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq 7n\big\|\mathbb{T}\big((t-t_k)^\alpha r\big)\big\|^2\sum_{k=1}^{n}\mathbb{E}\big\|\mathcal{I}_k\big(z(t_k^-)+x(t_k^-)\big)\big\|^2\left(\int_0^\delta \phi_\alpha(r)\,dr\right)^2\\
&\leq 7n\mathcal{M}^2H^2\sum_{k=1}^{n}\mathcal{M}_{\mathcal{I}_k}\widetilde{r}\left(\int_0^\delta \phi_\alpha(r)\,dr\right)^2.
\end{aligned}$$

By substituting the estimations (J_{22})-(J_{27}) into (3.11), we get

$$\begin{aligned}
&\mathbb{E}\big\|(\overline{\Upsilon}_2 z)(t)-\big(\overline{\Upsilon}_2^{\epsilon,\delta}z\big)(t)\big\|^2\\
&\quad\leq 7\mathcal{M}^2\left[\mathcal{M}_h\overline{r}+\mathcal{N}_0^2\mathcal{M}_{\mathscr{G}}\big(1+\|\varphi\|_{\mathcal{B}}^2\big)+nH^2\sum_{k=1}^{n}\mathcal{M}_{\mathcal{I}_k}\widetilde{r}\right]\left(\int_0^\delta \phi_\alpha(r)\,dr\right)^2\\
&\qquad+7\alpha T^\alpha\mathcal{M}^2\int_0^t (t-s)^{\alpha-1}\big[m_{\mathscr{F}}(s)\Psi_{\mathscr{F}}\big(r^*+\widetilde{\mathcal{M}}_0\big(1+r^*\big)T\big)\\
&\qquad+tr(Q)m_\Sigma(s)\Psi_\Sigma\big(r^*+\widetilde{\mathcal{M}}_1\big(1+r^*\big)T\big)\big]\,ds\left(\int_0^\delta r\phi_\alpha(r)\,dr\right)^2\\
&\qquad+7\alpha\left(\frac{\mathcal{M}}{\Gamma(1+\alpha)}\right)^2\epsilon^\alpha\int_{t-\epsilon}^t (t-s)^{\alpha-1}\big[m_{\mathscr{F}}(s)\Psi_{\mathscr{F}}\big(r^*+\widetilde{\mathcal{M}}_0\big(1+r^*\big)T\big)\\
&\qquad+tr(Q)m_\Sigma(s)\Psi_\Sigma\big(r^*+\widetilde{\mathcal{M}}_1\big(1+r^*\big)T\big)\big]\,ds.
\end{aligned}$$

This implies that there are relatively compact sets arbitrarily close to the set $\mathcal{V}(t)$, $t>0$. As a result, $\mathcal{V}(t)=\{(\overline{\Upsilon}_2 z)(t):z\in B_r\}$ is also relatively compact in $\mathcal{H}$.

Therefore, by the Arzelà-Ascoli theorem the operator $\overline{\Upsilon}_2$ is completely continuous. Consequently, by the Krasnoselskii fixed point theorem [1], Lemma 3.3, we get that $\overline{\Upsilon}$ has at least one fixed point $z_1^*\in\mathscr{B}_T^0$. Let $u(t)=z_1^*(t)+x(t)$ on $(-\infty,T]$. Along these lines, u is a fixed point of the operator Υ, which is a mild solution of problem (1.1)-(1.3). The proof is now finished. □

4 Application

In this section, we provide an illustration of the existence results for an IFNSIDS with SDD of the form

$$\begin{aligned}
&D_t^\alpha\left[u(t,x)-\int_{-\infty}^t \mu_1(s-t)u\big(s-\varrho_1(t)\varrho_2\big(\|u(t)\|\big),x\big)\,ds\right]\\
&\quad=\frac{\partial^2}{\partial x^2}u(t,x)+\int_{-\infty}^t \mu_2(t,x,s-t)P_1\big(u\big(s-\varrho_1(t)\varrho_2\big(\|u(t)\|\big),x\big)\big)\,ds\\
&\qquad+\int_0^t\int_{-\infty}^s k_1(s-\tau)P_2\big(u\big(\tau-\varrho_1(\tau)\varrho_2\big(\|u(\tau)\|\big),x\big)\big)\,d\tau\,ds\\
&\qquad+\left[\int_{-\infty}^t \mu_3(t,x,s-t)Q_1\big(u\big(s-\varrho_1(t)\varrho_2\big(\|u(t)\|\big),x\big)\big)\,ds\right.\\
&\qquad\left.+\int_0^t\int_{-\infty}^s k_2(s-\tau)Q_2\big(u\big(\tau-\varrho_1(\tau)\varrho_2\big(\|u(\tau)\|\big),x\big)\big)\,d\tau\,ds\right]\frac{d\beta(t)}{dt},\\
&\qquad x\in[0,\pi],0\leq t\leq T,
\end{aligned}\tag{4.1}$$

$$u(t,0)=0=u(t,\pi),\quad t\geq 0,\tag{4.2}$$

$$u(0,x) + \int_0^{\pi} k_3(x,z)u(t,z)\,dz = \varphi(t,x), \quad t \in (-\infty,0], 0 \le x \le \pi, \tag{4.3}$$

$$\Delta u(t_k,x) = \int_{-\infty}^{t_k} \eta_k(s-t_k)u(s,x)\,ds, \quad k = 1,2,\dots,n, \tag{4.4}$$

where $\beta(t)$ is a standard cylindrical Wiener process in $\mathcal{H}$ defined on a stochastic space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathcal{P})$, D_t^{α} is Caputo's fractional derivative of order $0<\alpha<1$, φ is continuous, and $0 < t_1 < t_2 < \cdots < t_n < T$ are prefixed numbers. We consider $\mathcal{H} = \mathcal{K} = L^2[0,\pi]$ with norm $\|\cdot\|_{\mathcal{L}^2}$ and define the operator $\mathscr{A} : D(\mathscr{A}) \subset \mathcal{H} \to \mathcal{H}$ by $\mathscr{A}w = w''$ with the domain

$$D(\mathscr{A}) = \bigl\{w \in \mathcal{H} : w, w' \text{ are absolutely continuous}, w'' \in \mathcal{H}, w(0) = w(\pi) = 0\bigr\}.$$

Then

$$\mathscr{A}w = \sum_{n=1}^{\infty} n^2 \langle w, w_n\rangle w_n, \quad w \in D(\mathscr{A}),$$

where $w_n(s) = \sqrt{\frac{2}{\pi}}\sin(ns)$, $n = 1,2,\dots$, is an orthogonal set of eigenvectors of $\mathscr{A}$. It is well known that $\mathscr{A}$ is the infinitesimal generator of an analytic semigroup $\{\mathbb{T}(t)\}_{t\ge 0}$ in $\mathcal{H}$ and is provided by

$$\mathbb{T}(t)w = \sum_{n=1}^{\infty} e^{-n^2 t}\langle w, w_n\rangle w_n \quad \text{for all } w \in \mathcal{H} \text{ and } t > 0.$$

If we fix $\beta = \frac{1}{2}$, then the operator $(\mathscr{A})^{\frac{1}{2}}$ is given by

$$(\mathscr{A})^{\frac{1}{2}}w = \sum_{n=1}^{\infty} n\langle w, w_n\rangle w_n, \quad w \in \bigl(D(\mathscr{A})^{\frac{1}{2}}\bigr),$$

where $(D(\mathscr{A})^{\frac{1}{2}}) = \{\omega(\cdot) \in \mathcal{H} : \sum_{n=1}^{\infty} n\langle \omega, w_n\rangle w_n \in \mathcal{H}\}$ and $\|(\mathscr{A})^{-\frac{1}{2}}\| = 1$. For $\gamma < 0$, define the phase space

$$\mathcal{B} = \Bigl\{\varphi \in C\bigl((-\infty,0],\mathcal{H}\bigr) : \lim_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } \mathcal{H}\Bigr\},$$

and let $\|\varphi\|_{\mathcal{B}} = \sup_{\theta\in(-\infty,0]}\{e^{\gamma\theta}\|\varphi(\theta)\|_{\mathcal{L}^2}\}$. Then $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space satisfying (P_1)-(P_3) with $H = 1$, $\mathscr{E}_1(t) = \max\{1, e^{-\gamma t}\}$, $\mathscr{E}_2(t) = e^{-\gamma t}$. Therefore, for $(t,\varphi) \in [0,T] \times \mathcal{B}$, where $\varphi(\theta)(x) = \varphi(\theta,x)$, $(\theta,x) \in (-\infty,0] \times [0,\pi]$, setting

$$u(t)(x) = u(t,x), \qquad \varrho(t,\varphi) = \varrho_1(t)\varrho_2\bigl(\|\varphi(0)\|\bigr),$$

we have

$$\mathscr{G}(t,\varphi)(x) = \int_{-\infty}^{0} \mu_1(\theta)\varphi(\theta)(x)\,d\theta,$$

$$\mathscr{F}(t,\varphi,\mathscr{H}\varphi)(x) = \int_{-\infty}^{0} \mu_2(t,x,\theta)P_1\bigl(\varphi(\theta)(x)\bigr)\,d\theta + \mathscr{H}\varphi(x),$$

$$\Sigma(t,\varphi,\overline{\mathscr{H}}\varphi)(x) = \int_{-\infty}^{0} \mu_3(t,x,\theta)Q_1\big(\varphi(\theta)(x)\big)\,d\theta + \overline{\mathscr{H}}\varphi(x),$$

and

$$\mathcal{I}_k(\varphi)(x) = \int_{-\infty}^{0} \eta_k(\theta)\varphi(\theta)(x)\,d\theta, \quad k = 1,2,\ldots,n,$$

where

$$\mathscr{H}\varphi(x) = \int_0^t \int_{-\infty}^{0} k_1(s-\theta)P_2\big(\varphi(\theta)(x)\big)\,d\theta\,ds,$$
$$\overline{\mathscr{H}}\varphi(x) = \int_0^t \int_{-\infty}^{0} k_2(s-\theta)Q_2\big(\varphi(\theta)(x)\big)\,d\theta\,ds.$$

Then using these configurations, system (4.1)-(4.4) is usually written in the theoretical form of problem (1.1)-(1.3).

Suppose further that:

(i) the functions $\varrho_i : [0,\infty) \to [0,\infty)$, $i = 1,2$, are continuous;

(ii) the function $\mu_1(\theta) \geq 0$ is continuous in $(-\infty,0]$ and satisfies

$$\int_{-\infty}^{0} \mu_1^2(\theta)\,d\theta < \theta \quad \text{and} \quad \mathcal{M}_{\mathscr{G}} = \left(\frac{-1}{2\gamma}\int_{-\infty}^{0} \mu_1^2(\theta)\,d\theta\right)^2;$$

(iii) the functions $\mu_i(t,x,\theta) \geq 0$, $i = 2,3$, are continuous in $[0,T] \times [0,\pi] \times (-\infty,0]$ and satisfy

$$\int_{-\infty}^{0} \mu_2(t,x,\theta)\,d\theta = p_1(t,x) < \infty, \qquad \left(\int_0^{\pi} p_1^2(t,x)\,dx\right)^{\frac{1}{2}} = m_1(t) < \infty,$$
$$\int_{-\infty}^{0} \mu_3(t,x,\theta)\,d\theta = p_2(t,x) < \infty, \qquad \left(\int_0^{\pi} p_2^2(t,x)\,dx\right)^{\frac{1}{2}} = \overline{m}_1(t) < \infty;$$

(iv) the functions $k_i(t-s)$, $i = 1,2$, are continuous in $[0,\pi]$, $k_i(t-s) \geq 0$, and

$$\int_0^t \int_{-\infty}^{0} k_1(s-\theta)\,d\theta\,ds = q_1(t) < \infty, \qquad \left(\int_0^{\pi} q_1^2(t)\,dx\right)^{\frac{1}{2}} = \sqrt{\pi}m_2(t) < \infty,$$
$$\int_0^t \int_{-\infty}^{0} k_2(s-\theta)\,d\theta\,ds = q_2(t) < \infty, \qquad \left(\int_0^{\pi} q_2^2(t)\,dx\right)^{\frac{1}{2}} = \sqrt{\pi}\overline{m}_2(t) < \infty,$$

with $m_{\mathscr{F}}(t) = m_1(t) + \sqrt{\pi}m_2(t)$ and $m_\Sigma(t) = \overline{m}_1(t) + \sqrt{\pi}\overline{m}_2(t)$.

(v) the functions P_i, Q_i, $i = 1,2$, are continuous, and for all $(\theta,x) \in (-\infty,0] \times [0,\pi]$,

$$0 \leq P_i\big(u(\theta)(x)\big) \leq \Theta_{\mathscr{F}}\big(\|u(\theta,\cdot)\|_{\mathcal{L}^2}\big) \quad \text{with } \liminf_{r\to\infty} \frac{\Theta_{\mathscr{F}}(r)}{r} = \Lambda < \infty,$$
$$0 \leq Q_i\big(u(\theta)(x)\big) \leq \Theta_{\Sigma}\big(\|u(\theta,\cdot)\|_{\mathcal{L}^2}\big) \quad \text{with } \liminf_{r\to\infty} \frac{\Theta_{\Sigma}(r)}{r} = \widetilde{\Lambda} < \infty,$$

where $\Theta_{\mathscr{F}}, \Theta_\Sigma : [0,\infty) \to (0,\infty)$ are continuous and nondecreasing functions;

(vi) the functions $\eta_k \in C(\mathcal{R}^+,\mathcal{R}^+)$, $k = 1,2,\ldots,n$, are finite.

Thus, under all these conditions, we have

$$\begin{aligned}
&\big\|\mathscr{A}^{\frac{1}{2}}\mathscr{G}(t,\varphi_1)-\mathscr{A}^{\frac{1}{2}}\mathscr{G}(t,\varphi_2)\big\|_{\mathcal{L}^2}\\
&\quad\le\left[\int_0^{\pi}\left(\int_{-\infty}^{0}\mu_1(\theta)\big(\varphi_1(\theta)-\varphi_2(\theta)\big)(x)\,d\theta\right)^2dx\right]^{\frac{1}{2}}\\
&\quad\le\left(\int_{-\infty}^{0}\mu_1^2(\theta)\,d\theta\right)^{\frac{1}{2}}\left(\int_0^{\pi}\int_{-\infty}^{0}\big(\varphi_1(\theta)-\varphi_2(\theta)\big)^2(x)\,d\theta\,dx\right)^{\frac{1}{2}}\\
&\quad\le\left(\int_{-\infty}^{0}\mu_1^2(\theta)\,d\theta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{0}\int_0^{\pi}\big(\varphi_1(\theta)-\varphi_2(\theta)\big)^2(x)\,dx\,d\theta\right)^{\frac{1}{2}}\\
&\quad\le\left(\int_{-\infty}^{0}\mu_1^2(\theta)\,d\theta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{0}e^{-2\gamma\theta}\cdot e^{2\gamma\theta}\int_0^{\pi}\big(\varphi_1(\theta)-\varphi_2(\theta)\big)^2(x)\,dx\,d\theta\right)^{\frac{1}{2}}\\
&\quad\le\left(\int_{-\infty}^{0}\mu_1^2(\theta)\,d\theta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{0}e^{-2\gamma\theta}\,d\theta\right)^{\frac{1}{2}}\cdot\sup_{-\infty<\theta\le0}\left\{e^{\gamma\theta}\left(\int_0^{\pi}\big(\varphi_1(\theta)-\varphi_2(\theta)\big)^2(x)\,dx\right)^{\frac{1}{2}}\right\}\\
&\quad\le\left(\frac{-1}{2\gamma}\int_{-\infty}^{0}\mu_1^2(\theta)\,d\theta\right)^2\cdot\sup_{-\infty<\theta\le0}\left\{e^{\gamma\theta}\big\|\varphi_1(\theta)-\varphi_2(\theta)\big\|_{\mathcal{L}^2}\right\}\\
&\quad\le\mathcal{M}_{\mathscr{G}}\|\varphi_1-\varphi_2\|_{\mathcal{B}}.
\end{aligned}$$

Therefore, hypothesis (H1) holds. Similarly, we have

$$\begin{aligned}
&\big\|\mathscr{F}(t,\varphi,\mathscr{H}\varphi)\big\|_{\mathcal{L}^2}\\
&\quad=\left[\int_0^{\pi}\left\{\int_{-\infty}^{0}\mu_2(t,x,\theta)P_1\big(\varphi(\theta)(x)\big)\,d\theta+\int_0^t\int_{-\infty}^{0}k_1(s-\theta)P_2\big(\varphi(\theta)(x)\big)\,d\theta\,ds\right\}^2dx\right]^{\frac{1}{2}}\\
&\quad\le\left[\int_0^{\pi}\left\{\int_{-\infty}^{0}\mu_2(t,x,\theta)\Theta_{\mathscr{F}}\big(\big\|\varphi(\theta)(\cdot)\big\|_{\mathcal{L}^2}\big)\,d\theta\right.\right.\\
&\qquad\left.\left.+\int_0^t\int_{-\infty}^{0}k_1(s-\theta)\Theta_{\mathscr{F}}\big(\big\|\varphi(\theta)(\cdot)\big\|_{\mathcal{L}^2}\big)\,d\theta\,ds\right\}^2dx\right]^{\frac{1}{2}}\\
&\quad\le\left[\int_0^{\pi}\left\{\int_{-\infty}^{0}\mu_2(t,x,\theta)\Theta_{\mathscr{F}}\Big(\sup_{-\infty<\theta\le0}\big\{e^{\gamma\theta}\big\|\varphi(\theta)\big\|_{\mathcal{L}^2}\big\}\Big)\,d\theta\right.\right.\\
&\qquad\left.\left.+\int_0^t\int_{-\infty}^{0}k_1(s-\theta)\Theta_{\mathscr{F}}\Big(\sup_{-\infty<\theta\le0}\big\{e^{\gamma\theta}\big\|\varphi(\theta)\big\|_{\mathcal{L}^2}\big\}\Big)\,d\theta\,ds\right\}^2dx\right]^{\frac{1}{2}}\\
&\quad\le\left[\int_0^{\pi}\left\{\int_{-\infty}^{0}\mu_2(t,x,\theta)\,d\theta\right\}^2dx\right]^{\frac{1}{2}}\Theta_{\mathscr{F}}\big(\|\varphi\|_{\mathcal{B}}\big)\\
&\qquad+\left[\int_0^{\pi}\left\{\int_0^t\int_{-\infty}^{0}k_1(s-\theta)\,d\theta\,ds\right\}^2dx\right]^{\frac{1}{2}}\Theta_{\mathscr{F}}\big(\|\varphi\|_{\mathcal{B}}\big)\\
&\quad\le\left(\int_0^{\pi}p_1^2(t,x)\,dx\right)^{\frac{1}{2}}\Theta_{\mathscr{F}}\big(\|\varphi\|_{\mathcal{B}}\big)+\left(\int_0^{\pi}q_1^2(t)\,dx\right)^{\frac{1}{2}}\Theta_{\mathscr{F}}\big(\|\varphi\|_{\mathcal{B}}\big)\\
&\quad\le\big[m_1(t)+\sqrt{\pi}\,m_2(t)\big]\Theta_{\mathscr{F}}\big(\|\varphi\|_{\mathcal{B}}\big)\\
&\quad=m_{\mathscr{F}}(t)\Theta_{\mathscr{F}}\big(\|\varphi\|_{\mathcal{B}}\big).
\end{aligned}$$

Since $\Theta_{\mathscr{F}} : [0,\infty) \to (0,\infty)$ is a continuous and nondecreasing function, we can take $\Psi_{\mathscr{F}}(r) = \Theta_{\mathscr{F}}(r)$ in (H2).

In the same way, we have

$$\begin{aligned}
&\big\|\Sigma(t,\varphi,\overline{\mathscr{H}}\varphi)\big\|_{\mathcal{L}^2} \\
&= \bigg[\int_0^\pi \bigg\{\int_{-\infty}^0 \mu_3(t,x,\theta)Q_1\big(\varphi(\theta)(x)\big)\,d\theta + \int_0^t \int_{-\infty}^0 k_2(s-\theta)Q_2\big(\varphi(\theta)(x)\big)\,d\theta\,ds\bigg\}^2\,dx\bigg]^{\frac{1}{2}} \\
&\le \bigg[\int_0^\pi \bigg\{\int_{-\infty}^0 \mu_3(t,x,\theta)\Theta_\Sigma\big(\big\|\varphi(\theta)(\cdot)\big\|_{\mathcal{L}^2}\big)\,d\theta \\
&\quad + \int_0^t \int_{-\infty}^0 k_2(s-\theta)\Theta_\Sigma\big(\big\|\varphi(\theta)(\cdot)\big\|_{\mathcal{L}^2}\big)\,d\theta\,ds\bigg\}^2\,dx\bigg]^{\frac{1}{2}} \\
&\le \bigg[\int_0^\pi \bigg\{\int_{-\infty}^0 \mu_3(t,x,\theta)\Theta_\Sigma\Big(\sup_{-\infty<\theta\le 0}\big\{e^{\gamma\theta}\big\|\varphi(\theta)\big\|_{\mathcal{L}^2}\big\}\Big)\,d\theta \\
&\quad + \int_0^t \int_{-\infty}^0 k_2(s-\theta)\Theta_\Sigma\Big(\sup_{-\infty<\theta\le 0}\big\{e^{\gamma\theta}\big\|\varphi(\theta)\big\|_{\mathcal{L}^2}\big\}\Big)\,d\theta\,ds\bigg\}^2\,dx\bigg]^{\frac{1}{2}} \\
&\le \bigg[\int_0^\pi \bigg\{\int_{-\infty}^0 \mu_3(t,x,\theta)\,d\theta\bigg\}^2\,dx\bigg]^{\frac{1}{2}}\Theta_\Sigma\big(\|\varphi\|_{\mathcal{B}}\big) \\
&\quad + \bigg[\int_0^\pi \bigg\{\int_0^t \int_{-\infty}^0 k_2(s-\theta)\,d\theta\,ds\bigg\}^2\,dx\bigg]^{\frac{1}{2}}\Theta_\Sigma\big(\|\varphi\|_{\mathcal{B}}\big) \\
&\le \bigg(\int_0^\pi p_2^2(t,x)\,dx\bigg)^{\frac{1}{2}}\Theta_\Sigma\big(\|\varphi\|_{\mathcal{B}}\big) + \bigg(\int_0^\pi q_2^2(t)\,dx\bigg)^{\frac{1}{2}}\Theta_\Sigma\big(\|\varphi\|_{\mathcal{B}}\big) \\
&\le \big[\overline{m}_1(t) + \sqrt{\pi}\,\overline{m}_2(t)\big]\Theta_\Sigma\big(\|\varphi\|_{\mathcal{B}}\big) \\
&= m_\Sigma(t)\Theta_\Sigma\big(\|\varphi\|_{\mathcal{B}}\big).
\end{aligned}$$

Since $\Theta_\Sigma : [0,\infty) \to (0,\infty)$ is a continuous and nondecreasing function, we can take $\Psi_\Sigma(r) = \Theta_\Sigma(r)$ in (H4). Therefore, hypotheses (H1)-(H4) are satisfied. Furthermore, if (H5), (H6), and the bounds in (3.1) are satisfied, then system (4.1)-(4.4) has a mild solution on $\mathscr{I}$.

5 Conclusion

In this paper, we have studied the existence results for impulsive stochastic fractional neutral integro-differential systems with nonlocal and state-dependent delay conditions in a Hilbert space. More precisely, by utilizing the stochastic analysis theory, fractional powers of operators, and the Krasnoselskii fixed point theorem, we investigate the IFNSIDS with NLCs and SDD in a Hilbert space. To validate the obtained theoretical results, we analyze one example. The FDEs are very efficient to describe the real-life phenomena; thus, it is essential to extend the present study to establish the other qualitative and quantitative properties such as stability and controllability.

There are two direct issues that require further study. First, we will investigate the approximate controllability of fractional neutral stochastic integro-differential systems with state-dependent delay in the cases of a noncompact operator and a normal topological space. Second, we will study the approximate controllability of a new class of impulsive fractional stochastic differential equations with state-dependent delay and noninstantaneous impulses as discussed in [4].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have made equal contributions. All authors read and approved the final manuscript.

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References

1. Sakthivel, R, Ganesh, R, Suganya, S: Approximate controllability of fractional neutral stochastic system with infinite delay. Rep. Math. Phys. **70**(3), 291-311 (2012)
2. Pazy, A: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983)
3. Balasubramaniam, P, Kumaresan, N, Ratnavelu, K, Tamilalagan, P: Local and global existence of mild solution for impulsive fractional stochastic differential equations. Bull. Malays. Math. Soc. **38**(2), 867-884 (2015)
4. Gautam, G, Dabas, J: Mild solutions for class of neutral fractional functional differential equations with not instantaneous impulses. Appl. Math. Comput. **259**, 480-489 (2015)
5. Baleanu, D, Machado, JAT, Luo, ACJ: Fractional Dynamics and Control. Springer, New York (2012)
6. Diethelm, D: The Analysis of Fractional Differential Equations. Springer, Berlin (2010)
7. Tarasov, VE: Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer, Heidelberg (2010)
8. Yan, Z, Lu, F: Existence of an optimal control for fractional stochastic partial neutral integro-differential equations with infinite delay. J. Nonlinear Sci. Appl. **8**(5), 557-577 (2015)
9. Tariboon, J, Ntouyas, SK, Sudsutad, W: Coupled systems of Riemann-Liouville fractional differential equations with Hadamard fractional integral boundary conditions. J. Nonlinear Sci. Appl. **9**(1), 295-308 (2016)
10. Dhage, BC, Dhage, SB, Ntouyas, SK: Existence and approximate solutions for fractional differential equations with nonlocal conditions. J. Fract. Calc. Appl. **7**(1), 24-35 (2016)
11. Tan, J, Cheng, C: Existence of solutions to nonlinear fractional differential equations with boundary conditions on an infinite interval in Banach spaces. Bound. Value Probl. **2015**, Article ID 153 (2015)
12. Benchohra, M, Bouriah, S: Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order. Moroccan J. Pure Appl. Anal. **1**(1), 1-16 (2015)
13. Agarwal, RP, Andrade, DB, Siracusa, G: On fractional integro-differential equations with state-dependent delay. Comput. Math. Appl. **62**, 1143-1149 (2011)
14. Benchohra, M, Litimein, S, Trujillo, JJ, Velasco, MP: Abstract fractional integro-differential equations with state-dependent delay. Int. J. Evol. Equ. **6**(2), 25-38 (2012)
15. Aissani, K, Benchohra, M: Fractional integro-differential equations with state-dependent delay. Adv. Dyn. Syst. Appl. **9**(1), 17-30 (2014)
16. Kavitha, V, Wang, P-Z, Murugesu, R: Existence results for neutral functional fractional differential equations with state dependent-delay. Malaya J. Mat. **1**(1), 50-61 (2012)
17. Carvalho dos Santos, JP, Mallika Arjunan, M, Cuevas, C: Existence results for fractional neutral integrodifferential equations with state-dependent delay. Comput. Math. Appl. **62**, 1275-1283 (2011)
18. Carvalho dos Santos, JP, Cuevas, C, De Andrade, D: Existence results for a fractional equation with state-dependent delay. Adv. Differ. Equ. **2011**, Article ID 642013 (2011)
19. Sakthivel, R, Yong, R: Approximate controllability of fractional differential equations with state-dependent delay. Results Math. **63**(3-4), 949-963 (2013)
20. Vijayakumar, V, Ravichandran, C, Murugesu, R: Approximate controllability for a class of fractional neutral integro-differential inclusions with state-dependent delay. Nonlinear Stud. **20**(4), 513-532 (2013)
21. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
22. Stamova, IM: Stability Analysis of Impulsive Functional Differential Equations. de Gruyter, Berlin (2009)
23. Graef, JR, Henderson, J, Ouahab, A: Impulsive Differential Inclusions: A Fixed Point Approach. de Gruyter, Berlin (2013)
24. Bainov, D, Covachev, V: Impulsive Differential Equations with a Small Parameter. World Scientific, Singapore (1995)
25. Benchohra, M, Henderson, J, Ntouyas, SK: Impulsive Differential Equations and Inclusions. Contemporary Mathematics and Its Applications, vol. 2. Hindawi Publishing Corporation, New York (2006)
26. Chadha, A, Pandey, DN: Existence results for an impulsive neutral stochastic fractional integro-differential equation with infinite delay. Nonlinear Anal., Theory Methods Appl. **128**, 149-175 (2015)
27. Aissani, K, Benchohra, M: Impulsive fractional differential inclusions with infinite delay. Electron. J. Differ. Equ. **2013**, 265 (2013)
28. Bonanno, G, Rodriguez-Lopez, R, Tersian, S: Existence of solutions to boundary value problem for impulsive fractional differential equations. Fract. Calc. Appl. Anal. **17**(3), 717-744 (2014)

29. Rodriguez-Lopez, R, Tersian, S: Multiple solutions to boundary value problem for impulsive fractional differential equations. Fract. Calc. Appl. Anal. **17**(4), 1016-1038 (2014)
30. Benchohra, M, Berhoun, F: Impulsive fractional differential equations with state-dependent delay. Commun. Appl. Anal. **14**(2), 213-224 (2010)
31. Dabas, J, Gautam, G: Impulsive neutral fractional integro-differential equation with state-dependent delay and integral boundary condition. Electron. J. Differ. Equ. **2013**, 273 (2013)
32. Yan, Z, Yan, X: Existence of solutions for impulsive partial stochastic neutral integro-differential equations with state-dependent delay. Collect. Math. **64**, 235-250 (2013)
33. Byszewski, L: Theorems about existence and uniqueness of solutions of a semi-linear evolution nonlocal Cauchy problem. J. Math. Anal. Appl. **162**, 494-505 (1991)
34. Byszewski, L, Lakshmikantham, V: Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. Appl. Anal. **40**, 11-19 (1991)
35. Zang, Y, Li, J: Approximate controllability of fractional impulsive neutral stochastic differential equations with nonlocal conditions. Bound. Value Probl. **2013**, Article ID 193 (2013)
36. Balasubramaniam, P, Tamilalagan, P: Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi's function. Appl. Math. Comput. **256**, 232-246 (2015)
37. Guendouzi, T, Benzatout, O: Existence of mild solutions for impulsive fractional stochastic differential inclusions with state-dependent delay. Chin. J. Math. **2014**, Article ID 981714 (2014)
38. Yan, Z, Zhang, H: Existence of solutions to impulsive fractional partial neutral stochastic integro-differential inclusions with state-dependent delay. Electron. J. Differ. Equ. **2013**, 81 (2013)
39. Guendouzi, T, Bousmaha, L: Existence of solutions for fractional partial neutral stochastic functional integro-differential inclusions with state-dependent delay and analytic resolvent operators. Vietnam J. Math. **43**(4), 687-704 (2015)
40. Yan, Z, Jia, X: Approximate controllability of partial fractional neutral stochastic functional integro-differential inclusions with state-dependent delay. Collect. Math. **66**, 93-124 (2015)
41. Sakthivel, R, Ren, Y: Exponential stability of second-order stochastic evolution equations with Poisson jumps. Commun. Nonlinear Sci. Numer. Simul. **17**(12), 867-884 (2012)
42. Sakthivel, R, Revathi, P, Anthoni, SM: Existence of pseudo almost automorphic mild solutions to stochastic fractional differential equations. Nonlinear Anal., Theory Methods Appl. **75**, 3339-3347 (2012)
43. Guendouzi, T, Bousmaha, L: Approximate controllability of fractional neutral stochastic functional integro-differential inclusions with infinite delay. Qual. Theory Dyn. Syst. **13**, 89-119 (2014)
44. Zhang, X, Zhu, C, Yuan, C: Approximate controllability of impulsive fractional stochastic differential equations with state-dependent delay. Adv. Differ. Equ. **2015**, Article ID 91 (2015)
45. Yan, Z, Zhang, H: Asymptotic stability of fractional impulsive stochastic partial integro-differential equations with state-dependent delay. Electron. J. Differ. Equ. **2013**, 206 (2013)
46. Sakthivel, R, Revathi, P, Ren, Y: Existence of solutions for nonlinear fractional stochastic differential equations. Nonlinear Anal., Theory Methods Appl. **81**, 70-86 (2013)
47. Revathi, P, Sakthivel, R, Ren, Y, Anthoni, SM: Existence of almost automorphic mild solutions to non-autonomous neutral stochastic differential equations. Appl. Math. Comput. **230**, 639-649 (2014)
48. Revathi, P, Sakthivel, R, Ren, Y: Stochastic functional differential equations of Sobolev-type with infinite delay. Stat. Probab. Lett. **109**, 68-77 (2016)
49. Yan, Z: Approximate controllability of fractional neutral integro-differential inclusions with state-dependent delay in Hilbert spaces. IMA J. Math. Control Inf. **30**, 443-462 (2013)
50. Hale, J, Kato, J: Phase space for retarded equations with infinite delay. Funkc. Ekvacioj **21**, 11-41 (1978)
51. Hino, Y, Murakami, S, Naito, T: Functional Differential Equations with Unbounded Delay. Springer, Berlin (1991)
52. Fu, X, Huang, R: Existence of solutions for neutral integro-differential equations with state-dependent delay. Appl. Math. Comput. **224**, 743-759 (2013)
53. Kilbas, A, Srivastava, H, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
54. Podlubny, I: Fractional Differential Equations. Academic Press, New York (1999)
55. Zhou, Y: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)
56. Shu, XB, Xu, F: The existence of solutions for impulsive fractional partial neutral differential equations. J. Math. **2013**, Article ID 147193 (2013)
57. Lunardi, A: Analytic Semigroups and Optimal Regularity in Parabolic Problems. Birkhäuser, Basel (1995)
58. Mainardi, F, Paradisi, P, Gorenflo, R: Probability distributions generated by fractional diffusion equations. In: Kertesz, J, Kondor, I (eds.) Econophysics: An Emerging Science. Kluwer Academic, Dordrecht (2000)
59. Zhou, Y, Jiao, F: Existence of mild solutions for fractional neutral evolution equations. Comput. Math. Appl. **59**(3), 1063-1077 (2010)
60. Shu, XB, Shi, Y: A study on the mild solution of impulsive fractional evolution equations. Appl. Math. Comput. **273**, 465-476 (2016)