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Fractional advection differential equation within Caputo and Caputo–Fabrizio derivatives

Dumitru Baleanu[1,2], Bahram Agheli[3] and Maysaa Mohamed Al Qurashi[4]

Abstract
In this research, we applied the variational homotopic perturbation method and q-homotopic analysis method to find a solution of the advection partial differential equation featuring time-fractional Caputo derivative and time-fractional Caputo–Fabrizio derivative. A detailed comparison of the obtained results was reported. All computations were done using *Mathematica*.

Keywords
Fractional integral operators, Caputo derivative, Caputo–Fabrizio derivative, VHPIM, q-homotopic analysis method

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Introduction

In this research work, we have suggested the q-homotopic analysis method (q.HAM) and the variational homotopic perturbation method (VHPIM) to find a solution for the advection partial differential equations (*PDEs*) with time-fractional derivatives. We applied the methods based on Caputo–Fabrizio and Caputo derivatives[1,2] in order to compare the reported results. This work at first has been conducted in order to use the homotopic analysis method (HAM) by Liao[3] and further to use it in order to solve *PDEs* featuring time-fractional derivative. El-Tawil and Huseen[4] proposed a method called q.HAM which is considered a more general method of HAM. There exists a supportive parameter n and h in q.HAM such that on concession that $n = 1$, the standard HAM can be obtained. Otherwise, we consider the VHPIM.[5] It should be noted that there are no accurate analytical solutions for most of the fractional differential equations. Consequently, for such equations we have to employ some direct and iterative methods. Researchers have used variant methods to solve fractional differential equations (FDEs) and fractional partial differential equations (FPDEs) in recent years. These methods include Abdomina's decomposition method,[6,7] variational iteration method (VIM),[8,9] HPM,[10–12] and HAM.[13–17]

There are some books and papers related to applications of fractional calculus fitting real data for interested readers.[18–22]

This work is arranged as follows: in section "Preliminaries," the preliminaries are introduced. In section "Fundamental notion of the q.HAM," the description of the q.HAM is offered. The VHPIM in section "Fundamental of VHPIM" is explained. In section "Application and consequences," the application of q.HAM and VHPIM to the advection differential equation featuring time-fractional derivative is

[1]Department of Mathematics, Çankaya University, Ankara, Turkey
[2]Institute of Space Science, Bucharest, Romania
[3]Department of Mathematics, Islamic Azad University (Qaemshahr Branch), Qaem Shahr, Iran
[4]Mathematics Department, King Saud University, Riyadh, Saudi Arabia

Corresponding author:
Dumitru Baleanu, Department of Mathematics, Çankaya University, Ankara 06530, Turkey.
Email: dumitru@cankaya.edu.tr

illustrated and makes a comparison of q.HAM and VHPIM featuring Caputo–Fabrizio and Caputo derivative, respectively. Finally, in section "Conclusion," some conclusions regarding the proposed method are drawn.

Preliminaries

In this section, we introduced briefly Caputo's fractional derivative[1] and Caputo–Fabrizio fractional derivative.[23–26]

Definition 2.1. A function $u(t)$ belongs to $\mathbf{R}$, $t>0$, is assumed to be in C_β ($\beta \in \mathbf{R}$), on the occasion that be $n \in \mathbf{R}(>\beta)$, that $u(t) = t^n u_1(t)$, which $u_1(t) \in C[0, \infty)$, which is considered to be in C_β^l iff $u^{(l)} \in C_\beta$, $l \in \mathbf{N}$.

Definition 2.2. The fractional integral of $u(t)$, $t>a$, stated below is named Niemann-Knoxville fractional integral operator featuring $\nu>0$, of a $u(t) \in C_\beta$, $\beta \geq -1$, will be stated in the form below

$$I_a^\nu u(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-r)^{\nu-1} u(r) dr$$

$$I^\nu u(t) = I_0^\nu u(t), \quad I^0 u(t) = u(t)$$

Definition 2.3. The fractional derivative of $u(t)$ stated below is named Caputo's fractional derivative

$$D^\nu u(t) = \frac{1}{\Gamma(l-\nu)} \int_a^t (t-r)^{l-\nu-1} u^{(l)}(r) dr$$

for $u(t) \in C_{-1}^l$, $l-1<\nu \leq l$, $t>a$, and $l \in \mathbf{N}$.[1]

Remark 2.4. The property below convinces[1]

$$I_a^\nu D_a^\nu u(t) = u(t) - \sum_{k=0}^{l-1} u^{(k)}(a^+) \frac{(t-a)^k}{k!} \tag{1}$$

where $u \in C_\beta^l$, $l \in \mathbf{N}$, $\beta \geq -1$, $l-1<\nu \leq l$, and $t>a$.

Definition 2.5. The fractional derivative of $u(t)$ stated below is named Caputo–Fabrizio's fractional derivative[23]

$$\mathfrak{D}_t^\nu u(t) = \frac{T(\nu)}{1-\nu} \int_a^t u'(r) \exp\left[-\frac{\nu(t-r)}{1-\nu}\right] dr \tag{2}$$

in which $t>a$, $0<\nu \leq 1$, and $T(\nu)$ is called the normalization function and it satisfies $T(0) = T(1) = 1$.

Definition 2.6. The fractional integral of $u(t)$ stated below is named Caputo–Fabrizio fractional integral

$$\mathfrak{J}_t^\nu u(t) = \frac{1-\nu}{T(\nu)} u(t) + \frac{\nu}{T(\nu)} \int_a^t u(r) dr, \; 0<\nu \leq 1 \tag{3}$$

Remark 2.7. When $0<\nu \leq 1$, the property below convinces

$$\mathfrak{J}_a^\nu \mathfrak{D}_a^\nu u(t) = u(t) - u(a) \tag{4}$$

Fundamental notion of the q.HAM

To explain the essential notions of the q.HAM for time-fractional *PDEs*, we consider

$$\mathcal{N}(u(x,t)) = f(x,t) \tag{5}$$

in which $u(x,t)$ is an unfamiliar function, $\mathcal{N}$ is an operator nonlinear and linear, t and x denote the independent variables, and D^ν denotes that Caputo fractional derivative or Caputo–Fabrizio fractional derivative featuring $0<\nu \leq 1$. First, we construct the zero-order modified equation as

$$\begin{aligned}&(1-nq)\mathcal{L}[\psi(x,t;q) - u_0(x,t)] = q h W(x,t) \times \\ &(\mathcal{N}[\psi(x,t;q)] - f(x,t))\end{aligned} \tag{6}$$

here $n>1$, $q \in [0, 1/n]$ is the embedded parameter, $h \neq 0$ is a supportive parameter, $W(x,t) \neq 0$ is a supportive function, $\mathcal{L}$ is a supportive linear operator, and $u_0(x,t)$ is primary speculation. Clearly, since $q=0$ and $q = 1/n$, equation (6) turns out to be

$$\psi(x,t;0) = u_0(x,t), \quad \psi\left(x,t;\frac{1}{n}\right) = u(x,t) \tag{7}$$

in the state order. Therefore, q goes up from 0 to $1/n$, the answer $\psi(x,t;q)$ changes to the primary speculation $u_0(x,t)$ to $u(x,t)$. If $u_0(x,t)$, $\mathcal{L}$, h, and $W(x,t)$ are selected suitably, answer of equation (7) exists for $q \in [0, 1/n]$.

Here, we consider the Taylor series expression of $\psi(x,t;q)$ with respect to q in

$$\phi(x,t;q) = \sum_{m=0}^{\infty} u_m(x,t) q^m \tag{8}$$

where

$$\phi_m(x,t) = \frac{1}{m!} \frac{d^m}{dq^m} \varphi_m(x,t;q)\Big|_{q=0} \tag{9}$$

It is supposed that the supportive linear operator, the primary speculation, the supportive parameter h, and the supportive function $W(x,t)$ are opted so, equation (8) is convergent when $q \to 1/n$. Following that the rough answer (8) can be represented as

$$u(x,t) = \phi\left(x,t;\frac{1}{n}\right) = \sum_{m=0}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m \tag{10}$$

Then, we can express the vector

$$\vec{u}_n(x,t) = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_n(x,t)\} \tag{11}$$

After mth-order derivation of equation (6) by attention to q, next with $q = 0$, the mth-order modified equation is given as

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1} u(x,t)] = hW(x,t)\mathcal{R}_m(\vec{u}_{m-1}(x,t)) \tag{12}$$

with primary concessions

$$u_m^{(n)}(x,t) = 0, \quad n = 0,1,2,3,\ldots,m-1 \tag{13}$$

where

$$\begin{aligned}\mathcal{R}_m(\vec{u}_{m-1}(x,t)) = &-\left(f(x,t) - \frac{\chi_m}{n} f(x,t)\right)\\ &+ \frac{1}{(m-1)!}\frac{d^{m-1}}{dq^{m-1}}\mathcal{N}(x,t;q)|_{q=0}\end{aligned} \tag{14}$$

and

$$\chi_m = \begin{cases} n, & m>1 \\ 0, & m \le 1 \end{cases} \tag{15}$$

Operating the Niemann-Knoxville integral operator I^ν on both sides of equation (12)

$$\begin{aligned}u_m(x,t) = &\chi_m u_{m-1}(x,t) - \chi_m\left(u_{m-1}(x,0^+)\right)\\ &+ hW(x,t)I^\nu \mathcal{R}_m(\vec{u}_{m-1}(x,t))\end{aligned} \tag{16}$$

With regard to the fact that $u_m(x,t), m \ge 1$ is controlled by equation (12) featuring linear boundary concessions that are resultant from the initial problem.

As a result of the existence of the factor $(1/n)^m$, there will be more chance for the occurrence of convergence or even we can achieve faster convergence in comparison with the standard HAM.

Fundamental of VHPIM

In this part, we assume VHPIM in two stages for equation (5), featuring

$$\mathcal{N}(u(x,t)) = D_t^\nu u(x,t) - \mathfrak{K}u(x,t) \tag{17}$$

in which $0 < \nu \le 1$ and $\mathfrak{K}$ in which derivatives concerning t and x, is an operator in t, and x. Now, VHPIM is introduced in the following two stages.

Stage 1

Conforming to VIM, we construct the rectification functional for formula (5)

$$\begin{aligned}u_{j+1}(x,t) = &u_j(x,t) + I^\nu[\theta(t)(D^\nu u_j(x,t) - \\ &\mathfrak{K}(\breve{u}_j, (\breve{u}_j)_x, (\breve{u}_j)_{xx}) - f(x,t))]\\ = &u_j(x,t)\\ &+ \frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}\theta(r)(D^\nu u_j(x,r)\\ &- \mathfrak{K}(\breve{u}_j, (\breve{u}_j)_x, (\breve{u}_j)_{xx}) - f(x,r))dr\end{aligned} \tag{18}$$

Here, I^ν implies the Niemann-Knoxville fractional integral, and θ is a Lagrange coefficient, that can be recognized as optimal by variational approach. The function $\breve{u}_k$ is supposed as a limited variation. In other words, $\delta\breve{u}_k = 0$.

Stage 2

By utilizing the HPM and VIM, we obtain the below formula

$$\begin{aligned}\sum_{j=0}^{\infty} p^n u_j(x,t) = &u_0(x,t) + p\Bigg\{\sum_{j=1}^{\infty} p^j u_j(x,t) + \\ &I^\nu(\theta(t)(\sum_{j=0}^{\infty} p^j D^\nu u_j(x,t) - \\ &\sum_{j=0}^{\infty} p^j \mathfrak{K}(\breve{u}_j, (\breve{u}_j)_x, (\breve{u}_j)_{xx}) - \\ &f(x,t)))\Bigg\}\end{aligned} \tag{19}$$

In equation (19), $p \in [0,1]$ is a secured parameter and u_0 is a primary estimate of formula (5).

Balancing the sentences featuring the same powers of p in two sides of the formula (18), we may obtain $u_j (j = 0,1,2,\ldots)$.

Eventually, conforming to HPM, when p tends to be 1, we can gain the answer with approximation

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t) \tag{20}$$

Differently, conforming to VIM, the rectification functional (19) that may be uttered by approximation is stated as

$$\begin{aligned}u_{j+1}(x,t) = u_j(x,t) + &\int_0^t \theta(r)(\frac{d}{dt}u(x,r) - \\ &\mathfrak{K}(\breve{u}_j, (\breve{u}_n)_x, (\breve{u}_j)_{xx}) - f(x,r))dr\end{aligned} \tag{21}$$

where $\breve{u}_j$ is a rectification functional; however, $\breve{u}_j$ is assumed as a surrounded variation, that is, $\delta\breve{u}_j(x,t) = 0$.

Afterward, by creating functional stationary

$$\delta u_{j+1}(x,t) = \delta u_j(x,t) + \delta \int_0^t \theta(r)\left(\frac{d^m}{dt^m}u(x,r) - f(x,r)\right)dr$$

the Lagrange multiplier will be equal to $\theta = -1$. Thus, the following repetition rule to be earned

$$u_{j+1}(x,t) = u_j(x,t) - I^\nu(D^\nu u_j(x,t) - \mathfrak{K}\, u_j(x,t) - f(x,t)) \tag{22}$$

Applying equation (19), we can create the repetition rule as follows

$$\sum_{j=0}^{\infty} p^j u_j(x,t) = u_0(x,t) + p\{\sum_{j=1}^{\infty} p^j u_j(x,t) + I^\nu(\sum_{j=0}^{\infty} p^j D^\nu u_j(x,t) - \sum_{j=0}^{\infty} p^j \mathfrak{K}(\breve{u}_j, (\breve{u}_j)_x, (\breve{u}_j)_{xx}) - f(x,t))\} \tag{23}$$

Balanced with the coefficient of some power of p in two hands of equation (23), we can get $u_i(x,t)$, $(i = 0,1,2,\dots)$. As a consequence HPM, we can gain an answer of equation (5)

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t)$$

Application and consequences

In this portion, we utilize q.HAM and VHPIM to solve time-fractional advection PDE

$$D_t^\nu u(x,t) + u_x(x,t)\, u(x,t) = x(1 + t^2) \tag{24}$$

where $0<\nu\leq 1$, $t>0$, and $x \in \mathbb{R}$ is chosen featuring the primary status

$$u(x,0) = 0 \tag{25}$$

With the replacement of the primary status $u(x,0)$ in the recurrent formula (16), the consequence with q.HAM featuring Caputo derivative is stated as

$$u_0(x,t) = 0$$

$$u_1(x,t) = -\frac{hxt^\nu(\nu^2 + 3\nu + 2 + 2t^2)}{\Gamma(\nu + 3)}$$

$$u_2(x,t) = -\frac{hnxt^\nu(\nu^2 + 3\nu + 2 + t^2)}{\Gamma(\nu + 3)} + \frac{h^4 t^{3\nu} x^2 \nu + 2(-16t^4 2\nu + 3(2(\nu + 3)(2\nu + 5)t^2}{3\Gamma(\nu + 3)^2\Gamma(3\nu + 3)\Gamma(3\nu + 7)} + \frac{3(\nu + 2)(3\nu + 5)(\nu^2 + 3\nu + 3)\Gamma(2\nu + 2)\Gamma(3\nu + 4)}{3\Gamma(\nu + 3)^2\Gamma(3\nu + 3)\Gamma(3\nu + 7)} - \frac{3(\nu + 1)^2\Gamma(2\nu + 1)\Gamma(3\nu + 7)(9\nu^2(\nu + 3) + 20\nu))}{3\Gamma(\nu + 3)^2\Gamma(3\nu + 3)\Gamma(3\nu + 7)} + \frac{2(2\nu + 1)(\nu(\nu + 3) + 6)t^2 + 4}{3\Gamma(\nu + 3)^2\Gamma(3\nu + 3)\Gamma(3\nu + 7)}$$

$$\dots$$

Then, we consider the first three sentences with $n = 1$ as estimates of answer for formula (24)

$$u(x,t) \approx -\frac{hxt^\nu(\nu^2 + 3\nu + 2 + 2t^2)}{\Gamma(\nu + 3)} - \frac{hnxt^\nu(\nu^2 + 3\nu + 2 + 2t^2)}{\Gamma(\nu + 3)} + \frac{h^4 t^{3\nu} x^2 \nu + 2(-16t^4 2\nu + 3(2(\nu + 3)(2\nu + 5)t^2}{3\Gamma(\nu + 3)^2\Gamma(3\nu + 3)\Gamma(3\nu + 7)} + \frac{3(\nu + 2)(3\nu + 5)(\nu(\nu + 3) + 3))\Gamma(2\nu + 2)\Gamma(3\nu + 4)}{3\Gamma(\nu + 3)^2\Gamma(3\nu + 3)\Gamma(3\nu + 7)} - \frac{3(\nu + 1)^2\Gamma(2\nu + 1)\Gamma(3\nu + 7)(9\nu^2(\nu + 3) + 20\nu))}{3\Gamma(\nu + 3)^2\Gamma(3\nu + 3)\Gamma(3\nu + 7)} + \frac{2(2\nu + 1)(\nu(\nu + 3) + 6)t^2 + 4}{3\Gamma(\nu + 3)^2\Gamma(3\nu + 3)\Gamma(3\nu + 7)} \tag{26}$$

Now, when the incipient value $u(x,0)$ is substituted into the recursive equation (16), with q.HAM featuring Caputo–Fabrizio derivative, we can obtain

$$u_0(x,t) = 0$$

$$u_1(x,t) = -\frac{1}{3}hx(-3\nu + t(\nu(t^2 - 3t + 3) + 3t) + 3)$$

$$u_2(x,t) = \frac{1}{315}hx(-105ht(\nu(t^2 - 3t + 3) + 3t) - h^2\left(-315(\nu - 1)^3 + \nu(\nu(15\nu - 26) + 13)t^5 + 945(\nu - 1)^2\nu t - 70(\nu - 1)\nu^2 t^6 + 5\nu^3 t^7 - 630(\nu - 1)(2(\nu - 1)\nu + 1)t^2 - 21105\nu(\nu(11\nu - 20) + 10)t^3)\right) + 105(\nu - 1)(\nu(7\nu - 6) + 3)t^4 + 105nt(\nu((t - 3)t + 3) + 3t)$$

$$\dots$$

The first three statements for approximate answer for equation (24) will be stated as

$$u(x,t) \approx -\frac{1}{3}hx(-3\nu + t(\nu(t^2 - 3t + 3) + 3t) + 3) +$$
$$\frac{1}{315}hx(-105ht(\nu(t^2 - 3t + 3) + 3t)-$$
$$105nt(\nu((t-3)t + 3) + 3t) +$$
$$h^2\left(-315(\nu-1)^3 + 945(\nu-1)^2\nu t-\right.$$
$$70(\nu-1)\nu^2 t^6 + 5\nu^3 t^7-$$
$$630(\nu-1)(2(\nu-1)\nu+1)t^2-$$
$$105(\nu-1)(\nu(7\nu-6)+3)t^4 +$$
$$105\nu(\nu(11\nu-20)+10)t^3 +$$
$$\left.21\nu(\nu(15\nu-26)+13)t^5\right)) \tag{27}$$

Substituting the incipient value $u(x,0)$ within the recurrent formula (16), the consequence with VHPIM featuring Caputo derivative will be

$$u_0(x,t) = 0$$
$$u_1(x,t) = \frac{xt^\nu(\nu^2 + 3\nu + 2t^2 + 2)}{\nu(\nu^2 + 3\nu + 2)\Gamma(\nu)}$$
$$u_2(x,t) = \frac{xt^\nu(\nu^2 + 3\nu + 2t^2 + 2)}{\nu(\nu^2 + 3\nu + 2)\Gamma(\nu)}-$$
$$\frac{t^\nu x}{\Gamma(\nu)^2}\left(\frac{\pi\csc(\pi\nu)\Gamma(\nu)(\Gamma(\nu+3) + 2t^2\Gamma(\nu+1))}{\Gamma(1-\nu)\Gamma(\nu+1)\Gamma(\nu+3)} +\right.$$
$$\left(t^{2\nu}\left(4t^4\Gamma(2\nu+5)\Gamma(3\nu+1)\Gamma(3\nu+3) +\right.\right.$$
$$\nu^2 + 3\nu + 2\left(4t^2\Gamma(2\nu+3)\Gamma(3\nu+1) +\right.$$
$$\left.\left.\left.\left(\nu^2 + 3\nu + 2\right)\Gamma(2\nu+1)\Gamma(3\nu+3)\right)\Gamma(3\nu+5)\right)\right)/$$
$$\nu^2\left(\nu^2 + 3\nu\right)^2\Gamma(3\nu+1)\Gamma(3\nu+3)\Gamma(3\nu+5))$$
$$\ldots$$

Then, consider the first three sentences with $n = 1$ as estimates of answer for formula (24) are

$$u(x,t) \approx \frac{xt^\nu(\nu^2 + 3\nu + 2t^2 + 2)}{\nu(\nu^2 + 3\nu + 2)\Gamma(\nu)} +$$
$$\frac{xt^\nu(\nu^2 + 3\nu + 2t^2 + 2)}{\nu(\nu^2 + 3\nu + 2)\Gamma(\nu)}-$$
$$\frac{t^\nu x}{\Gamma(\nu)^2}\left(\frac{\pi\csc(\pi\nu)\Gamma(\nu)(\Gamma(\nu+3) + 2t^2\Gamma(\nu+1))}{\Gamma(1-\nu)\Gamma(\nu+1)\Gamma(\nu+3)} +\right.$$
$$\left(t^{2\nu}\left(4t^4\Gamma(2\nu+5)\Gamma(3\nu+1)\Gamma(3\nu+3) +\right.\right.$$
$$\nu^2 + 3\nu + 2\left(4t^2\Gamma(2\nu+3)\Gamma(3\nu+1) +\right.$$
$$\left.\left.\left.\left(\nu^2 + 3\nu + 2\right)\Gamma(2\nu+1)\Gamma(3\nu+3)\right)\Gamma(3\nu+5)\right)\right)/$$
$$\nu^2\left(\nu^2 + 3\nu + 2\right)^2\Gamma(3\nu+1)\Gamma(3\nu+3)\Gamma(3\nu+5))$$

Now, when the initial amount $u(x,0)$ is substituted into the iteration (16), with VHPIM featuring Caputo–Fabrizio derivative

$$u_0(x,t) = 0$$
$$u_1(x,t) = \frac{1}{3}\nu t^3 x - \nu t^2 x + t^2 x + \nu t x - \nu x + x$$
$$u_2(x,t) = (\nu-2)(\nu-1)\nu x - 3(\nu-1)^2\nu t x +$$
$$\frac{2}{9}(\nu-1)\nu^2 t^6 x - \frac{1}{3}(\nu-1)(\nu(7\nu-6)+3)t^4 x$$
$$\frac{1}{63}\nu^3 t^7 x + 2(\nu-1)(2(\nu-1)\nu+1)t^2 x-$$
$$\frac{1}{3}\nu(\nu(11\nu-20)+10)t^3 x-$$
$$\frac{1}{15}\nu(\nu(15\nu-26)+13)t^5 x$$
$$\ldots$$

Following that, the third sequence term approximate answer for formula (24) is stated as

$$\begin{aligned} u(x,t) \approx{} & \frac{1}{3}\nu t^3 x - \nu t^2 x + t^2 x + \nu t x - \nu x + x + \\ & (\nu-2)(\nu-1)\nu x - 3(\nu-1)^2\nu t x + \\ & \frac{2}{9}(\nu-1)\nu^2 t^6 x - \frac{1}{3}(\nu-1)(\nu(7\nu-6)+3)t^4 x + \\ & \frac{1}{63}\nu^3 t^7 x + 2(\nu-1)(2(\nu-1)\nu+1)t^2 x- \\ & \frac{1}{3}\nu(\nu(11\nu-20)+10)t^3 x- \\ & \frac{1}{15}\nu(\nu(15\nu-26)+13)t^5 x \end{aligned} \tag{28}$$

In Tables 1 and 2, we may observe the rough answers for $\nu = 1.0$, that is taken for several values of x and t using q.HAM and VHPIM with two fractional derivatives, involving singular differential operator which is named Caputo and involving nonsingular differential operator which is named Caputo–Fabrizio.

We can see the accrue and estimate answers with q.HAM toward $\nu = 1$, in Figure 1 with VHPIM.

In Figure 2, we can see the accrue and estimate answers toward $h = -1$, $n = 1$, and $\nu = 1$ with q.HAM.

In Table 3, the list of the times in seconds for every iteration in two methods used by *CPU* has been shown.

Conclusion

In this work, we have prosperously applied q.HAM and VHPIM to compare between Caputo and Caputo–Fabrizio derivatives for the time-fractional advection partial differential equation. The results indicate that rough answers for both derivatives for both methods are similar. And the Caputo–Fabrizio derivative is faster than the Caputo derivative in terms of *CPU* speed in calculation in *Mathematica*.

Table 1. Estimate values with VHPIM when $\nu = 1$ for equation (24).

Approximate answer				
t	x	u_{Exact}	Caputo	Caputo–Fabrizio
0.3	0.50	0.15	0.149836	0.149836
	0.75	0.225	0.224754	0.224754
	1.00	0.30	0.299673	0.299673
0.5	0.50	0.25	0.247855	0.247855
	0.75	0.375	0.371782	0.371782
	1.00	0.50	0.495709	0.495709
0.7	0.50	0.35	0.338142	0.338142
	0.75	0.525	0.507213	0.507213
	1.00	0.70	0.676283	0.676283

Table 2. Estimate values with q.HAM when $\nu = 1$, $h = -1$, and $n = 1$ for equation (24).

Approximate answer				
t	x	u_{Exact}	Caputo	Caputo–Fabrizio
0.3	0.50	0.15	0.150334	0.150334
	0.75	0.225	0.225500	0.225500
	1.00	0.30	0.299673	0.299673
0.5	0.50	0.25	0.248333	0.248333
	0.75	0.375	0.372499	0.372499
	1.00	0.50	0.496666	0.496666
0.7	0.50	0.35	0.348534	0.348534
	0.75	0.525	0.507832	0.507832
	1.00	0.70	0.677110	0.677110

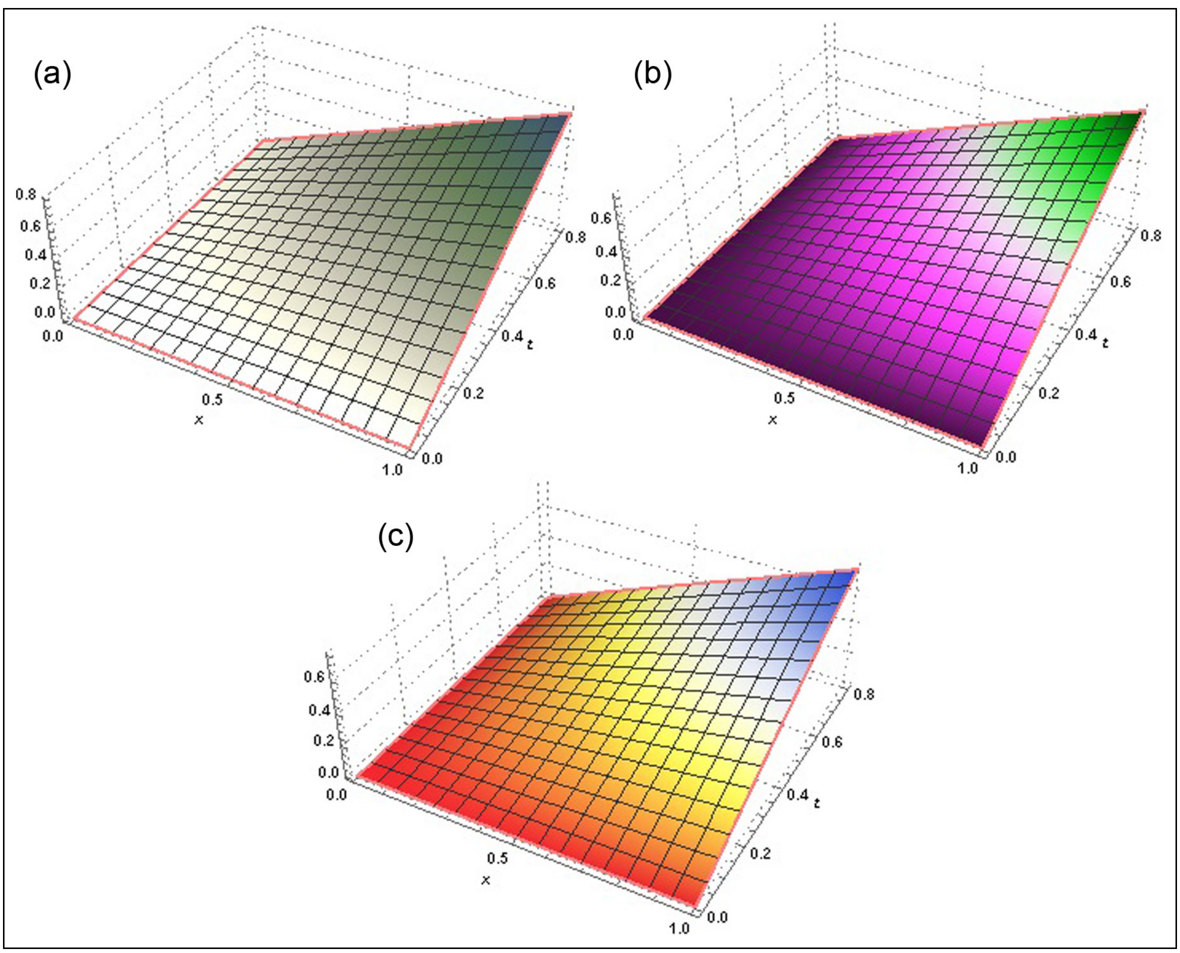

Figure 1. (a) The estimate answer, (b) approximate answer featuring Caputo-Fabrizio derivative with VHPIM, and (c) approximate answer featuring Caputo derivative with VHPIM.

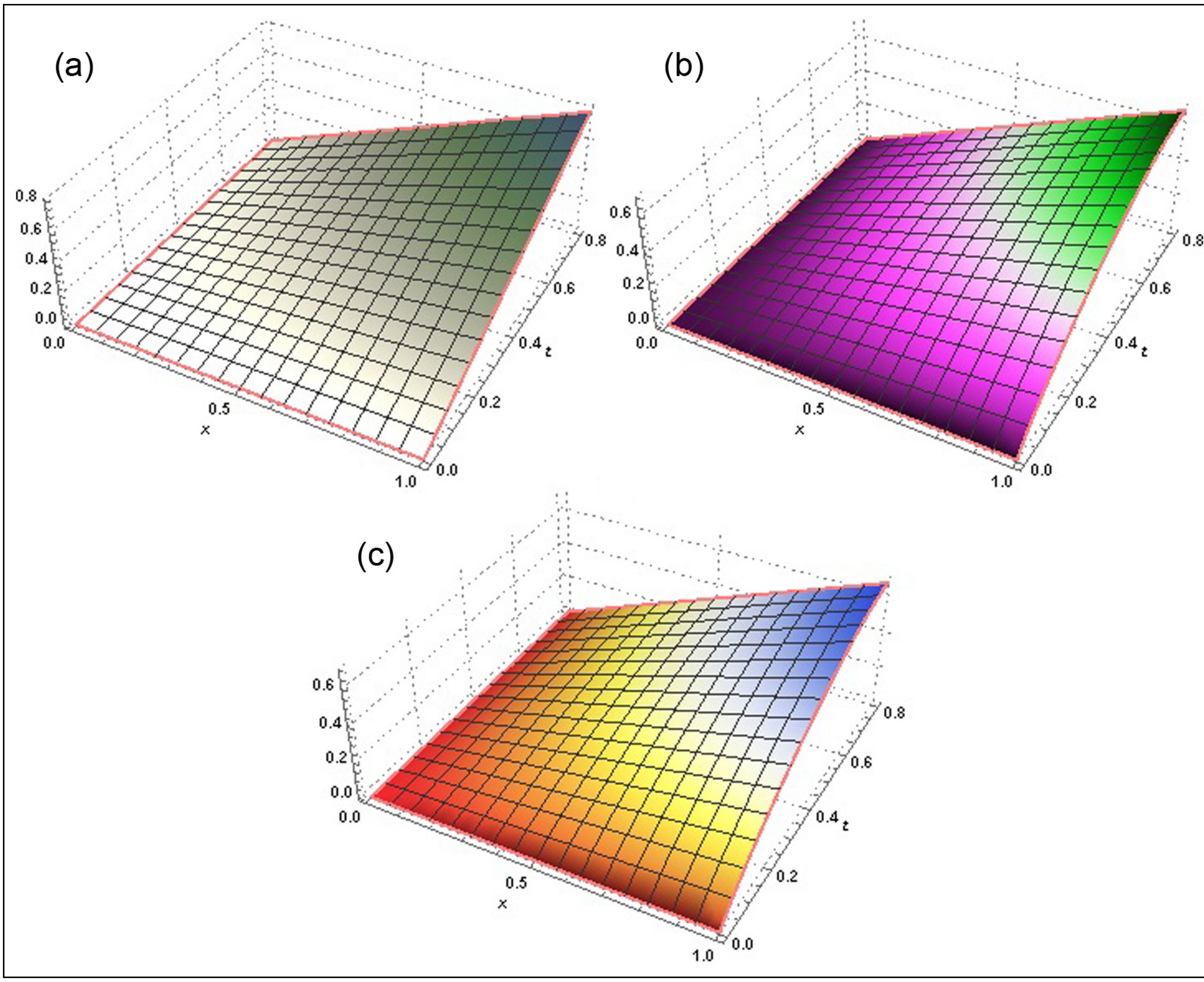

Figure 2. (a) The estimate answer, (b) approximate answer featuring Caputo-Fabrizio derivative with q.HAM, and (c) approximate answer featuring Caputo derivative with q.HAM.

Table 3. List of the times in seconds used.

	VHPIM		q.HAM	
	Caputo	Caputo–Fabrizio	Caputo	Caputo–Fabrizio
$u_0(x,t)$	0	0	0	0
$u_1(x,t)$	2.359375	0.015625	2.359375	0.015625
$u_2(x,t)$	16.312500	3.968750	18.625000	4.328125

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References

1. Kilbas AA, Srivastava HM and Trujillo JJ. *Theory and applications of fractional differential equations*. Amsterdam: Elsevier B.V., 2006.
2. Miller KS and Ross B. *An introduction to the fractional calculus and fractional differential equations*. New York: John Wiley & Sons, 1993.
3. Liao SJ. *The proposed homotopy analysis technique for the solution of nonlinear problems*. Doctoral Dissertation, PhD Thesis, Shanghai Jiao Tong University, Shanghai, China, 1992.
4. El-Tawil MA and Huseen SN. The q-homotopy analysis method (q.HAM). *Int J Appl Math Mech* 2012; 8: 51–75.
5. Neamaty A, Agheli B and Darzi R. Numerical solution of high-order fractional Volterra integro-differential

equations by variational homotopy perturbation iteration method. *J Comput Nonlin Dyn* 2015; 10: 061023.
6. Momani S and Shawagfeh NT. Decomposition method for solving fractional Riccati differential equations. *Appl Math Comput* 2006; 182: 1083–1092.
7. Wang Q. Numerical solutions for fractional KdV–Burgers equation by Adomian decomposition method. *Appl Math Comput* 2006; 182: 1048–1055.
8. Mustafa Inc. The approximate and exact solutions of the space- and time-fractional Burgers equations with initial conditions by variational iteration method. *J Math Anal Appl* 2008; 345: 476–484.
9. Yang XJ, Baleanu D, Khan Y, et al. Local fractional variational iteration method for diffusion and wave equations on Cantor sets. *Rom J Phys* 2014; 59: 36–48.
10. Momani S and Odibat Z. Homotopy perturbation method for nonlinear partial differential equations of fractional order. *Phys Lett A* 2007; 365: 345–350.
11. Odibat Z and Momani S. Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order. *Chaos Soliton Fract* 2008; 36: 167–174.
12. Hosseinnia S, Ranjbar A and Momani S. Using an enhanced homotopy perturbation method in fractional differential equations via deforming the linear part. *Comput Math Appl* 2008; 56: 3138–3149.
13. Hashim I, Abdulaziz O and Momani S. Homotopy analysis method for fractional IVPs. *Commun Nonlinear Sci* 2009; 14: 674–684.
14. Zurigat M, Momani S and Alawneh A. Analytical approximate solutions of systems of fractional algebraic–differential equations by homotopy analysis method. *Comput Math Appl* 2010; 59: 1227–1235.
15. Kumar P and Agrawal OP. An approximate method for numerical solution of fractional differential equations. *Signal Process* 2006; 86: 2602–2610.
16. Yang XJ, Baleanu D and Zhong WP. Approximation solution for diffusion equation on Cantor time-space. *Proc Rom Acad Ser A* 2013; 14: 127–133.
17. Yuste SB. Weighted average finite difference methods for fractional diffusion equations. *J Comput Phys* 2006; 216: 264–274.
18. Atangana A. Application of fractional calculus to epidemiology. In: Cattani C, Srivastava HM and Yang X-J (eds) *Fractional dynamics*. Warsaw: Walter de Gruyter GmbH & Co KG, 2015, pp.174–187.
19. Ding Y and Ye H. A fractional-order differential equation model of HIV infection of $CD4^+$ T-cells. *Math Comput Model* 2009; 50: 386–392.
20. Sierociuk D, Dzieliski A, Sarwas G, et al. Modelling heat transfer in heterogeneous media using fractional calculus. *Philos T Roy Soc A* 2013; 371: 20120146.
21. Baleanu D, Gven ZB and Machado JT (eds). *New trends in nanotechnology and fractional calculus applications*. New York: Springer, 2010, pp.xii + -531.
22. Ray SS and Sahoo S. Comparison of two reliable analytical methods based on the solutions of fractional coupled Klein–Gordon–Zakharov equations in plasma physics. *Comp Math Math Phys* + 2016; 56: 1319–1335.
23. Caputo M and Fabrizio M. A new definition of fractional derivative without singular kernel. *Prog Fract Differ Appl* 2015; 1: 73–85.
24. Losada J and Nieto JJ. Properties of a new fractional derivative without singular kernel. *Prog Fract Differ Appl* 2015; 1: 87–92.
25. Atangana A and Baleanu D. Caputo-Fabrizio derivative applied to groundwater flow within confined aquifer. *J Eng Mech: ASCE* 2016; 2016: D4016005.
26. Alsaedi A, Baleanu D, Etemad S, et al. On coupled systems of time-fractional differential problems by using a new fractional derivative. *J Funct Space* 2016; 2016: 4626940.