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Solutions of the time fractional reaction–diffusion equations with residual power series method

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Abstract

In this article, the residual power series method for solving nonlinear time fractional reaction–diffusion equations is introduced. Residual power series algorithm gets Maclaurin expansion of the solution. The algorithm is tested on Fitzhugh–Nagumo and generalized Fisher equations with nonlinearity ranging. The solutions of our equation are computed in the form of rapidly convergent series with easily calculable components using Mathematica software package. Reliability of the method is given by graphical consequences, and series solutions are used to illustrate the solution. The found consequences show that the method is a powerful and efficient method in determination of solution of the time fractional reaction–diffusion equations.

Keywords

Residual power series method, time fractional Fitzhugh–Nagumo equation, time fractional non-homogeneous reaction–diffusion equation, two-dimensional time fractional Fisher equation, series solution

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Introduction

In the last few years, there has been considerable interest in fractional calculus used in many fields, such as regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, viscoelasticity, electrical circuits, electro-analytical chemistry, biology, and control theory.[1–4] Besides, there has been a significant theoretical development in fractional differential equations and its applications.[5–10] However, fractional derivatives supply an important implement for the definition of hereditary characteristics of different necessaries and treatments. This is the fundamental advantage of fractional differential equations in return to classical integer-order problems.

In this article, we apply the residual power series method (RPSM) to find series solution for nonlinear time fractional reaction–diffusion equations. The RPSM was developed as an efficient method for fuzzy differential equations.[11] The RPSM is constituted with a repeated algorithm. It has been successfully put into practice to handle the approximate solution of the generalized Lane-Emden equation,[12] the solution of composite and non-composite fractional differential equations,[13] predicting and representing the multiplicity of solutions to boundary value problems of fractional order,[14] constructing and predicting the solitary pattern solutions for nonlinear time fractional

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dispersive partial differential equations,[15] the approximate solution of the nonlinear fractional KdV–Burgers equation,[16] the approximate solutions of fractional population diffusion model,[17] and the numerical solutions of linear non-homogeneous partial differential equations of fractional order.[18] In addition, K Moaddy et al.[19] used this method to obtain analytical approximate solution for different types of differential algebraic equations system. The proposed method is an alternative process for getting analytic Maclaurin series solution of problems.

In this article, we consider the following one- and two-dimensional fractional nonlinear reaction–diffusion equations of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = K\frac{\partial^2 u}{\partial x^2} + r(u) \tag{1}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = K\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + r(u) \tag{2}$$

where $t>0, 0<\alpha\leq 1, x, y \in R, K$ is the diffusion coefficient, $r(u)$ is some reasonable nonlinear function of u which is chosen as reaction kinetics, and α is a parameter defining the order of the time fractional derivative. If we write $r(u) = u(1-u)(u-\theta)$ and $K=1$, equation (1) leads to the time fractional Fitzhugh–Nagumo equation, which is an important nonlinear reaction–diffusion equation.[20,21] If we write $r(u) = u(1-u) + \sin x + 2\sin x\,(t^\alpha/\Gamma(1+\alpha)) + \sin^2 x\,(t^{2\alpha}/(\Gamma(1+\alpha))^2)$ and $K=1$, equation (1) leads to the time fractional non-homogeneous reaction–diffusion equation. If we write $r(u) = u^2(1-u)$ and $K = 1/2$, equation (2) leads to two-dimensional time fractional Fisher equation.[22] This equations are as follows

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-\theta),\ 0<\theta<1 \tag{3}$$

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} = {} & \frac{\partial^2 u}{\partial x^2} + u(1-u) + \sin x \\ & + 2\sin x\frac{t^\alpha}{\Gamma(1+\alpha)} + \sin^2 x\frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}\end{aligned} \tag{4}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + u^2(1-u) \tag{5}$$

In Baranwal et al.,[22] an analytic algorithm for time fractional nonlinear reaction–diffusion equations (3),(4) and (5) based on a new iterative method (NIM). In Bhrawy,[20] the authors used Jacobi collocation method in order to find the approximate solutions of equation (3). SZ Rida et al.[21] used generalized differential transform method for numerical solutions of equation (3). Khan et al.[23] applied homotopy analysis method (HAM) and Merdan[24] applied fractional variational iteration method (FVIM) for series solutions of equation (3).

In these equations, the function $u(x,t)$ is assumed to be a function of time and space, which means that $u(x,t)$ is disappearing for $t<0$ and $x<0$, and this function is considered to be analytic for $t>0$. Also, the function $f(x)$ is considered to be analytic for $x>0$.

In section "Basic definitions of fractional calculus theory" of this work, some preliminary results related to the Caputo derivative and the fractional power series (PS) are described. In section "Applications for RPSM algorithm and graphical results," the base opinion of the RPSM is constituted to construct the solution of the time fractional nonlinear reaction–diffusion equations and some graphical consequences are included to demonstrate the reliability and efficiency of the method. Finally, consequences are introduced in section "Conclusion."

Basic definitions of fractional calculus theory

We first illustrate the main descriptions and various features of the fractional calculus theory[2] in this section.

Definition 1. The Riemann–Liouville fractional integral operator of order $\alpha(\alpha\geq 0)$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(v)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha>0,\ x>0 \tag{6}$$

$$J^0 f(x) = f(x)$$

Definition 2. The Caputo fractional derivatives of order α are defined as

$$\begin{aligned} & D^\alpha f(x) = J^{m-\alpha}D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \\ & \int_0^x (x-t)^{m-\alpha-1}\frac{d^m}{dt^m}f(t)dt \\ & m-1<\alpha\leq m,\ x>0 \end{aligned} \tag{7}$$

where D^m is the classical differential operator of order m.

For the Caputo derivative we have

$$D^\alpha x^\beta = 0,\ \ \beta<\alpha$$
$$D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha},\ \ \beta\geq\alpha$$

Definition 3. For n to be the smallest integer that exceeds α, the Caputo time fractional derivative operator of order α of $u(x,t)$ is defined as[13,16]

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial t^n} d\tau, \qquad n-1<\alpha<n \tag{8}$$
$$D_t^n u(x,t) = \frac{\partial^n u(x,t)}{\partial t^n}, \quad n \in N$$

and the space fractional derivative of order β of $u(x,t)$ is defined as

$$D_x^\beta u(x,t) = \frac{\partial^\beta u(x,t)}{\partial x^\beta} = \frac{1}{\Gamma(n-\beta)} \int_0^x (x-\tau)^{n-\beta-1} \frac{\partial^n u(\tau,t)}{\partial t^n} d\tau \qquad n-1<\beta<n \tag{9}$$
$$D_x^n u(x,t) = \frac{\partial^n u(x,t)}{\partial x^n}, \quad n \in N$$

Definition 4. A PS expansion of the form

$$\sum_{m=0}^{\infty} c_m(t-t_0)^{m\alpha} = c_0 + c_1(t-t_0)^\alpha + c_2(t-t_0)^{2\alpha} + \cdots$$
$$0 \le m-1<\alpha \le m, \ t \ge t_0$$

is named fractional PS at $t = t_0$.[13]

Definition 5. A PS of the form

$$\sum_{m=0}^{\infty} f_m(x)(t-t_0)^{m\alpha} = f_0(x) + f_1(x)(t-t_0)^\alpha + f_2(x)(t-t_0)^{2\alpha} + \cdots \tag{10}$$
$$0 \le m-1<\alpha \le m, t \ge t_0$$

is named fractional PS at $t = t_0$.[13]

Theorem 1. (see El-Ajou et al.[16] for proof). Only if $u(x,t)$ has a multiple fractional PS representing at $t = t_0$ of the form

$$u(x,t) = \sum_{m=0}^{\infty} f_m(x)(t-t_0)^{m\alpha} \tag{11}$$
$$0 \le m-1<\alpha \le m, \quad x \in I, \quad t_0 \le t < t_0 + R$$

If $D_t^{m\alpha} u(x,t)$ are continuous on $I \times (t_0, t_0+R)$, $m = 0,1,2,\ldots,$ then coefficients $f_m(x)$ are given as

$$f_m(x) = \frac{D_t^{m\alpha} u(x,t_0)}{\Gamma(m\alpha+1)}, \quad m = \overline{0,\infty}$$

where $D_t^{m\alpha} = (\partial^{m\alpha}/\partial t^{m\alpha}) = (\partial^\alpha/\partial t^\alpha).(\partial^\alpha/\partial t^\alpha)\ldots(\partial^\alpha/\partial t^\alpha)$ (m-times) and $R = \min_{c \in I} R_c$, in which R_c is the radius of convergence of the fractional PS $\int_{m=0}^{\infty} f_m(c)(t-t_0)^{m\alpha}$

Result 1. The fractional PS expansion of $u(x,t)$ at t_0 should be of the form

$$u(x,t) = \sum_{m=0}^{\infty} \frac{D_t^{m\alpha} u(x,t_0)}{\Gamma(m\alpha+1)}(t-t_0)^{m\alpha} \tag{12}$$
$$0 \le m-1<\alpha \le m, x \in I, t_0 \le t < t_0 + R$$

which is a generalized Taylor's series formula. To specify, if one set $\alpha = 1$ in equation (12), then the classical Taylor's series formula

$$u(x,t) = \sum_{m=0}^{\infty} \frac{\partial^m u(x,t_0)}{\partial t^m} \frac{(t-t_0)}{m!}, \qquad x \in I, t_0 \le t < t_0 + R$$

is obtained.[16]

Applications for RPSM algorithm and graphical results

Example 1. First, we consider time fractional Fitzhugh–Nagumo equation.[21,23]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-\theta), \quad t>0, \quad 0<\alpha \le 1, \quad x \in R, \quad 0<\theta<1 \tag{13}$$

by the initial condition

$$u(x,0) = \frac{1}{1+e^{-\frac{x}{\sqrt{2}}}} \tag{14}$$

The exact solution for equation (13) for $\alpha = 1$ is[25] $u(x,y) = (1+e^{-(1/2)(x+(1+2\theta/\sqrt{2})t)})^{-1}$

We apply the RPSM to find out series solution for the time fractional Fitzhugh–Nagumo equation subject to given initial conditions by replacing its fractional PS expansion with its truncated residual function. From this equation, a repetition formula for the calculation of coefficients is supplied, while coefficients in fractional PS expansion can be calculated repeatedly by repeated fractional differentiation of the truncated residual function.[13,26]

The RPSM propose the solution for equations (13) and (14) with a fractional PS at $t = 0$.[11] Suppose that the solution takes the expansion form

$$u = \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \ 0<\alpha \le 1, \quad x \in I, \quad 0 \le t < R \tag{15}$$

Then, we let u_k to denote k. The truncated series of u

$$u_k = \sum_{n=0}^{k} f_n(x)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \quad 0<\alpha\leq 1, \quad x\in I, \quad 0\leq t<R \tag{16}$$

where $u_0 = f_0(x) = u(x,0) = f(x)$.

Also, equation (16) can be written as

$$u_k = f(x) + \sum_{n=1}^{k} f_n(x)\frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \tag{17}$$

where $0<\alpha\leq 1, \quad 0\leq t<R, \quad x\in I, \quad k=\overline{1,\infty}$

At first, to find the value of coefficients $f_n(x), n=1,2,3,\ldots,k$ in series expansion of equation (17), we define the residual function Res for equation (3) as

$$Res = \frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} - u(1-u)(u-\theta)$$

and the kth residual function, Res_k as follows

$$Res_k = \frac{\partial^\alpha u_k}{\partial t^\alpha} - \frac{\partial^2 u_k}{\partial x^2} - u_k(1-u_k)(u_k-\theta), \quad \text{where } k=1,2,3,\ldots \tag{18}$$

As in the literature,[11–14] it is clear that $Res=0$ and $\lim_{k\to\infty} Res_k = Res$ for each $x\in I$ and $t\geq 0$. Then, $D_t^{r\alpha}Res=0$, fractional derivative of a stationary in the Caputo's idea is zero and the fractional derivative $D_t^{r\alpha}$ of Res and Res_k are pairing at $t=0$ with each $r=\overline{0,k}$. To give residual PS algorithm: First, we replace the kth truncated series of u into equation (13). Second, we find the fractional derivative formula $D_t^{(k-1)\alpha}$ of both $Res_{u,k}$, where $k=\overline{1,\infty}$, and finally, we can solve found system

$$D_t^{(k-1)\alpha}Res_{u,k}=0, \quad 0<\alpha\leq 1, \quad x\in I, \quad t=0, \quad k=\overline{1,\infty} \tag{19}$$

to get the required coefficients $f_n(x)$ for $n=\overline{1,k}$ in equation (17).

Hence, to determine $f_1(x)$, we write $k=1$ in equation (18)

$$Res_1 = \frac{\partial^\alpha u_1}{\partial t^\alpha} - \frac{\partial^2 u_1}{\partial x^2} - u_1(1-u_1)(u_1-\theta) \tag{20}$$

where

$$u_1 = \frac{t^\alpha}{\Gamma(1+\alpha)}f_1(x) + f(x)$$

for

$$u_0 = f_0(x) = f(x) = u(x,0) = \frac{1}{1+e^{-\frac{x}{\sqrt{2}}}}$$

Therefore

$$Res_1 = f_1(x) - f''(x) - \frac{t^\alpha}{\Gamma(1+\alpha)}f_1''(x) - \left(\frac{t^\alpha}{\Gamma(1+\alpha)}f_1(x)+f(x)\right)\left(1-\left(\frac{t^\alpha}{\Gamma(1+\alpha)}f_1(x)+f(x)\right)\right)\left(\left(\frac{t^\alpha}{\Gamma(1+\alpha)}f_1(x)+f(x)\right)-\theta\right)$$

From equation (19) we deduce that $Res_1=0$ $(t=0)$ and, thus

$$f_1(x) = -\frac{e^{\frac{x}{\sqrt{2}}}(-1+2\theta)}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2} \tag{21}$$

Therefore, the first residual power series (RPS) approximate solutions are

$$u_1 = -\frac{e^{\frac{x}{\sqrt{2}}}(-1+2\theta)}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{1}{1+e^{-\frac{x}{\sqrt{2}}}} \tag{22}$$

Similarly, to find out the form of the second unknown coefficient $f_2(x)$, we write $k=2$ in equation (18)

$$Res_2 = \frac{\partial^\alpha u_2}{\partial t^\alpha} - \frac{\partial^2 u_2}{\partial x^2} - u_2(1-u_2)(u_2-\theta)$$

where

$$u_2 = f(x) + \frac{t^\alpha}{\Gamma(1+\alpha)}f_1(x) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}f_2(x)$$

Therefore

$$Res_2 = f_1(x) + \frac{t^\alpha}{\Gamma(1+\alpha)}f_2(x) - f''(x) - \frac{t^\alpha}{\Gamma(1+\alpha)}f_1''(x) - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}f_2''(x) - \left(f(x)+\frac{t^\alpha}{\Gamma(1+\alpha)}f_1(x)+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}f_2(x)\right)\left(1-f(x)-\frac{t^\alpha}{\Gamma(1+\alpha)}f_1(x)-\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}f_2(x)\right)\left(\left(\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}f_2(x)+\frac{t^\alpha}{\Gamma(1+\alpha)}f_1(x)+f(x)\right)-\theta\right)$$

From equation (19), we deduce that $D_t^\alpha Res_2=0$ $(t=0)$ and thus

$$f_2(x) = -\frac{e^{\frac{x}{\sqrt{2}}}(-1+e^{\frac{x}{\sqrt{2}}})(-1+2\theta)^2}{4(1+e^{\frac{x}{\sqrt{2}}})^3} \tag{23}$$

Therefore, the second RPS approximate solutions are

$$u_2 = \frac{1}{1+e^{-\frac{x}{\sqrt{2}}}} - \frac{e^{\frac{x}{\sqrt{2}}}(-1+2\theta)}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{e^{\frac{x}{\sqrt{2}}}(-1+e^{\frac{x}{\sqrt{2}}})(-1+2\theta)^2}{4\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{24}$$

Similarly, to determine $f_3(x)$, we write $k=3$ in equation (18)

$$Res_3 = \frac{\partial^{\alpha} u_3}{\partial t^{\alpha}} - \frac{\partial^2 u_3}{\partial x^2} - u_3(1-u_3)(u_3-\theta)$$

where

$$u_3 = f(x) + \frac{t^{\alpha}}{\Gamma(1+\alpha)}f_1(x) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}f_2(x) + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}f_3(x)$$

Therefore

$$\begin{aligned}Res_3(x,t) &= f_1(x) + \frac{t^{\alpha}}{\Gamma(1+\alpha)}f_2(x) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}f_3(x)\\ &- \left(f''(x) + \frac{t^{\alpha}}{\Gamma(1+\alpha)}f_1''(x) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}f_2''(x) + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}f_3''(x)\right)\\ &- \left(f(x) + \frac{t^{\alpha}}{\Gamma(1+\alpha)}f_1(x) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}f_2(x) + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}f_3(x)\right)\\ &\left(1 - \left(f(x) + \frac{t^{\alpha}}{\Gamma(1+\alpha)}f_1(x) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}f_2(x) + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}f_3(x)\right)\right)\\ &\left(\left(f(x) + \frac{t^{\alpha}}{\Gamma(1+\alpha)}f_1(x) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}f_2(x) + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}f_3(x)\right) - \theta\right)\end{aligned}$$

From equation (19), we deduce that $D_t^{2\alpha}Res_3 = 0 \ \ (t=0)$ and thus

$$f_3(x) = -\frac{e^{\frac{x}{\sqrt{2}}}(1+4e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x})(-1+2\theta)^3}{16\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4} \tag{25}$$

Therefore, the third RPS approximate solutions are

$$u_3 = \frac{1}{1+e^{-\frac{x}{\sqrt{2}}}} - \frac{e^{\frac{x}{\sqrt{2}}}(-1+2\theta)}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{e^{\frac{x}{\sqrt{2}}}(-1+e^{\frac{x}{\sqrt{2}}})(-1+2\theta)^2}{4\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{e^{\frac{x}{\sqrt{2}}}(1+4e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x})(-1+2\theta)^3}{16\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \tag{26}$$

Similarly, applying the same procedure for $k=4$ and taking into account the form of $f_0(x), f_1(x), f_2(x),$ and $f_3(x)$, respectively, will lead after easy calculations to the following form of $f_4(x)$

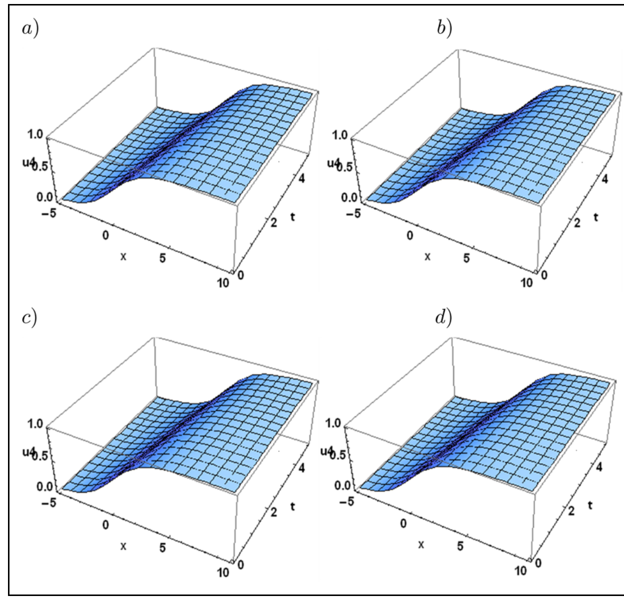

Figure 1. The surface graph of the exact solution u and the u_4 approximate solution of the time fractional Fitzhugh–Nagumo equation ($\theta = 0.8$): (a) $u_4(x,t)$ when $\alpha = 0.1$, (b) $u_4(x,t)$ when $\alpha = 0.3$, (c) $u_4(x,t)$ when $\alpha = 0.9$, and (d) $u(x,t)$ when $\alpha = 1$.

$$f_4(x) = -\frac{e^{\frac{x}{\sqrt{2}}}(-1+11e^{\frac{x}{\sqrt{2}}}+e^{\frac{3x}{\sqrt{2}}}-11e^{\sqrt{2}x})(-1+2\theta)^4}{96\left(1+e^{\frac{x}{\sqrt{2}}}\right)^5} \tag{27}$$

Therefore, the fourth RPS approximate solutions are (Figure 1)

$$u_4 = \frac{1}{1+e^{-\frac{x}{\sqrt{2}}}} - \frac{e^{\frac{x}{\sqrt{2}}}(-1+2\theta)}{2(1+e^{\frac{x}{\sqrt{2}}})^2}\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{e^{\frac{x}{\sqrt{2}}}(-1+e^{\frac{x}{\sqrt{2}}})(-1+2\theta)^2}{4(1+e^{\frac{x}{\sqrt{2}}})^3}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{e^{\frac{x}{\sqrt{2}}}(1+4e^{\frac{x}{\sqrt{2}}}+e^{\sqrt{2}x})(-1+2\theta)^3}{16(1+e^{\frac{x}{\sqrt{2}}})^4}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{e^{\frac{x}{\sqrt{2}}}(-1+11e^{\frac{x}{\sqrt{2}}}+e^{\frac{3x}{\sqrt{2}}}-11e^{\sqrt{2}x})(-1+2\theta)^4}{96(1+e^{\frac{x}{\sqrt{2}}})^5}\frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \tag{28}$$

In where, we plot the RPS approximate solution $u_k(x,t)$ for $k = 1, 2, 3,$ and 4 which are closing the axis $y=0$ as the number of iterations increase. Figure 2 clears that the exact error is being smaller as the number of k is increasing. It is clear that the value of kth truncated series $u_k(x,t)$ affects the RPS approximate solutions.

Figure 3 clears that $u_4(x,t)$ solution are closing the exact solution as the number of α increase.

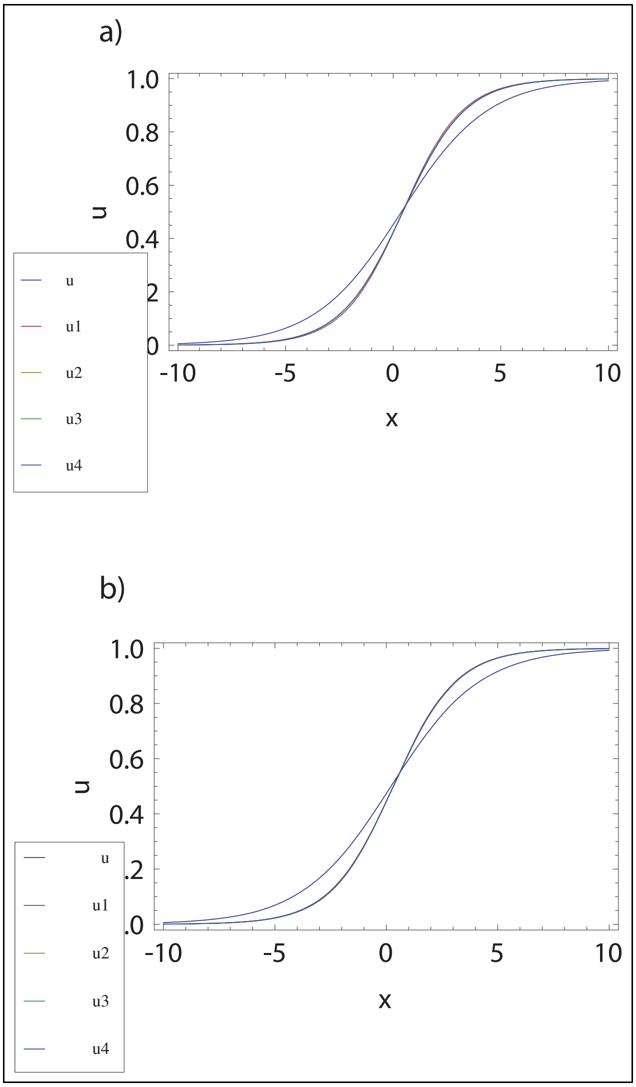

Figure 2. $u_k(x, t)$ solution of the time fractional Fisher equation when $k = 1, 2, 3$, and 4 versus its exact solution for $\theta = 0.8$: (a) $\alpha = 0.1, t = 0.9$ and (b) $\alpha = 0.6, t = 0.5$.

In Table 1, comparison among approximate solutions with known results is made. These results are obtained using RPSM, HAM,[21] FVIM,[24] and an NIM.[22]

These table clarify the exact error is being smallest in the value of the $t = 0.01$.

Example 2. We consider time fractional non-homogeneous reaction–diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(1-u) + \sin x + 2\sin x \frac{t^\alpha}{\Gamma(1+\alpha)} + \sin^2 x \frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}, \quad t>0, \quad 0<\alpha\leq 1, x \in R \tag{29}$$

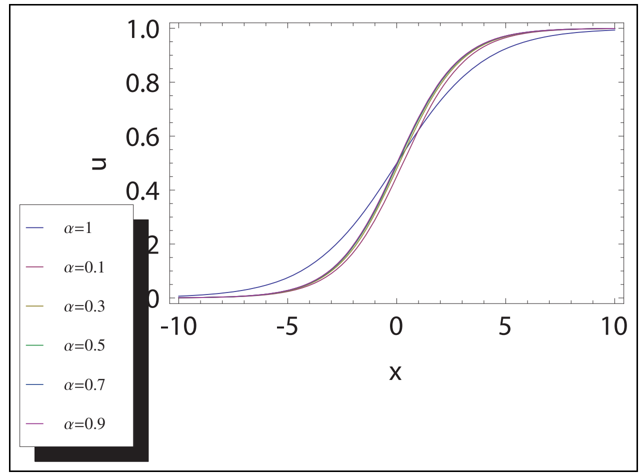

Figure 3. $u_4(x, t)$ solution of the time fractional Fitzhugh–Nagumo equation when $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9,$ and $1 (t = 0.01)$.

by the initial condition

$$u(x, 0) = 1 \tag{30}$$

For equation (29), the kth residual function, Res_k as follows

$$Res_k = \frac{\partial^\alpha u_k}{\partial t^\alpha} - \frac{\partial^2 u_k}{\partial x^2} - u_k(1-u_k) - \sin x - 2\sin x \frac{t^\alpha}{\Gamma(1+\alpha)} - \sin^2 x \frac{t^{2\alpha}}{(\Gamma(1+\alpha))^2}, \quad k = 1, 2, 3, \ldots \tag{31}$$

We apply repeating process as in the former application

$$f_1(x) = \sin x, f_n(x) = 0, \quad n = 2, 3, 4, \ldots \tag{32}$$

Therefore, the first RPS approximate solutions are

$$u_1 = 1 + \sin x \frac{t^\alpha}{\Gamma(1+\alpha)} \tag{33}$$

which, in fact, is the exact solution of equation (29) (Figures 4 and 5).

Example 3. We study two-dimensional time fractional Fisher equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + u^2(1-u), \quad t>0, 0<\alpha\leq 1, 3.22 \quad \Phi = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\} \tag{34}$$

by the initial condition

Table 1. Comparison between approximate solutions u_{RPSM}, u_{HAM}, u_{NIM}, and exact solution ($x = 0.01,\ \theta = 0.8$).

	$u_2(\alpha = 0.8)$				$u_4(\alpha = 1)$		
t	u_{RPSM}	u_{HAM}	u_{FVIM}	u_{NIM}	u_{RPSM}	u_{Exact}	$\lvert u_{Exact} - u_{RPSM}\rvert$
0.01	0.499745	0.499765	0.499774	0.497779	0.501018	0.50072	0.000298096
0.05	0.494437	0.494699	0.494541	0.487317	0.498018	0.498598	0.000580611
0.1	0.489004	0.489798	0.489186	0.476613	0.494268	0.495947	0.00167905
0.15	0.484113	0.485631	0.484366	0.46698	0.490518	0.493295	0.0027775
0.2	0.479543	0.481948	0.479864	0.457985	0.486769	0.490644	0.00387588

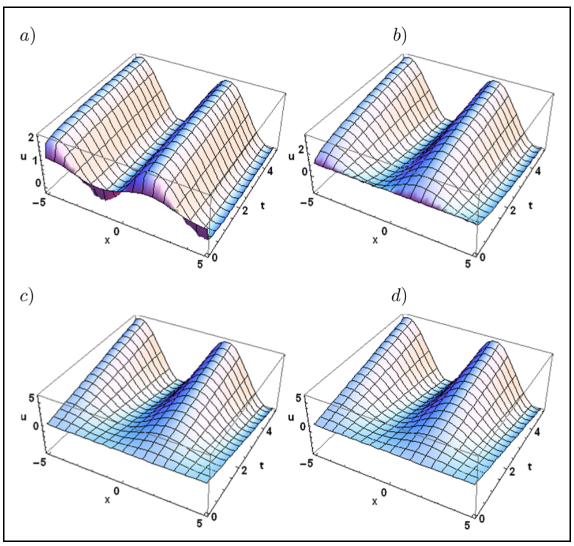

Figure 4. The surface graph of the exact solution u and the u_4 approximate solution of time fractional non-homogeneous reaction–diffusion equation: (a) $u_4(x,t)$ when $\alpha = 0.1$, (b) $u_4(x,t)$ when $\alpha = 0.5$, (c) $u_4(x,t)$ when $\alpha = 0.9$, and (d) $u(x,t)$ when $\alpha = 1$.

$$u(x,y,0) = \frac{1}{1 + e^{\frac{x+y}{\sqrt{2}}}} \tag{35}$$

The exact solution for equation (35) for $\alpha = 1$ is[26] $u(x,y,t) = (1/1 + e^{(x+y-(t/\sqrt{2})/\sqrt{2})})$

For equation (35), the kth residual function, Res_k as follows

$$Res_k = \frac{\partial^\alpha u_k}{\partial t^\alpha} - \frac{1}{2}\left(\frac{\partial^2 u_k}{\partial x^2} + \frac{\partial^2 u_k}{\partial y^2}\right) - u_k^2(1-u_k),$$
$$k = 1, 2, 3, \ldots$$

We apply repeating process as in the former application

$$f_1(x,y) = \frac{e^{\frac{x+y}{\sqrt{2}}}}{2\left(1 + e^{\frac{x+y}{\sqrt{2}}}\right)^2}$$

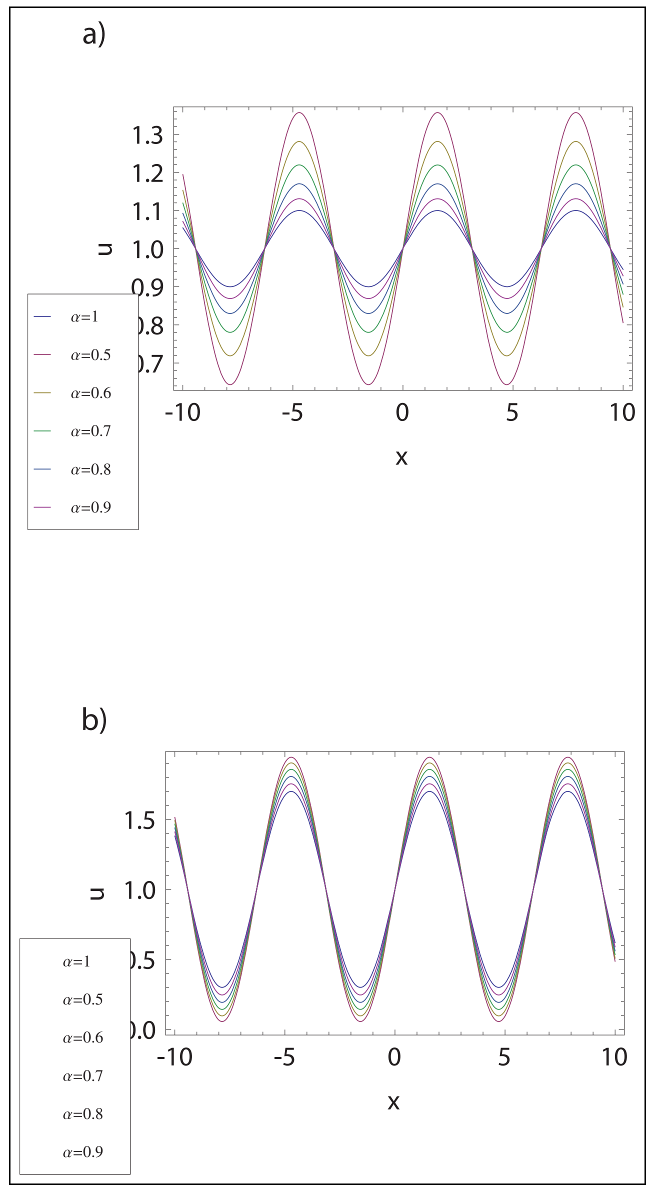

Figure 5. $u_4(x,t)$ solution of the time fractional non-homogeneous reaction–diffusion when $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9, 1$: (a) $t = 0.1$ and (b) $t = 0.7$.

$$f_2(x,y) = \frac{e^{\frac{x+y}{\sqrt{2}}}\left(-1 + e^{\frac{x+y}{\sqrt{2}}}\right)}{4\left(1 + e^{\frac{x+y}{\sqrt{2}}}\right)^3} \tag{36}$$

$$f_3(x,y) = \frac{e^{\frac{x+y}{\sqrt{2}}}\left(1 - 4e^{\frac{x+y}{\sqrt{2}}} + e^{\sqrt{2}(x+y)}\right)}{16\left(1 + e^{\frac{x+y}{\sqrt{2}}}\right)^4}$$

$$f_4(x,y) = \frac{e^{\frac{x+y}{\sqrt{2}}}\left(-1 + 11e^{\frac{x+y}{\sqrt{2}}} + e^{\frac{3(x+y)}{\sqrt{2}}} - 11e^{\sqrt{2}(x+y)}\right)}{96\left(1 + e^{\frac{x+y}{\sqrt{2}}}\right)^5}$$

Therefore, the fourth RPS approximate solutions are

$$\begin{aligned} u_4 = {} & \frac{1}{1 + e^{\frac{x+y}{\sqrt{2}}}} + \frac{e^{\frac{x+y}{\sqrt{2}}}}{2\left(1 + e^{\frac{x+y}{\sqrt{2}}}\right)^2}\frac{t^{\alpha}}{\Gamma(1+\alpha)} \\ & + \frac{e^{\frac{x+y}{\sqrt{2}}}\left(-1 + e^{\frac{x+y}{\sqrt{2}}}\right)}{4\left(1 + e^{\frac{x+y}{\sqrt{2}}}\right)^3}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\ & + \frac{e^{\frac{x+y}{\sqrt{2}}}\left(1 - 4e^{\frac{x+y}{\sqrt{2}}} + e^{\sqrt{2}(x+y)}\right)}{16\left(1 + e^{\frac{x+y}{\sqrt{2}}}\right)^4}\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \\ & + \frac{e^{\frac{x+y}{\sqrt{2}}}\left(-1 + 11e^{\frac{x+y}{\sqrt{2}}} + e^{\frac{3(x+y)}{\sqrt{2}}} - 11e^{\sqrt{2}(x+y)}\right)}{96\left(1 + e^{\frac{x+y}{\sqrt{2}}}\right)^5} \\ & \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \end{aligned} \tag{37}$$

Figures 6 and 7 clear that the exact error is being smaller as the number of k is increasing. Therefore, $u_k(x,t)$ affects the RPS approximate solutions.

In Figure 8, we plot the RPS approximate solution $u_4(x,t)$ which are closing the exact solution as the number of α increase. These figures clear that the convergence of the approximate solutions to the exact solution related to the order of the solution.

In Table 2, comparison among approximate solutions with known results is made. These results are obtained using RPSM and an NIM.[22] This table clarifies the convergence of the approximate solutions to the exact solution, and exact error is being smaller as the value of the t is decreasing.

Conclusion

The RPSM is applied successfully for solving the nonlinear fractional differential equations. The fundamental objective of this article is to introduce an algorithmic form and implement a new analytical repeated algorithm derived from the RPS to find numerical solutions for the time fractional reaction–diffusion equation. Graphical and numerical

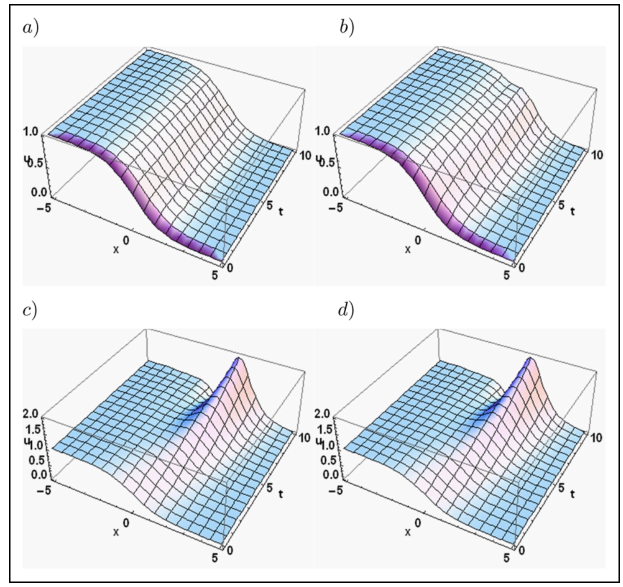

Figure 6. The surface graph of the exact solution $u(x,t)$ and the $u_4(x,t)$ approximate solution of the two-dimensional time fractional Fisher equation for $y = x$ (a) $u_4(x,t)$ when $\alpha = 0.1$, (b) $u_4(x,t)$ when $\alpha = 0.5$, (c) $u_4(x,t)$ when $\alpha = 0.9$, and (d) $u(x,t)$ when $\alpha = 1$.

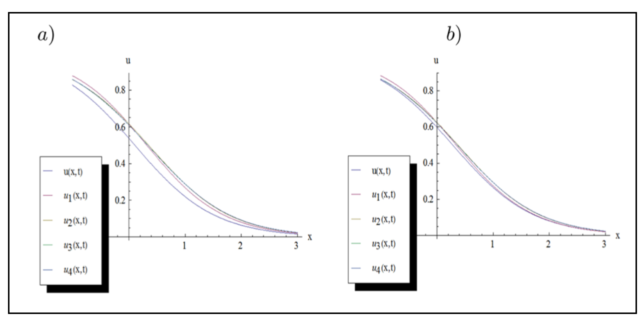

Figure 7. $u_k(x,t)$ solution of the two-dimensional time fractional Fisher equation when $k = 1, 2, 3$, and 4 versus its exact solution for $y = x$: (a) $\alpha = 0.1$, $t = 0.3$ and (b) $\alpha = 0.5$, $t = 0.8$.

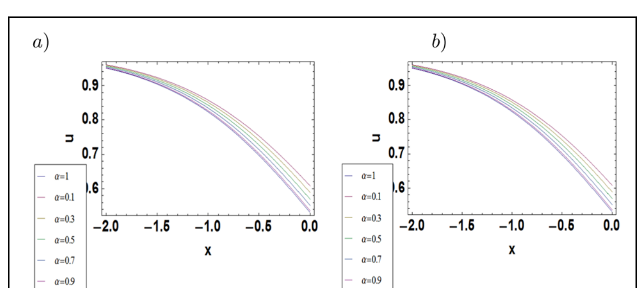

Figure 8. $u_4(x,t)$ solution of the two-dimensional time fractional fisher equation when $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$, and 1 for $y = x$: (a) $t = 0.05$ and (b) $t = 0.25$.

Table 2. Comparison between approximate solutions u_{RPSM}, u_{NIM}, and exact solution ($x = y = 0.5$).

	$u_2(\alpha = 0.8)$		$u_4(\alpha = 1)$		
t	u_{RPSM}	u_{NIM}	u_{RPSM}	u_{Exact}	$\lvert u_{Exact} - u_{RPSM}\rvert$
0.01	0.333229	0.333229	0.331345	0.331345	7.56291×10^{-10}
0.05	0.341156	0.341156	0.335791	0.335791	9.588×10^{-8}
0.1	0.349387	0.349387	0.341391	0.34139	7.8041×10^{-7}
0.15	0.356899	0.356899	0.347036	0.347033	2.67873×10^{-6}
0.2	0.364004	0.364004	0.352726	0.352719	6.4552×10^{-6}

consequences are introduced to illustrate the solutions. Thus, it is concluded that the RPSM becomes powerful and efficient in finding numerical solutions for a wide class of linear and nonlinear fractional differential equations. From the results, it is clear that the RPSM yields very accurate and convergent approximate solutions using only a few iterates in fractional problems. The work emphasized our belief that the present method can be applied as an alternative to get analytic solutions for different kinds of fractional linear and nonlinear partial differential equations applied in mathematics, physics, and engineering.

Declaration of conflicting interests

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