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On the Existence and Uniqueness of Solutions for Local Fractional Differential Equations

Hossein Jafari [1,2,*], Hassan Kamil Jassim [3], Maysaa Al Qurashi [4] and Dumitru Baleanu [5,6]

[1] Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar 47416, Iran
[2] Department of Mathematical Sciences, University of South Africa (UNISA), Pretoria 0003, South Africa
[3] Department of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar, Nasiriyah 64001, Iraq; hassan.kamil28@yahoo.com
[4] Department of Mathematics, King Saud University, P.O. Box 22452, Riyadh 11495, Saudi Arabia; Maysaa@ksu.edu.sa
[5] Department of Mathematics, Çankaya University, Ankara 06530, Turkey; dumitru@cankaya.edu.tr
[6] Institute of Space Sciences, P.O. Box MG-23, Magurele-Bucharest RO-76911, Romania
* Correspondence: jafari@umz.ac.ir; Tel: +98-11-3530-2466

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Abstract: In this manuscript, we prove the existence and uniqueness of solutions for local fractional differential equations (LFDEs) with local fractional derivative operators (LFDOs). By using the contracting mapping theorem (CMT) and increasing and decreasing theorem (IDT), existence and uniqueness results are obtained. Some examples are presented to illustrate the validity of our results.

Keywords: existence and uniqueness solutions; local fractional differential equations; contracting mapping theorem; local fractional operators

1. Introduction

Differential equations (DEs) with fractional order are generalizations of ordinary differential equations to non-integer order. Indeed, fractional differential equations have been subjected to many studies due to their frequent occurrence in different applications in physics, fluid mechanics, physiology, engineering, electrochemistry, and signals [1–8]. Therefore, numerical and analytical techniques have been developed to deal with fractional differential equations [9–11].

The existence and uniqueness of solutions of differential equations with the Riemann-Liouville fractional derivative and the Caputo fractional derivative using the Schauder fixed point theorem, the lower and upper solution method, the contracting mapping principle and the Leray-Schauder theory have been investigated in some papers [12–15].

Very recently in [16], the author studied the existence and uniqueness of solutions of some classes of differential equations with local fractional derivative operators. In this paper, we are interested in the existence and uniqueness of DEs with LFDOs of the form:

$$D_{\chi}^{2\alpha}\gamma(\chi) = \Omega(\chi,\gamma)\,,\, \gamma(\upsilon) = y_0\,,\; D_{\chi}^{\alpha}\gamma(\upsilon) = \gamma_1 \tag{1}$$

$$D_{\chi}^{3\alpha}\gamma(\chi) = \Omega(\chi,\gamma)\,,\, \gamma(\upsilon) = y_0\,,\; D_{\chi}^{\alpha}\gamma(\upsilon) = \gamma_1\,, D_{\chi}^{2\alpha}\gamma(\upsilon) = \gamma_2 \tag{2}$$

and a system of DEs with LFDOs of the form

$$D_{\tau}^{\alpha}X = \Phi(\tau, X)\,,\; X(\mu) = X_0 \tag{3}$$

where

$$X = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{bmatrix}, \; \Phi(\tau, X) = \begin{bmatrix} \Phi_1(\tau, \chi_1, \chi_2, \dots, \chi_n) \\ \Phi_2(\tau, \chi_1, \chi_2, \dots, \chi_n) \\ \vdots \\ \Phi_n(\tau, \chi_1, \chi_2, \dots, \chi_n) \end{bmatrix}, \; X_0 = \begin{bmatrix} \chi_{10} \\ \chi_{20} \\ \vdots \\ \chi_{n0} \end{bmatrix}$$

and D^{α}, $D^{2\alpha}$, and $D^{3\alpha}$ are the LFDOs of order α, 2α, and 3α respectively and υ, $\mu \in [\sigma, \omega]$. By using a variety of tools including the (CMT) and (IDT), existence and uniqueness results are obtained.

The rest of this paper is organized as follows. In Section 2, we give some necessary notations, definitions, and theorems. In Section 3, we study the existence and uniqueness of solutions of local fractional differential Equations (1)–(3) by using the contracting mapping theorem. Examples are given to illustrate our results in Section 4. Finally, in Section 5 we outline the main conclusions.

2. Basic Definitions and Preliminaries

In this section, we present some basic definitions and theorems that are used to prove our new results (see [16]).

Definition 1. *Let us consider that* $\Psi : [\sigma, \omega] \times R^{\alpha} \to R^{\alpha}$ *is LF continuous. We say that* $\Psi(\chi, \gamma)$ *satisfies a Lipschitz continuous (LC) if exists* $0 < \eta < 1$ *such that for all* $\chi \in [\sigma, \omega]$

$$|\Psi(\chi, \gamma_1) - \Psi(\chi, \gamma_2)| \leq \eta^{\alpha} |\gamma_1 - \gamma_2| , \qquad 0 < \alpha \leq 1$$

Definition 2. *A generalized normed linear space on* Λ *of fractional dimension* α, *is a mapping* $|| \cdot ||_{\alpha} : \Lambda \to R^{\alpha}$, *if it satisfies the following properties:*

1. $||\chi^{\alpha}||_{\alpha} \geq 0$ *&* $||\chi^{\alpha}||_{\alpha} = 0$ *if and only if* $\chi^{\alpha} = 0^{\alpha}$
2. $||\kappa^{\alpha}\chi^{\alpha}||_{\alpha} = |\kappa^{\alpha}| \; ||\chi^{\alpha}||_{\alpha}$
3. $||\chi^{\alpha} + \gamma^{\alpha}||_{\alpha} \leq \; ||\chi^{\alpha}||_{\alpha} + \; ||\gamma^{\alpha}||_{\alpha}$, *for* χ^{α}, $\gamma^{\alpha} \in \Lambda$ *and* $\kappa \in R$

Definition 3. *Let* $(X, \; || \cdot ||_{\alpha})$ *be a generalized Banach space (GBS), and let* $T : X \to X$ *be a map. If a number* $\beta^{\alpha} \in (0^{\alpha}, 1^{\alpha})$ *and*

$$||T(\chi^{\alpha}) - T(\gamma^{\alpha})||_{\alpha} \leq \beta^{\alpha} ||\chi^{\alpha} - \gamma^{\alpha}||_{\alpha}$$

for χ^{α}, $\gamma^{\alpha} \in X$. *Then* T *is called contraction mapping (CM) on GBS* $(X, \; || \cdot ||_{\alpha})$.

Definition 4. *Let* $(X, \; || \cdot ||_{\alpha})$ *be a GBS and let* $T : X \to X$. *If* $\chi^{\alpha} \in X$ *and* $||T\chi^{\alpha} - \chi^{\alpha}||_{\alpha} = 0$, *then* χ^{α} *is called a fixed point (FP) of* T.

Theorem 1. (*CMT*)*: A contracting mapping* T *defined on a complete GBS* $(X, \; || \cdot ||_{\alpha})$ *has a unique FP.*

Proof. See [14].

Theorem 2. *Assume that* $\Omega \in C_{\alpha}[\sigma, \omega]$ *(*C_{α} *is called a LF continuous set) and* $\Omega \in D_{\alpha}(\sigma, \omega)$ *(*D_{α} *is called a LF derivative set). Then a point* $\tau \in (\sigma, \omega)$ *with*

$$\Omega(\omega) - \Omega(\sigma) = \Omega^{(\alpha)}(\tau)(\omega - \sigma)^{\alpha}, \; 0 < \alpha \leq 1$$

Proof. See [14].

Theorem 3. (Increasing and Decreasing Theorem)

1. *If* $\Omega^{(\alpha)}(\chi) > 0$ *on* $[\sigma, \omega]$, *then* $\Omega(\chi)$ *is an increasing on that interval.*
2. *If* $\Omega^{(\alpha)}(x) < 0$ *on* $[\sigma, \omega]$, *then* $\Omega(\chi)$ *is a decreasing on that interval.*

Proof. See in [14].

3. Main Results

Here, we investigate the existence and uniqueness of solutions of the LFDEs (1), (2) and (3). First, we prove the existence and uniqueness of solutions of the LFDEs by applying (the CMT).

Theorem 4. *Let us consider that* $T : X \to X$ *is a map on the complete GBS* $(X, \ ||\cdot||_{\alpha})$ *such that for some* $m \geq 1$, T^m *is contracting. Then T has a unique FP.*

Proof. Since T^m is CM on X, then we have T^m has a unique FP ξ^{α}.

$$\begin{aligned} ||T\xi^{\alpha} - \xi^{\alpha}||_{\alpha} &= ||T^{m+1}\xi^{\alpha} - T^m\xi^{\alpha}||_{\alpha} \\ &= ||T^m(T\xi^{\alpha}) - T^m(\xi^{\alpha})||_{\alpha} \\ &\leq \beta^{\alpha}||T\xi^{\alpha} - \xi^{\alpha}||_{\alpha} \end{aligned}$$

Moreover, since $\beta^{\alpha} \in (0^{\alpha}, 1^{\alpha})$, then $||T\xi^{\alpha} - \xi^{\alpha}||_{\alpha} = 0$, in other words, $T\xi^{\alpha} = \xi^{\alpha}$. Therefore, ξ^{α} is FP of T. For uniqueness, assume that ξ_1^{α}, ξ_2^{α} are FPs of T such that $T\xi_1^{\alpha} = \xi_1^{\alpha}$, $T\xi_2^{\alpha} = \xi_2^{\alpha}$ then ξ_1^{α}, ξ_2^{α} FPs of T^m. Therefore $\xi_1^{\alpha} = \xi_2^{\alpha}$.

Theorem 5. *Assume that* $\Omega : [\sigma, \omega] \times R^{\alpha} \to R^{\alpha}$ *is LF continuous map. Then* Ω *is LC.*

Proof. Since $\frac{\partial^{\alpha}\Omega}{\partial\gamma^{\alpha}}$ is LF continuous, then it attains a maximum value, denoted by

$$\eta^{\alpha} = \max_{(\chi,\gamma)\in Dom\,\Omega} \left| \frac{\partial^{\alpha}\Omega(\chi,\gamma)}{\partial\gamma^{\alpha}} \right|$$

Now, let us consider (χ, γ_1), $(\chi, \gamma_2) \in Dom\,\Omega$.
Using Theorem 2, there is a point $(\chi, \tau) \in Dom\,\Omega$ such that:

$$|\Omega(\chi,\gamma_1) - \Omega(\chi,\gamma_2)| = \left| \frac{\partial^{\alpha}\Omega(\chi,\tau)}{\partial\gamma^{\alpha}} \right| |\gamma_1 - \gamma_2| \leq \eta^{\alpha}|\gamma_1 - \gamma_2|$$

Theorem 6. *If* $\upsilon \in [\sigma, \omega]$ *closed interval,* β_0, $\beta_1 \in R^{\alpha}$ *and* $\Omega : [\sigma, \omega] \times R^{\alpha} \to R^{\alpha}$ *is an LF continuous function and satisfies a LC, then the LFDE:*

$$D_{\chi}^{2\alpha}\gamma(\chi) = \Omega(\chi, \gamma) \tag{4}$$

subject to the initial conditions

$$\gamma(\upsilon) = \beta_0, \gamma^{(\alpha)}(\upsilon) = \beta_1 \tag{5}$$

has a unique solution in $C_{\alpha}[\sigma, \omega]$.

Proof. Let the map $T : C_{\alpha}[\sigma, \omega] \to C_{\alpha}[\sigma, \omega]$ be defined by

$$T\,\zeta(\chi) = \beta_0 + \frac{(\chi-\upsilon)^{\alpha}}{\Gamma(1+\alpha)}\beta_1 + \frac{1}{\Gamma(1+\alpha)}\int_{\upsilon}^{\chi} \frac{(\chi-\tau)^{\alpha}}{\Gamma(1+\alpha)}\Omega(\tau, \zeta(\tau))\,(d\tau)^{\alpha} \tag{6}$$

and $\zeta_1(\chi)$, $\zeta_2(\chi) \in C_\alpha[\sigma, \omega]$.

We first prove by induction that

$$\begin{aligned} ||\, T^n\zeta_1(\chi) - T^n\zeta_2(\chi)\,||_\alpha &\le \frac{\eta^{n\alpha}\,|\chi-\upsilon|^{2n\alpha}}{\Gamma^n(1+2\alpha)}\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha \\ &\le \frac{\eta^{n\alpha}\,|\omega-\sigma|^{2n\alpha}}{\Gamma^n(1+2\alpha)}\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha, \qquad n = 1, 2, \ldots\ldots \end{aligned} \tag{7}$$

In fact, for $n = 1$, we obtain

$$\begin{aligned} ||\, T\zeta_1(\chi) - T\zeta_2(\chi)\,||_\alpha &= |T\zeta_1(\chi) - T\zeta_2(\chi)| \\ &= \left|\frac{1}{\Gamma(1+\alpha)}\int_\upsilon^\chi \frac{(\chi-\tau)^\alpha}{\Gamma(1+\alpha)}\,[\Omega(\tau,\zeta_1(\tau)) - \Omega(\tau,\zeta_2(\tau))]\,(d\tau)^\alpha\right| \\ &\le \left|\frac{1}{\Gamma(1+\alpha)}\int_\upsilon^\chi \frac{(\chi-\tau)^\alpha}{\Gamma(1+\alpha)}\eta^\alpha\,|\zeta_1(\chi) - \zeta_2(\chi)|\,(d\tau)^\alpha\right| \\ &\le \eta^\alpha\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha\left|\frac{1}{\Gamma(1+\alpha)}\int_\upsilon^\chi \frac{(\chi-\tau)^\alpha}{\Gamma(1+\alpha)}(d\tau)^\alpha\right| \\ &\le \frac{\eta^\alpha\,|\chi-\upsilon|^{2\alpha}}{\Gamma(1+2\alpha)}\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha \\ &\le \frac{\eta^\alpha\,|\omega-\sigma|^{2\alpha}}{\Gamma(1+2\alpha)}\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha \end{aligned}$$

For $n = 2$,

$$\begin{aligned} ||\, T^2\zeta_1(\chi) - T^2\zeta_2(\chi)\,||_\alpha &= |T^2\zeta_1(\chi) - T^2\zeta_2(\chi)| \\ &= \left|\frac{1}{\Gamma(1+\alpha)}\int_\upsilon^\chi \frac{(\chi-\tau)^\alpha}{\Gamma(1+\alpha)}\,[\Omega(\tau,T\zeta_1(\tau) - \Omega(\tau,T\zeta_2(\tau))]\,(d\tau)^\alpha\right| \\ &\le \left|\frac{1}{\Gamma(1+\alpha)}\int_\upsilon^\chi \frac{(\chi-\tau)^\alpha}{\Gamma(1+\alpha)}\eta^\alpha\,|T\zeta_1(\chi) - T\zeta_2(\chi)|\,(d\tau)^\alpha\right| \\ &\le \frac{\eta^{2\alpha}\,|\chi-\upsilon|^{2\alpha}}{\Gamma(1+2\alpha)}\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha\left|\frac{1}{\Gamma(1+\alpha)}\int_\upsilon^\chi \frac{(\chi-\tau)^\alpha}{\Gamma(1+\alpha)}(d\tau)^\alpha\right| \\ &\le \frac{\eta^{2\alpha}\,|\chi-\upsilon|^{4\alpha}}{\Gamma^2(1+2\alpha)}\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha \\ &\le \frac{\eta^{2\alpha}\,|\omega-\sigma|^{4\alpha}}{\Gamma^2(1+2\alpha)}\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha \end{aligned}$$

We suppose the desired inequality holds for $n = k$.

$$\begin{aligned} ||\, T^k\zeta_1(\chi) - T^k\zeta_2(\chi)\,||_\alpha &\le \frac{\eta^{k\alpha}\,|\chi-\upsilon|^{2k\alpha}}{\Gamma^n(1+2\alpha)}\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha \\ &\le \frac{\eta^{k\alpha}\,|\omega-\sigma|^{2k\alpha}}{\Gamma^n(1+2\alpha)}\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha \end{aligned}$$

Then,

$$\begin{aligned} ||\, T^{k+1}\zeta_1(\chi) - T^{k+1}\zeta_2(\chi)\,||_\alpha &= \left|T^{k+1}\zeta_1(\chi) - T^{k+1}\zeta_2(\chi)\right| \\ &= \left|\frac{1}{\Gamma(1+\alpha)}\int_\upsilon^\chi \frac{(\chi-\tau)^\alpha}{\Gamma(1+\alpha)}\left[\Omega(\tau,T^k\zeta_1(\tau) - \Omega(\tau,T^k\zeta_2(\tau))\right]\,(d\tau)^\alpha\right| \\ &\le \left|\frac{1}{\Gamma(1+\alpha)}\int_\upsilon^\chi \frac{(\chi-\tau)^\alpha}{\Gamma(1+\alpha)}\eta^\alpha\left|T^k\zeta_1(\chi) - T^k\zeta_2(\chi)\right|\,(d\tau)^\alpha\right| \\ &\le \left|\frac{1}{\Gamma(1+\alpha)}\int_\upsilon^\chi \frac{(\chi-\tau)^\alpha}{\Gamma(1+\alpha)}\eta^\alpha\,||\,T^k\zeta_1(\chi) - T^k\zeta_2(\chi)\,||_\alpha\,(d\tau)^\alpha\right| \\ &\le \frac{\eta^{(k+1)\alpha}\,|\chi-\upsilon|^{2k\alpha}}{\Gamma^k(1+2\alpha)}\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha\left|\frac{1}{\Gamma(1+\alpha)}\int_\upsilon^\chi \frac{(\chi-\tau)^\alpha}{\Gamma(1+\alpha)}(d\tau)^\alpha\right| \\ &\le \frac{\eta^{(k+1)\alpha}\,|\chi-\upsilon|^{2(k+1)\alpha}}{\Gamma^{k+1}(1+2\alpha)}\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha \\ &\le \frac{\eta^{(k+1)\alpha}\,|\omega-\sigma|^{2(k+1)\alpha}}{\Gamma^{k+1}(1+2\alpha)}\,||\,\zeta_1(\chi) - \zeta_2(\chi)\,||_\alpha \end{aligned}$$

Hence, the estimates (7) hold.

Now, we have

$$\frac{\eta^{n\alpha}\ |\omega-\sigma|^{2n\alpha}}{\Gamma^{n}(1+2\alpha)}\,||\,\zeta_1(\chi)-\zeta_2(\chi)\,||_{\alpha}\to 0\quad \text{as}\quad n\to\infty$$

So far n sufficiently large

$$0<\frac{\eta^{n\alpha}\ |\omega-\sigma|^{2n\alpha}}{\Gamma^{n}(1+2\alpha)}<1$$

and T^n is a contraction on $C_\alpha[\sigma,\omega]$.

Therefore T has a unique fixed point in $C_\alpha[\sigma,\omega]$, which gives a unique solution to the local fractional differential.

Theorem 7. *If $\upsilon\in[\sigma,\omega]$, β_0, β_1, $\beta_2\in R^\alpha$ and $\Omega:[\sigma,\omega]\times R^\alpha\to R^\alpha$ is a local fractional continuous function and satisfies a Lipschitz continuous, then the local fractional differential equation:*

$$D_\chi^{3\alpha}\gamma(\chi)=\Omega(\chi,\gamma) \tag{8}$$

$$\gamma(\upsilon)=\beta_0,\ \gamma^{(\alpha)}(\upsilon)=\beta_1,\ \gamma^{(2\alpha)}(\upsilon)=\beta_2 \tag{9}$$

has a unique solution in $[\sigma,\omega]$.

Proof. Let the map $T:C_\alpha[\sigma,\omega]\to C_\alpha[\sigma,\omega]$ be defined by

$$T\,\zeta(\chi)=\beta_0+\frac{(\chi-\upsilon)^{\alpha}}{\Gamma(1+\alpha)}\beta_1+\frac{(\chi-\upsilon)^{2\alpha}}{\Gamma(1+2\alpha)}\beta_2+\frac{1}{\Gamma(1+\alpha)}\int_{\upsilon}^{\chi}\frac{(\chi-\tau)^{2\alpha}}{\Gamma(1+2\alpha)}\Omega(\tau,\zeta(\tau))\,(d\tau)^{\alpha} \tag{10}$$

and $\zeta_1(\chi)$, $\zeta_2(\chi)\in C_\alpha[\sigma,\omega]$.

We claim that for all n,

$$\begin{aligned} ||\,T^n\zeta_1(\chi)-T^n\zeta_2(\chi)\,||_{\alpha} &\le \tfrac{\eta^{n\alpha}\,|\chi-\upsilon|^{3n\alpha}}{\Gamma^{n}(1+3\alpha)}\,||\,\zeta_1(\chi)-\zeta_2(\chi)\,||_{\alpha}\\ &\le \tfrac{\eta^{n\alpha}\,|\omega-\sigma|^{3n\alpha}}{\Gamma^{n}(1+3\alpha)}\,||\,\zeta_1(\chi)-\zeta_2(\chi)\,||_{\alpha}, \qquad n=1\,,2\,,\ldots\ldots \end{aligned} \tag{11}$$

The case is $n=1$ has already shown. The induction step is as follows:

$$\begin{aligned} ||\,T^{n+1}\zeta_1(\chi)-T^{n+1}\zeta_2(\chi)\,||_{\alpha} &= \left|T^{n+1}\zeta_1(\chi)-T^{n+1}\zeta_2(\chi)\right|\\ &= \left|\tfrac{1}{\Gamma(1+\alpha)}\int_{\upsilon}^{\chi}\tfrac{(\chi-\tau)^{2\alpha}}{\Gamma(1+2\alpha)}\left[\Omega(\tau,T^n\zeta_1(\tau)-\Omega(\tau,T^n\zeta_2(\tau))\right]\,(d\tau)^{\alpha}\right|\\ &\le \left|\tfrac{1}{\Gamma(1+\alpha)}\int_{\upsilon}^{\chi}\tfrac{(\chi-\tau)^{2\alpha}}{\Gamma(1+2\alpha)}\eta^{\alpha}\left|T^n\zeta_1(\chi)-T^n\zeta_2(\chi)\right|\,(d\tau)^{\alpha}\right|\\ &\le \left|\tfrac{1}{\Gamma(1+\alpha)}\int_{\upsilon}^{\chi}\tfrac{(\chi-\tau)^{2\alpha}}{\Gamma(1+2\alpha)}\eta^{\alpha}\,||\,T^n\zeta_1(\chi)-T^n\zeta_2(\chi)\,||_{\alpha}\,(d\tau)^{\alpha}\right|\\ &\le \tfrac{\eta^{(n+1)\alpha}\,|\chi-\upsilon|^{3n\alpha}}{\Gamma^{n}(1+3\alpha)}\,||\,\zeta_1(\chi)-\zeta_2(\chi)\,||_{\alpha}\left|\tfrac{1}{\Gamma(1+\alpha)}\int_{\upsilon}^{\chi}\tfrac{(\chi-\tau)^{2\alpha}}{\Gamma(1+2\alpha)}(d\tau)^{\alpha}\right|\\ &\le \tfrac{\eta^{(n+1)\alpha}\,|\chi-\upsilon|^{3(n+1)\alpha}}{\Gamma^{n+1}(1+3\alpha)}\,||\,\zeta_1(\chi)-\zeta_2(\chi)\,||_{\alpha}\\ &\le \tfrac{\eta^{(k+1)\alpha}\,|\omega-\sigma|^{3(n+1)\alpha}}{\Gamma^{n+1}(1+3\alpha)}\,||\,\zeta_1(\chi)-\zeta_2(\chi)\,||_{\alpha} \end{aligned}$$

Hence, the estimates (11) hold.

Now, we have

$$\frac{\eta^{n\alpha}\ |\omega-\sigma|^{3n\alpha}}{\Gamma^{n}(1+3\alpha)}\,||\,\zeta_1(\chi)-\zeta_2(\chi)\,||_{\alpha}\ \to 0\ \text{ as }\ n\to\infty$$

So far n is sufficiently large

$$0<\frac{\eta^{n\alpha}\ |\omega-\sigma|^{3n\alpha}}{\Gamma^{n}(1+3\alpha)}<1$$

and T^n is a contraction on $C_\alpha[\sigma,\omega]$.

Therefore T has a unique fixed point in $C_\alpha[\sigma,\omega]$. This gives an unique solution to the local fractional differential.

Next, we apply the increasing and decreasing test to prove the uniqueness of the solution of the system of LFDEs.

Theorem 8. *If $\mu\in[\sigma,\omega]$, and $\Phi:\ [\sigma,\omega]\times R_n^\alpha\to R_n^\alpha$ is LF continuous and satisfies a Lipschitz continuous, then there is at most one solution $X(\tau)$ of the local fractional differential system*

$$D_\tau^\alpha X=\Phi(\tau,X) \tag{12}$$

that satisfies a given initial condition $X(\mu)=X_0\ \in R_n^\alpha$.

Proof. Suppose that $\Phi(\tau,X)$ satisfies the LC.

$$||\,\Phi(\tau,\,X)-\Phi(\tau,\,Y)\,||_{\alpha}\le\eta^\alpha\,||\,X-Y\,||_{\alpha}$$

For any $(\tau,\ X),\,(\tau,\,Y)\ \in\ Dom\,\Phi$. Let $X(\tau)\ =\ [\chi_1(\tau)\,,\ \chi_2(\tau)\,,\ \ldots,\ \chi_n(\tau)]$ and $Y(\tau)\ =\ [\gamma_1(\tau)\,,\gamma_2(\tau)\,,\ldots,\gamma_n(\tau)]$ be two solutions of (12) such that $X(\mu)=Y(\mu)=X_0$.

Let $\vartheta(\tau)=\,||\,X(\tau)-Y(\tau)\,||_{\alpha}^{2}=\sum\limits_{i=1}^{n}\left[\chi_i(\tau)-\gamma_i(\tau)\right]^2$, then

$$\begin{aligned}\tfrac{d^\alpha}{d\tau^\alpha}\vartheta(\tau)\ &=\sum_{i=1}^{n}2\left[\chi_i(\tau)-\gamma_i(\tau)\right]\ \left[\chi_i^{(\alpha)}(\tau)-\gamma_i^{(\alpha)}(\tau)\right]\\&=\sum_{i=1}^{n}2\left[\chi_i(\tau)-\gamma_i(\tau)\right]\ \left[\Phi_i(\tau,\,X(\tau)-\Phi_i(\tau,\,Y(\tau))\right]\\&=2\left[X(\tau)-Y(\tau)\right]\left[\Phi(\tau,\,X(\tau)-\Phi(\tau,\,Y(\tau))\right]\end{aligned}$$

Since $\frac{d^\alpha}{d\tau^\alpha}\vartheta(\tau)\le\left|\frac{d^\alpha}{d\tau^\alpha}\vartheta(\tau)\right|$, we obtain

$$\begin{aligned}\tfrac{d^\alpha}{d\tau^\alpha}\vartheta(\tau)\ &\le 2\left|\Phi(\tau,\,X(\tau)-\Phi(\tau,\,Y(\tau))\right|\,|X-Y|\\&\le 2\,||\,\Phi(\tau,\,X)-\Phi(\tau,\,Y)\,||_{\alpha}\,||\,X-Y\,||_{\alpha}\\&\le 2\eta^\alpha\,||\,X-Y\,||_{\alpha}^{2}=2\eta^\alpha\vartheta(\tau)\,.\end{aligned}$$

Hence, we have

$$\vartheta^{(\alpha)}(\tau)-2\eta^\alpha\vartheta(\tau)\le 0$$

$$\left(\vartheta^{(\alpha)}(\tau)-2\eta^\alpha\vartheta(\tau)\right)\ E_\alpha(-2\eta^\alpha\tau^\alpha)\le 0$$

$$\frac{d^\alpha}{d\tau^\alpha}\left[\vartheta(\tau)\,E_\alpha(-2\eta^\alpha\tau^\alpha)\right]\le 0$$

Thus, $\vartheta(\tau)\,E_\alpha(-2\eta^\alpha\tau^\alpha)$ is a decreasing function.

Therefore $\vartheta(\tau)\,E_\alpha(-2\eta^\alpha\tau^\alpha)\le\vartheta(\sigma)\,E_\alpha(-2\eta^\alpha\sigma^\alpha)=0$, for $\tau>\sigma$.

Since $\vartheta(\tau)\ \ge 0$, we obtain $\vartheta(\tau)\ =0$, for $\tau>\sigma$. Hence, $X(\tau)=Y(\tau)$, for $\tau>\sigma$.

Now, let us consider the case where $\tau < \sigma$.
Since $-\frac{d^\alpha}{d\tau^\alpha}\vartheta(\tau) \leq \left|\frac{d^\alpha}{d\tau^\alpha}\vartheta(\tau)\right|$, we have

$$-\frac{d^\alpha}{d\tau^\alpha}\vartheta(\tau) \leq 2\eta^\alpha\vartheta(\tau)$$

$$\vartheta^{(\alpha)}(\tau) + 2\eta^\alpha\vartheta(\tau) \geq 0$$

$$\left(\vartheta^{(\alpha)}(\tau) + 2\eta^\alpha\vartheta(\tau)\right)\, E_\alpha(2\eta^\alpha\tau^\alpha) \geq 0$$

$$\frac{d^\alpha}{d\tau^\alpha}\left[\vartheta(\tau)\, E_\alpha(2\eta^\alpha\tau^\alpha)\right] \geq 0$$

So $\vartheta(\tau)\, E_\alpha(2\eta^\alpha\tau^\alpha)$ is an increasing function.
Therefore $\vartheta(\tau)\, E_\alpha(2\eta^\alpha\tau^\alpha) \leq \vartheta(\sigma)\, E_\alpha(2\eta^\alpha\sigma^\alpha) = 0$, for $\tau < \sigma$.
Since $\vartheta(\tau) \geq 0$, we obtain $\vartheta(\tau) = 0$, for $\tau < \sigma$.
Hence, $X(\tau) = Y(\tau)$, for $\tau < \sigma$.

4. Applications

To illustrate the application of our results, let us consider the following examples.

Example 1. *The local fractional IVP*

$$\gamma^{(2\alpha)} + \gamma = \frac{\chi^\alpha}{\Gamma(1+\alpha)},\ \gamma(0) = 1\,,\ \gamma^{(\alpha)}(0) = 0\,. \tag{13}$$

has a unique solution.

For this initial value problem, the integral operator T is defined as

$$\begin{aligned} T\,\zeta(\chi) &= 1 + \tfrac{1}{\Gamma(1+\alpha)}\int_0^\chi \tfrac{(\chi-\eta)^\alpha}{\Gamma(1+\alpha)}\left[\tfrac{\eta^\alpha}{\Gamma(1+\alpha)} - \zeta(\eta)\right](d\eta)^\alpha \\ &= 1 + \tfrac{\eta^{3\alpha}}{\Gamma(1+3\alpha)} - \tfrac{1}{\Gamma(1+\alpha)}\int_0^\chi \tfrac{(\chi-\eta)^\alpha}{\Gamma(1+\alpha)}\,\zeta(\eta)(d\eta)^\alpha \end{aligned}$$

It is clear that $T\,\zeta(\chi)$ is CM.
Now let $\zeta(\chi) = cos_\alpha(\chi^\alpha) - sin_\alpha(\chi^\alpha) - \frac{\chi^\alpha}{\Gamma(1+\alpha)}$. Then

$$\begin{aligned} T\,\zeta(\chi) &= 1 + \tfrac{\chi^{3\alpha}}{\Gamma(1+3\alpha)} - \tfrac{1}{\Gamma(1+\alpha)}\int_0^\chi \tfrac{(\chi-\eta)^\alpha}{\Gamma(1+\alpha)}\left[\cos_\alpha(\eta^\alpha) - \sin_\alpha(\eta^\alpha) - \tfrac{\eta^\alpha}{\Gamma(1+\alpha)}\right](d\eta)^\alpha \\ &= \cos_\alpha(\chi^\alpha) - \sin_\alpha(\chi^\alpha) - \tfrac{\chi^\alpha}{\Gamma(1+\alpha)} \end{aligned}$$

In other words, $T\,\zeta(\chi) = \zeta(\chi)$, so $\zeta(\chi)$ is a unique fixed point of T, which gives a unique solution to the local fractional IVP (13).

Example 2. *The LFDE*

$$\gamma^{(3\alpha)} - \gamma + \frac{\chi^\alpha}{\Gamma(1+\alpha)} = 0\,,\gamma(0) = 1\,,\ \gamma^{(\alpha)}(0) = 2\,,\ \gamma^{(2\alpha)}(0) = 1 \tag{14}$$

has a unique solution.

For this initial value problem, the integral operator T is defined as

$$T\,\zeta(\chi) = 1 + \frac{2\chi^\alpha}{\Gamma(1+\alpha)} + \frac{\chi^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{1}{\Gamma(1+\alpha)}\int_0^\chi \frac{(\chi-\eta)^{2\alpha}}{\Gamma(1+2\alpha)}\left[\zeta(\eta) - \frac{\eta^\alpha}{\Gamma(1+\alpha)}\right](d\eta)^\alpha$$

Then $T\,\zeta(\chi)$ *is a contracting map.*
Now let $\zeta(\chi) = E_\alpha(\chi^\alpha) + \frac{\chi^\alpha}{\Gamma(1+\alpha)}$
Therefore,

$$\begin{aligned} T\,\zeta(\chi) &= 1 + \tfrac{2\chi^\alpha}{\Gamma(1+\alpha)} + \tfrac{\chi^{2\alpha}}{\Gamma(1+2\alpha)} + \tfrac{1}{\Gamma(1+\alpha)} \int_0^\chi \tfrac{(\chi-\eta)^{2\alpha}}{\Gamma(1+2\alpha)} \left[E_\alpha(\eta^\alpha) + \tfrac{\eta^\alpha}{\Gamma(1+\alpha)} - \tfrac{\eta^\alpha}{\Gamma(1+\alpha)} \right] (d\eta)^\alpha \\ &= E_\alpha(\chi^\alpha) + \tfrac{\chi^\alpha}{\Gamma(1+\alpha)} \end{aligned}$$

In other words, $T\,\zeta(\chi) = \zeta(\chi)$*, so* $\zeta(\chi)$ *is a unique FP of* T*, which gives a unique solution to the LFDE (14).*

5. Conclusions

We have presented some existence and uniqueness results for an initial value problem of local fractional differential equations (LFDEs) and a system of LFDEs with local fractional derivative operators. The proof of the existence and uniqueness of the solutions is proved by applying the contracting mapping theorem while the uniqueness of solutions for system of LFDEs is proved by applying the increasing and decreasing theorem. The present work can be extended to nonlinear differential equations with local fractional derivative operators.

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