ON LEFT-DEFINITE STURM-LIOUVILLE EQUATIONS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES OF
ÇANKAYA UNIVERSITY

BY
ATİLLA ARAS

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
IN
MATHEMATICS AND COMPUTER SCIENCE

JUNE 2017

Title of the Thesis: **On Left-Definite Sturm-Liouville Equations.**

Submitted by **Atilla ARAS**

Approval of the Graduate School of Natural and Applied Sciences, Çankaya University.

Prof. Dr. Can ÇOĞUN
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Fahd JARAD
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assist. Prof. Dr. Ekin UĞURLU
Supervisor

Examination Date: 01.06.2017

Examining Committee Members

Prof. Dr. Elgiz BAYRAM	(Ankara Univ.)	
Assist. Prof. Dr. Ekin UĞURLU	(Çankaya Univ.)	
Assoc. Prof. Dr. Fahd JARAD	(Çankaya Univ.)	

STATEMENT OF NON-PLAGIARISM PAGE

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name	:	Atilla ARAS
Signature	:	
Date	:	01.06.2017

ABSTRACT

ON LEFT-DEFINITE STURM-LIOUVILLE EQUATIONS

ARAS, Atilla

M.Sc., Department of Mathematics and Computer Science

Supervisor: Assist. Prof. Dr. Ekin UĞURLU

June 2017, 61 pages

Sturm-Liouville equations are very important to understand the nature of the real-world problems and have been investigated by many authors. To investigate the spectral properties of these problems it is convenient to construct the Hilbert space. Such a construction is done with the help of the weight function. In 1992, A.M. Krall studied on the second order equation $-(pg')' + qg = \lambda wg$, where p, q, w are real-valued functions with 1/p, q, w > 0 on the given interval [c,d] subject to some boundary conditions in the Sobolev space. Such equations are called left-definite equations. He also investigated the left definite fourth order equations and Hamiltonian systems on the finite intervals. Moreover, Race and Krall studied on the Weyl theory for a left-definite second order equation. Using these obtained results second-order, fourth-order equations and Hamiltonian systems are studied on finite and infinite intervals in this thesis.

Keywords: Sturm-Liouville equations, Hilbert space, Sobolev space, left-definite equations.

ÖZ

SOL BELİRLİ STURM-LIOUVILLE DENKLEMLERİ ÜZERİNE

ARAS, Atilla
Yüksek Lisans, Matematik-Bilgisayar Anabilim Dalı
Tez Yöneticisi: Yrd. Doç. Dr. Ekin UĞURLU
Haziran 2017, 61 sayfa

Sturm-Liouville denklemleri gerçek dünya problemlerinin yapısını anlamada çok önemlidir ve birçok yazar tarafından araştırılmıştır. Bu problemlerin spektral özelliklerini araştırmak için Hilbert uzayını inşa etmek uygun olmaktadır. Bu inşa ağırlık fonksiyonunun yardımıyla yapılmaktadır. 1992'de, A.M. Krall verilmiş [c,d] aralığında Sobolev uzayında bazı sınır şartlarına tabi 1/p, q, w > 0 ile p, q, w'nin reel değerli fonksiyonlar olduğu ikinci mertebeden $-(pg')' + qg = \lambda wg$ diferansiyel denklemi üzerine çalışmıştır. Bu denklemler sol belirli denklemler olarak adlandırılmaktadır. Krall sonlu aralıklarda sol belirli dördüncü mertebeden denklemler ve Hamilton sistemleri üzerine de araştırmalar yapmıştır. Bundan başka, Race ve Krall sol belirli ikinci mertebeden bir denklem için Weyl teorisi üzerine çalışmıştır. Bu tezde, elde edilmiş bu sonuçları kullanarak, ikinci mertebeden, dördüncü mertebeden denklemler ve Hamilton sistemleri sonlu ve sonsuz aralıklarda çalışılmıştır.

Anahtar Kelimeler: Sturm-Liouville denklemleri, Hilbert uzayı, Sobolev uzayı, Sol belirli denklemler

ACKNOWLEDGEMENTS

The author wishes to express his deepest gratitude to his supervisor Assist. Prof. Dr. Ekin UĞURLU for his guidance, advice, criticism, encouragements and insight throughout the research.

TABLE OF CONTENTS

LIST OF SYMBOLS AND ABBREVIATIONS

AC_{loc}	The Set of Absolutely Continuous Functions
BVP	Boundary Value Problem
$\mathbb{C}$	Complex Numbers
H^1	Sobolev Space
L_w^2	Hilbert Space
L^*	Adjoint of Operator L

PRELIMINARY

Definition 1.1 [1]. A vector space R is Euclidean if for all g, u belonging to R there exists a defined complex-valued function, denoted by (g, u), which satisfies the following conditions:

i) $(g, g) \geq 0$; $(g, g) = 0$ only if $g = 0$;

ii) $(g, u) = \overline{(u, g)}$;

iii) $(\lambda g, u) = \lambda(g, u)$;

iv) $(g_1 + g_2, u) = (g_1, u) + (g_2, u)$.

The function (g, u) is said to be the inner product of g and u.

Definition 1.2 [1]. The Cauchy-Schwartz inequality exists in any Euclidean space:

$$|(g, u)|^2 \leq (g, g)\,(u, u).$$

Definition 1.3 [1]. The number $\|g\| = \sqrt{(g, g)}$ is called the norm of the vector g. The norm has the following properties

i) $\|g\| \geq 0$;

ii) $\|g\| = 0$ if and only if $g = 0$;

iii) $\|\lambda g\| = |\lambda|\,\|g\|$;

iv) $\|g + u\| \leq \|g\| + \|u\|$.

Definition 1.4 [1]. A sequence g_1, g_2, g_3, ... is called a fundamental sequence (or Cauchy sequence) if for any positive number ε, there is an integer D>0 such that the inequality

$$\|g_m - g_n\| < \varepsilon, \quad m,n > D,$$

is satisfied.

Definition 1.5 [1]. Two vectors g, u are called orthogonal if (g,u) = 0.

Definition 1.6 [1]. A vector g is said to be normalized if $\|g\|$ = 1. For g ≠0 the vector u = g / $\|g\|$ is normalized. An orthogonal system of which all the vectors are normalized is called an orthonormal system.

Definition 1.7 [1]. A Euclidean space in which there exists a denumerable, complete, orthonormal system is called a Hilbert space.

Definition 1.8 [1]. It is denoted by L_w^2 (c,d) the aggregate of all complex-valued functions g(x) that are measurable and quadratically summable over the fixed interval (c,d) (which can be finite or infinite) with respect to a positive function w(x). The inner product is denoted as

$$(g,u) = \int_c^d g(x)\,\overline{u(x)}\,w(x)\,dx.$$

L_w^2 (c,d) is said to be a Hilbert space.

Definition 1.9 [2]. The space

$$W_2^l(c,d) = \{s \in L_W^2\ (c,d) : \forall h \in \{1,\ldots\ldots m\},\ s^{(h)} \in L_W^2\ (c,d)\},$$

where m∈ N_0 is said to be a Sobolev space. For s ∈ W_2^l(c,d) it is set

$$\|s\|_{2,l} = \left(\sum_{h=0}^{m} \left\|s^{(h)}\right\|^{2}\right)^{1/2},$$

where $\|.\|$ is the norm on L_w^2 (c,d).

Definition 1.10 [2]. W_2^l(c,d) is said to be a Banach space with respect to the norm $\|.\|_{2,l}$

Definition 1.11 [1]. An operator V which is defined on the whole Banach space D is called bounded if there is a positive number J such that

$$\|Vg\| \leq J\,\|g\| \text{ for all } g \in D.$$

The smallest number J is said to be the norm of the bounded operator V and is shown as $\|V\|$.

Definition 1.12 [1]. An operator V is called Hermitian if for all g,u ∈ Ɗ (V), domain of V,

$$(Vg, u) = (g, Vu)$$

takes place. A Hermitian operator is a symmetric operator if its domain of definition is dense in the Hilbert space F. An operator whose a domain of definition is dense in F is called self-adjoint if V= V*.

Lemma 1.13 [1]. When E is a symmetric extension of a symmetric operator D, we possess

$$D \subset E$$

and so

$$E^{*} \subset D^{*}.$$

Since E is a symmetric operator,

$$E \subset E^{*}$$

holds and so we possess

$$D \subset E \subset E^{*} \subset D^{*}.$$

Definition 1.14 [1]. A symmetric operator D is said to be maximal if it does not possess proper symmetric extension.

Lemma 1.15 [1]. Every operator D which possesses self-adjointness is a maximal symmetric operator.

As is shown in [1] that any differential expression which possesses self-adjointness with real, sufficiently often differentiable, coefficients can be put in the form

$$l(u) = [(-1)^{n}(j_0 u^{(n)})^{(n)} + (-1)^{n-1}(j_1 u^{(n-1)})^{(n-1)} + \ldots + j_n u] / w(x),$$

where w(x) > 0.

Definition 1.16 [1]. An expression l(u) is said to be regular if the interval (c,d) is finite and the function $1/j_0(x)$, $j_1(x), \ldots, j_n(x)$, w(x) are summable in the whole interval (c,d); otherwise l(u) is called singular.

Definition 1.17 [1]. The quasi-derivatives of a function u related to the expression l(u) are defined as follows:

$$u^{[s]} = \frac{d^s u}{dx^s} \quad , \qquad s = 1,2,\ldots,(n-1);$$

$$u^{[n]} = j_0 \frac{d^n u}{dx^n}$$

$$u^{[n+s]} = j_s \frac{d^{n-s} u}{dx^{n-s}} - \frac{d}{dx}(u^{[n+s-1]}).$$

For convenience it is written $u^{[0]} = u$.

Lemma 1.18 [1]. For the functions m and t for which the expression l(.) makes sense the following Lagrange's identity is obtained on the interval $[\alpha, \beta] \subset (a,b)$

$$\int_\alpha^\beta l(m)\,\bar{t}\,w(x)\,dx - \int_\alpha^\beta m\,\overline{l(t)}\,w(x)\,dx = [m,t]_\alpha^\beta$$

where

$$[m,t] = \sum_{k=1}^{n} \{ m^{[k-1]}\,\bar{t}^{\,[2n-k]} - m^{[2n-k]}\,\bar{t}^{\,[k-1]} \}$$

and

$$[m,t]_\alpha^\beta = [m,t]\,(\beta) - [m,t]\,(\alpha).$$

Suppose that the domain of L is B and for all m ∈ B, let

$$Lm = l(m).$$

It is denoted by B_0' the set of all functions m in B which is identically zero outside a finite interval $[\alpha, \beta] \subset$ (a,b). The resctriction of the operator L to B_0' is denoted by L_0'. In other words,

$$L_0' \subset L.$$

Lemma 1.19 [1]. For arbitrary functions m ∈ B_0', s ∈ B one has

$$(L_0' m, s) = (m, Ls),$$

that is

$$L \subset {L_0'}^*.$$

Lemma 1.20 [1]. The operator L_0' is Hermitian.

We consider that l(u) is regular in [c,d].

B_0 denotes the set of all functions u in B which fulfils

$$u^{[l]}(a) = u^{[l]}(b) = 0, \quad l=0,1,2\ldots,(2n-1).$$

The restriction of the operator L to B_0 is denoted by L_0.

Lemma 1.21 [1]. For arbitrary functions m ∈ B_0, s ∈ B one has

$$(L_0 m, s) = (m, Ls).$$

Lemma 1.22 [1]. The operator L_0 is Hermitian, i.e., for arbitrary elements g,u ∈ B_0 , one has

$$(L_0 g, u) = (g, L_0 u).$$

Theorem 1.23 [1]. The domain of definition B_0 of the operator L_0 is dense in L_w^2 (c,d).

Theorem 1.24 [1]. The operator L is adjoint to the operator L_0 , i.e.,

$$L = L_0{}^*.$$

Theorem 1.25 [1]. The operator L_0 is adjoint to the operator, L, i.e.,

$$L_0 = L^*.$$

Definition 1.26 [3]. An n×n matrix function ζ is called a fundamental matrix for the vector differential equation

$$g' = S(x)g$$

as long as ζ is a solution of the matrix equation

$$G' = S(x)G$$

on the interval [c,d] and det $\zeta(x) \neq 0$ on the interval [c,d], where S is a n×n matrix, G is a n×n matrix and g is a n×1 matrix.

Theorem 1.27 [3]. An n×n matrix function ζ is called a fundamental matrix for the vector differential equation

$$g' = S(x)g$$

if and only if the columns of ζ are n linearly independent solutions of

$$g' = S(x)g$$

on the interval [c,d].

Definition 1.28 [3]. Suppose that e, h are differentiable functions on an interval [c,d]. Then the Wronskian of e and h are defined as

$$W[e(x), h(x)] = e(x)h'(x)-e'(x)h(x)$$

for $x \in$ interval [c,d].

CHAPTER 1

INTRODUCTION

1.1 BACKGROUND

See the boundary value problem (bvp) over a finite interval [c,d],

$$-(pg')' + qg = \lambda wg + wf \quad (1),$$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} g(c) \\ pg'(c) \end{pmatrix} + \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \begin{pmatrix} g(d) \\ pg'(d) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2),$$

where 1/p, q, w > 0 are integrable functions over [c,d] and $q > \varepsilon w$ for $\varepsilon > 0$. The coefficients matrices meet the conditions of

$$\text{the rank} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \end{pmatrix} = 2$$

and the self-adjointness criterion

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}.$$

The equation (1) in this bvp in a left-definite background was discussed by lots of authors. The conditions $\alpha_{21} = \alpha_{22} = 0$, ${\alpha_{11}}^2 + {\alpha_{12}}^2 \neq 0$, $\beta_{11} = \beta_{12} = 0$, ${\beta_{21}}^2 + {\beta_{22}}^2 \neq 0$ are studied by the authors. These conditions can be found after the mathematical manipulations of (2).

Pleijel [4,5] introduced the subject, Everitt [6] considered the regular problem for the first time in a diffuse way, Bennewitz and Everitt [7] extended the theory and Krall [8] developed the theory of differential operators in a left-definite context.

The study of bvp in left definite setting begins with Weyl in 1910. Weyl did the classification of the limit-point and limit-circle cases for singular spectral problems of second-order formally self-adjoint linear differential equations [9]. After then a paper by Atkinson, Everitt and Ong has been introduced in the literature [10]. The paper investigates left-definite square integrable homogeneous solutions. Several years later, Schneider and Niessen worked on left-definite S-Hermitian problems [11-12]. Onyango-Otieno considered Jacobi, Laguerre and Hermite equations in 1980 [13]. By the way a discussion of differential operators in a left-definite setting did not exist. Work was begun by Everitt and Littlejohn who examined the fourth order Legendre-type equation [14]. There has been a little work on differential operators corresponding with left definite bvps in the suitable Hilbert spaces in the literature. One such study was performed by Krall in 1990. He investigated the spectral properties of a second order regular problem [8]. This work was followed by Krall and Littlejohn who examined the left-definite Legendre operator [15]. Moreover, Hajmirzaahmad introduced some results on the left-definite Jacobi and Laguerre operators [16-18].

1.2 OBJECTIVES

The objective of this thesis is to study of the way of constructing of the inner product associated with the Sobolev space with p,q>0. Then we collect the results on the operators in this new inner product space. Moreover, we continue the similar study for the left-definite Hamiltonian systems.

In addition, another important problem in the literature is to determine the number of the linearly independent solutions of the Sturm-Liouville equation belonging to the Lebesgue space L_w^2(c,d). We introduce the known results on the problem in right and left-definite cases.

1.3 ORGANIZATION OF THE THESIS

This thesis contains five chapters.

Chapter 1 is an introduction to the bvp (1), (2) in left definite context and includes objectives of this thesis.

Chapter 2 is comprised of L_w^2-theory, the Dirichlet formula and H^1 theory of Sturm-Liouville equation.

Chapter 3 we study the $L_{\mathcal{A}}^2$ theory, the Dirichlet formula and H^1 theory for left definite Hamiltonian Systems.

Chapter 4 consists of Weyl's Theory.

Chapter 5 includes the conclusion and discussion part.

CHAPTER 2

SECOND ORDER DIFFERENTIAL OPERATORS

2.1 INTRODUCTION

It is suitable to study a differential equation on the proper Hilbert function space and it is also possible to define differential operators whose eigenvectors can be corresponded with solutions of the differential equation fulfilling certain boundary conditions. It is essential to define an appropriate Hilbert function space in terms of one or more of the coefficients of the differential equation to permit the possibility of possessing the solution in operator theoretic terms. Boundary value problems can be considered by a uniquely determined unbounded self-adjoint operator in this function space where eigenfunctions and eigenvalues of the bvp are tantamount to the eigenvalues and eigenfunctions of the operator [6].

We now investigate L_w^2-theory, the Dirichlet formula and H^1 theory respectively.

Note that throughout this section it is assumed that 1/p, q, w > 0 and integrable on [c,d], q>εw, where ε is a positive constant.

2.2 L_w^2-THEORY

Let us make clear the definition of the L_M, L_m and L.

Definition 2.2.1 (The Maximal Operator) [19]. It is denoted by D_M the set consisting of those elements g in L_w^2 (c,d) which fulfils

1.g is absolutely continuous on every closed subinterval of [c,d].

2.pg′ is absolutely continuous on every closed subinterval of [c,d].

3.lg= (-(pg′)′+qg) / w exists a.e. and is in L_w^2 (c,d).

The maximal operator L_M is defined by making $L_M\ g = lg$ for all g in D_M .

Definition 2.2.2 (The Minimal Operator) [19]. It is denoted by D_m consisting of those elements g which fulfils

1.g is in D_M .

2.g(c)=0, pg′(c)=0, g(d)=0, pg′(d)=0.

The minimal operator L_m is defined by making $L_m g = lg$ for all g in D_m.

Definition 2.2.3 (The operator L) [19]. It is denoted by D consisting of those elements g which fulfils

1.g is in D_M.

2.α_{11} g(c)+ α_{12} pg′(c)+β_{11}g(d)+ β_{12} pg′(d) = 0.

α_{21} g(c)+ α_{22} pg′(c)+β_{21}g(d)+ β_{22} pg′(d) = 0.

The operator L is defined by making $Lg = lg$ for all g in D.

Theorem 2.2.4 [19]. The domains D_M and D_m are dense in L_w^2 (c, d).

$$L_m^* = L_M, L_M^* = L_m.$$

Proof [20]. The proof of this theorem can be embedded into the proof of the Theorem 3.2.1.

□

Since we know the form of the adjoint operator, its domain can be calculated with the help of Green's formula. Suppose that g is in D and u is in the domain of adjoint D* [20]. Then

$$0 = (Lg, u) - (g, L^*u)$$

$$= \int_c^d \bar{u}\,(-(pg')' + qg)\,dx - \int_c^d \overline{(-(pu')' + qu)}\, g\, dx$$

$$= p(g\bar{u}' - g'\bar{u})\,\big|_c^d$$

$$= (\bar{u}\ \ \overline{pu'}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g \\ pg' \end{pmatrix} \big|_c^d.$$

$$\text{Let } J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\ G=\begin{pmatrix}g\\ pg'\end{pmatrix},\ U=\begin{pmatrix}u\\ pu'\end{pmatrix}.$$

Then it is found $0=U^{*}JG|_{c}^{d}$ or

$$0=\left(U^{*}(c)\ U^{*}(d)\right)\begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}G(c)\\ G(d)\end{pmatrix}.$$

Let E and F be chosen so that the square matrix $\begin{pmatrix}K & S\\ E & F\end{pmatrix}$ is nonsingular. Then $\begin{pmatrix}K'^{*} & S'^{*}\\ E'^{*} & F'^{*}\end{pmatrix}$ require $\begin{pmatrix}K'^{*} & E'^{*}\\ S'^{*} & F'^{*}\end{pmatrix}\begin{pmatrix}K & S\\ E & F\end{pmatrix}=\begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix}$. Therefore, it is found

$$K'^{*}K + E'^{*}E= -J,$$

$$K'^{*}S + E'^{*}F= 0,$$

$$S'^{*}K + F'^{*}E= 0,$$

$$S'^{*}S + F'^{*}F= J.$$

The Green’s formula afterwards is

$$0=\left(U^{*}(c)\ U^{*}(d)\right)\begin{pmatrix}K'^{*} & E'^{*}\\ S'^{*} & F'^{*}\end{pmatrix}\begin{pmatrix}K & S\\ E & F\end{pmatrix}\begin{pmatrix}G(c)\\ G(d)\end{pmatrix}$$

or

$$0= (K'U(c)+S'U(d))^{*}(KG(c)+ SG(d)) +(E'U(c)+F'U(d))^{*}(EG(c)+FG(d)). \quad (3)$$

KG(c)+SG(d)=0, while EG(c)+FG(d) is arbitrary; if u is in D* then $E'U(c)+F'U(d) = 0$. (3) is called adjoint boundary condition.

The adjoint parametric conditions are tantamount to the originals. These adjoint parametric conditions are developed as follows [20]:

$$KG(c) + SG(d) = 0,$$

$$EG(c) + FG(d) = \alpha.$$

$$\begin{pmatrix}J & 0\\ 0 & -J\end{pmatrix}\begin{pmatrix}K'^{*} & E'^{*}\\ S'^{*} & F'^{*}\end{pmatrix}\begin{pmatrix}K & S\\ E & F\end{pmatrix}\begin{pmatrix}G(c)\\ G(d)\end{pmatrix}=\begin{pmatrix}J & 0\\ 0 & -J\end{pmatrix}\begin{pmatrix}K'^{*} & E'^{*}\\ S'^{*} & F'^{*}\end{pmatrix}\begin{pmatrix}0\\ \alpha\end{pmatrix},$$

where

$$\begin{pmatrix}K' & S'\\ E' & F'\end{pmatrix}\begin{pmatrix}K & S\\ E & F\end{pmatrix}=\begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix}.$$

Then

$$\begin{pmatrix}J & 0\\ 0 & -J\end{pmatrix}\begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix}\begin{pmatrix}G(c)\\ G(d)\end{pmatrix}=\begin{pmatrix}J & 0\\ 0 & -J\end{pmatrix}\begin{pmatrix}E'^{*}\alpha\\ F'^{*}\alpha\end{pmatrix}$$

and

$$\begin{pmatrix} -J^2 & 0 \\ 0 & -J^2 \end{pmatrix} \begin{pmatrix} G(c) \\ G(d) \end{pmatrix} = \begin{pmatrix} TE'^{*}\alpha \\ -TF'^{*}\alpha \end{pmatrix},$$

$$\text{where } -J^2 = I,$$

$$\Rightarrow \begin{pmatrix} G(c) \\ G(d) \end{pmatrix} = \begin{pmatrix} TE'^{*}\alpha \\ -TF'^{*}\alpha \end{pmatrix}. \quad (4)$$

Likewise, as above the parametric boundary condition for U(c) and U(d) can be developed as

$$\begin{pmatrix} U(c) \\ U(d) \end{pmatrix} = \begin{pmatrix} -TK'^{*}\alpha \\ TS'^{*}\alpha \end{pmatrix} \quad (5)$$

with boundary conditions

$$K'U(c) + S'U(d) = \alpha,$$

$$E'U(c) + F'U(d) = 0.$$

(4) and (5) are critical for proving the Theorem 2.2.5.

Now let us consider the following MG(c)+NG(d)=0 for g in D and PU(c)+QU(d)= α for u in D* boundary conditions.

Theorem 2.2.5 [20]. L=L*if and only if MJM*=NJN*.

Proof [20]. For self-adjointness occur, both form and domain must be the same. Let L=L* exist. Then D =D*. U(c)= -JM*ϕ and U(d)= JN*ϕ are achieved for D*. If the condition that MG(c)+NG(d)=0 is written by changing G(c) and G(d) with U(c) and U(d), M(-JM*ϕ) +N(JN*ϕ) =0 will be found. Then -MJM*ϕ+NJN*ϕ= 0 and (-MJM*+ NJN*) ϕ =0 are encountered. This means that MJM*= NJN*.

Conversely, let MJM*=NJN* exist. Then -MJM*+NJN*= 0 is found. This equality may be shown in matrix multiplication form as $(-MJ \quad NJ)\begin{pmatrix} M^{*} \\ N^{*} \end{pmatrix}$= 0 where M* and N* are adjoints of M and N respectively.

On the other side, $\begin{pmatrix} M & N \\ P & Q \end{pmatrix}\begin{pmatrix} G(c) \\ G(d) \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \Rightarrow \begin{pmatrix} M & N \\ P & Q \end{pmatrix}\begin{pmatrix} JP'^{*}\alpha \\ -JQ'^{*}\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$

$\Rightarrow \begin{pmatrix} MJP'^{*}\alpha - NJQ'^{*}\alpha \\ -PJP'^{*}\alpha - QJQ'^{*}\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \quad \Rightarrow \quad (MJP'^{*} - NJQ'^{*})\alpha = 0 \Rightarrow$

$(-MJ \quad NJ) \begin{pmatrix} P'^{*} \\ Q'^{*} \end{pmatrix} = 0.$

Then there takes place such an B* matrix so that $\begin{pmatrix} P'^{*} \\ Q'^{*} \end{pmatrix}$ B* = $\begin{pmatrix} M^{*} \\ N^{*} \end{pmatrix}$ is achieved or so is (M N)= B$\begin{pmatrix} P' \\ Q' \end{pmatrix}$. This leads that the conditions of MG(c)+NG(d)= 0 and P′G(c)+Q′G(d)=0 are tantamount. This leads to conclude that D= D*. Because the forms are the same, one gets L= L*.

□

Theorem 2.2.6 [20]. Eigenfunctions corresponded to different eigenvalues are mutually orthogonal. Eigenfunctions corresponded to each eigenvalue λ_j can be made mutually orthogonal.

Proof [20]. Let g_1 be an eigenfunction associated with λ_1, g_2 be an eigenfunction associated with λ_2. Then using (3) with g $=h_1$, u $=h_2$

$$Lg_1 = \lambda_1 g_1,\ Lg_2 = \lambda_2 g_2\ .$$

$$\lambda_1(h_1, h_2) = (\lambda_1 h_1, h_2) = (Lh_1, h_2) = (h_1, Lh_2) = (h_1, \lambda_2 h_2) = \lambda_2(h_1, h_2).$$

Since

$$\lambda_1 \neq \lambda_2\ ,\ (h_1, h_2) = 0.$$

Now let $g_1, \dots, g_M$ are eigenfunctions corresponded to λ. Suppose

$$t_1 = g_1 / \|g_1\|.$$

It is then defined inductively

$$v_K = g_K - \sum_{i=1}^{k-1} t_i\,(t_i, g_K) \quad \text{and } t_k = v_K / \|v_K\|.$$

Therefore t_k is orthogonal to $t_1, \dots, t_{k-1}$.

□

Now let us consider the construction of Green's function [21].

Let

$$1/p,\ q, w,\ f \in L^1\ (B, \mathbb{C}),\ B = (c,d),\ -\infty < c < d < \infty,\ \lambda \in \mathbb{C},$$

$$P=\begin{pmatrix}0 & 1/p\\ q & 0\end{pmatrix},\ W=\begin{pmatrix}0 & 0\\ w & 0\end{pmatrix},\ F=\begin{pmatrix}0\\ f\end{pmatrix},\ G=\begin{pmatrix}g\\ pg'\end{pmatrix},\ M,N \in M_2(\mathbb{C}).$$

The bvp

$$-(pg')' + qg = \lambda wg + f,\ \ MG(c) + NG(d) = 0,$$

is tantamount to the system

$$G' = (P-\lambda W)G + F,\ MG(c) + NG(d) = 0.$$

Suppose that $\zeta = \zeta(.,.,\lambda)$ is the primary fundamental matrix of the homogeneous system

$$G' = (P-\lambda W)G.$$

Keep in mind that

$$\zeta(t,u,\lambda) = \zeta(t,c,\lambda)\ \zeta(c,u,\lambda) \text{ for } c\leq t,\ u\leq d.$$

This comes from

$$\zeta(t,u,\lambda) = G(t)\ G^{-1}(u)$$

for any fundamental matrix solution G of $G' = (P-\lambda W)G$.

Theorem 2.2.7 [21]. Let

$$1/p,\ q,w,\ f \in L^1(B,\mathbb{C}),\ B=(c,d),\ -\infty\leq c<d\leq\infty,\ \lambda \in \mathbb{C},$$

$$P=\begin{pmatrix}0 & 1/p\\ q & 0\end{pmatrix},\ W=\begin{pmatrix}0 & 0\\ w & 0\end{pmatrix},\ F=\begin{pmatrix}0\\ f\end{pmatrix},\ G=\begin{pmatrix}g\\ pg'\end{pmatrix},\ M,N \in M_2(\mathbb{C}).$$

Let us consider the scalar bvp

$$-(pg')' + qg = \lambda wg + f,\ \ MG(c) + NG(d) = 0 \quad (6)$$

and vector bvp

$$G' = (P-\lambda W)G + F,\ \ MG(c) + NG(d) = 0. \quad (7)$$

Then these three statements are tantamount:

1)When f= 0 on B, the bvps (6) and (7) possess only the trivial solution.

2)The matrix

$$[M+N\zeta(d,c,\lambda)]$$

has an inverse.

3)For every $f \in L^1(B,\mathbb{C})$ each of the problems (6) and (7) possess a unique solution.

Furthermore, if

$$[M + N\zeta(d,c,\lambda)]^{-1}$$

exists, the matrix function V is defined by

$$V(t,u,\lambda) = \begin{cases} \zeta(t,c,\lambda)U(\lambda)\zeta(d,u,\lambda) & c \le t < u \le d, \\ \zeta(t,c,\lambda)U(\lambda)\zeta(d,u,\lambda) + \zeta(t,u,\lambda) & c \le t < u \le d \end{cases}$$

where

$$U(\lambda) = -[M + N\zeta(d,c,\lambda)]^{-1}N.$$

Theorem 2.2.8 [21]. For any f $\in L^1$ (B,ℂ), the unique solution g of (6) and the unique solution G of (7) respectively are given by

$$g(t) = \int_c^d V_{12}(t,u,\lambda)f(u)du, \qquad c \le t \le d,$$

$$G(t) = \int_c^d V(t,u,\lambda)f(u)du, \qquad c \le t \le d,$$

where

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}.$$

Proof [21]. G is a solution of

$$G' = (P-\lambda W)G + F \text{ on } B$$

if and only if g is a solution of

$$-(pg')' + qg = \lambda wg + f \text{ on } B,$$

where

$$G = \begin{pmatrix} g \\ pg' \end{pmatrix}.$$

For

$$D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \; d_j \in \mathbb{C},$$

determine a solution G of G' = (P-λW)G +F on B by the initial condition

$$G(c, \lambda) = D.$$

Then g is a solution of

$$-(pg')' + qg = \lambda wg + f$$

constrained by the initial conditions

$$g(c,\lambda) = d_1,$$

$$(pg')(c,\lambda) = d_2.$$

Keep in mind that

$$G(t,\lambda) = \zeta(t,c,\lambda)D, \qquad c \le t \le d,$$

and from the variation of parameters formula one obtains

$$G(t, \lambda) = \zeta(t, c, \lambda)D + \int_c^t \zeta(t, s, \lambda)F(s)ds, \qquad c \leq t \leq d.$$

In particular,

$$G(d, \lambda) = \zeta(d, c, \lambda)D + \int_c^d \zeta(d, s, \lambda)F(s)ds,$$

Keep in mind that the existence of the first integral above (f $\in L^1$ (B,ℂ)) shows that the solutions have finite limits at the end points and are bounded in a neighborhood of each point.

Let

$$E(\lambda) = [M+N\, \zeta(d, c, \lambda)]$$

then one may see that

$$MG(c,\lambda)+ NG(d,\lambda) = E(\lambda)D + N\int_c^d \zeta(d, s, \lambda)F(s)ds.$$

When f= 0 on B, G and g are nontrivial solutions if and only if D is not the zero vector. See from the above equation that, when f= 0 on B, there takes place a nontrivial solution G which fulfils the boundary condition MG(c,λ)+ NG(d,λ) = 0 if and only if E(λ) is singular. This is also true for the nontrivial solution g which satisfies the boundary condition MG(c,λ)+ NG(d,λ) = 0. It is also true that there takes place a unique solution G which fulfils the boundary condition MG(c)+NG(d) = 0 for every f $\in L^1$ (B,ℂ) if and only if E(λ) is nonsingular. This is again true for the nontrivial solution g satisfying the boundary condition MG(c)+ NG(d) = 0 for every f $\in L^1$ (B,ℂ) if and only if E(λ) is nonsingular.

Now suppose that E(λ) is nonsingular. Let

$$H=E^{-1}(\lambda)\ (-N)\int_c^d \zeta(d, s, \lambda)F(s)ds.$$

Then

$$MG(c,\lambda)+ NG(d,\lambda) = 0$$

and

$$G(t, \lambda) = \zeta(t, c, \lambda)[E^{-1}(\lambda)\ (-N)\int_c^d \zeta(d, s, \lambda)F(s)ds] + \int_c^t \zeta(t, s, \lambda)F(s)ds$$

$$=\int_c^d \zeta(t, c, \lambda)[E^{-1}(\lambda)\ (-N)\ \zeta(d, s, \lambda)F(s)ds] + \int_c^t \zeta(t, s, \lambda)F(s)ds$$

$$=\int_c^d V(t,s,\lambda)\, F(s)ds, \qquad c \le t \le d.$$

The property of $\zeta(d,s,\lambda) = \zeta(d,c,\lambda)\zeta(c,s,\lambda)$ and definition of V are used in the last equation.

□

Theorem 2.2.9 [20]. $(L-\lambda I)^{-1}$ is a bounded operator and holds for all nonreal λ. It is a bounded operator and takes place also for all real λ for which

$$\det[M + NG(d, \lambda)] \neq 0.$$

Proof [20]. $(L-\lambda I)^{-1}$ is obtained by the formula

$$G = \int_a^b B(\lambda, x, \xi)\, A(\xi)\, F(\xi)\, d\xi$$

as long as

$$\det[M + NG(d, \lambda)] \neq 0.$$

Since L possesses self-adjointness it must hold for all complex λ. It definitely holds for all real λ except the zeros of

$$\det[M + NG(d, \lambda)] = 0.$$

To see that $(L-\lambda I)^{-1}$ is bounded, suppose that

$$f(\xi) = A(\xi)^{1/2}\, F(\xi)$$

holds and

$$M(\lambda, x, \xi) = A(\xi)^{1/2} B(\lambda, x, \xi)\, A(x)^{1/2}.$$

Then

$$\|G\|^2 = \int_c^d G^*(x)\, A(x)\, G(x)\, dx$$

$$=\int_c^d \left[\int_c^d f^*(\xi) M^*(\lambda,x,\xi)\, d\xi\right]\left[\int_c^d M(\lambda,x,\eta)\, f(\eta)\, d\eta\right] dx.$$

When Schwarz's inequality is applied to both terms,

$$\|G\|^2 \le \|M\|^2\, \|F\|^2$$

is found where

$$\|M\|^2 = \int_c^d \int_c^d \sum_{i=1}^n \sum_{j=1}^n \left| M_{ij}^2(\lambda, x, \xi)\right| d\xi dx$$

with

$$M=M_{ij}.$$

□

2.3 THE DIRICHLET FORMULA

The Dirichlet formula is a formula that forms a new setting for the bvp of the previous section [19]. If the second derivative term of

$$\int_c^d (lg)\,\bar{u}\ \mathrm{w}\ \mathrm{d}x \quad (8)$$

is integrated by parts, one has [5]

$$-(\mathrm{p}g')\bar{u})\ |\ _c^d + \int_c^d [\, pg'\,\bar{u}' + \mathrm{q}g\bar{u}\ ]\ dx. \quad (9)$$

If the boundary conditions are thought to be separated, then pg′ is said to be eliminated at c and d [19]. When the coefficients have proper signs, a new form (a Sobolev space) is generated where the bvp remains self-adjoint [19]. If c and d are singular points, then under various conditions a Sobolev space may be constructed in which the boundary value problem remains self-adjoint [19].

The regular case when the boundary terms at c and d are mixed together will be considered here. First the g-terms together and the pg′-terms together are put with a minus sign with the term pg′(d). Then it is found that

$$\begin{pmatrix} \alpha_{11} & \beta_{11} \\ \alpha_{21} & \beta_{21} \end{pmatrix} \begin{pmatrix} g(c) \\ g(d) \end{pmatrix} - \begin{pmatrix} -\alpha_{12} & \beta_{12} \\ -\alpha_{22} & \beta_{22} \end{pmatrix} \begin{pmatrix} pg\prime(c) \\ -pg\prime(d) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Three situations may arise as follows:

1- $\begin{pmatrix} -\alpha_{12} & \beta_{12} \\ -\alpha_{22} & \beta_{22} \end{pmatrix}$ is nonsingular.

2- $\begin{pmatrix} -\alpha_{12} & \beta_{12} \\ -\alpha_{22} & \beta_{22} \end{pmatrix}$ is singular but not zero.

3- $\begin{pmatrix} -\alpha_{12} & \beta_{12} \\ -\alpha_{22} & \beta_{22} \end{pmatrix}$ is zero.

1-The nonsingular case [19]: It is assumed that $\alpha_{22}\beta_{12}$ -$\alpha_{12}\beta_{22}$ = 1 and then from the original coefficient matrix above

$$\begin{pmatrix} pg'(c) \\ -pg'(d) \end{pmatrix} = \begin{pmatrix} \alpha_{11}\beta_{22} - \alpha_{21}\beta_{12} & \beta_{11}\beta_{22} - \beta_{12}\beta_{21} \\ \alpha_{11}\alpha_{22} - \alpha_{12}\beta_{21} & \alpha_{22}\beta_{11} - \alpha_{12}\beta_{21} \end{pmatrix} \begin{pmatrix} g(c) \\ g(d) \end{pmatrix}$$

is found.

If the matrix is positive, then using the (8) and (9) Dirichlet formula becomes

$$\int_c^d (lg)\, \bar{u}\ \mathrm{w}\ \mathrm{d}x =$$

$$\int_c^d [\, pg'\,\bar{u}' + qg\bar{u}]\, dx\ + (\overline{u(c)}\ \ \overline{u(d)}\ )$$

$$\begin{pmatrix} \alpha_{11}\beta_{22} - \alpha_{21}\beta_{12} & \beta_{11}\beta_{22} - \beta_{12}\beta_{21} \\ \alpha_{22}\alpha_{11} - \alpha_{12}\alpha_{21} & \alpha_{22}\beta_{11} - \alpha_{12}\beta_{21} \end{pmatrix} \begin{pmatrix} g(c) \\ g(d) \end{pmatrix}.$$

If the matrix is positive, the right side of the Dirichlet formula may be used to define an Sobolev inner product.

Example 1: α_{11} = 1, α_{12} = 0, α_{21} = 1, α_{22}= 1, β_{11} = 1, β_{12} = 0, β_{21} = 1, β_{22} = 1

Then from the above formula

$$\begin{pmatrix} pg'(c) \\ -pg'(d) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} g(c) \\ g(d) \end{pmatrix}$$

is found.

2-The Singular, Non-zero Case [19]: When the matrix $\begin{pmatrix} -\alpha_{12} & \beta_{12} \\ -\alpha_{22} & \beta_{22} \end{pmatrix}$ is singular, the rows are said to be dependent. Then there takes place a number k such that

$$\mathrm{k}\ (-\alpha_{12}\ \beta_{12}) = (-\alpha_{22}\ \beta_{22}).$$

Hence k$\alpha_{12} = \alpha_{22}$, k$\beta_{12} = \beta_{22}$. The following is found by row manipulation:

$$\begin{pmatrix} \alpha_{11} & \beta_{11} \\ \alpha_{21} - k\alpha_{11} & \beta_{21} - k\beta_{11} \end{pmatrix} \begin{pmatrix} \mathrm{g(c)} \\ \mathrm{g(d)} \end{pmatrix} - \begin{pmatrix} -\alpha_{12} & \beta_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} pg'(c) \\ -pg'(d) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Assume that $\alpha_{12}^2 + \beta_{12}^2 = 1$ and define g_c, g_d, $g_c{}'$, $g_d{}'$ by

$$\begin{pmatrix} -\alpha_{12} & \beta_{12} \\ \beta_{12} & \alpha_{12} \end{pmatrix} \begin{pmatrix} \mathrm{g(c)} \\ \mathrm{g(d)} \end{pmatrix} = \begin{pmatrix} g_c \\ g_d \end{pmatrix},$$

$$\begin{pmatrix} -\alpha_{12} & \beta_{12} \\ \beta_{12} & \alpha_{12} \end{pmatrix} \begin{pmatrix} pg'(c) \\ -pg'(d) \end{pmatrix} = \begin{pmatrix} g_{c'} \\ g_{d'} \end{pmatrix}.$$

Because $\alpha_{12}^2 + \beta_{12}^2 = 1$,

$$\begin{pmatrix} -\alpha_{12} & \beta_{12} \\ \beta_{12} & \alpha_{12} \end{pmatrix} \begin{pmatrix} g_c \\ g_d \end{pmatrix} = \begin{pmatrix} \mathrm{g(c)} \\ \mathrm{g(d)} \end{pmatrix},$$

$$\begin{pmatrix} -\alpha_{12} & \beta_{12} \\ \beta_{12} & \alpha_{12} \end{pmatrix} \begin{pmatrix} g_{c'} \\ g_{d'} \end{pmatrix} = \begin{pmatrix} pg'(c) \\ -pg'(d) \end{pmatrix}.$$

The coefficient equation becomes

$$\begin{pmatrix} -\alpha_{11}\alpha_{12} + \beta_{11}\beta_{12} & \alpha_{11}\beta_{12} + \beta_{11}\alpha_{12} \\ -\alpha_{12}(\alpha_{21} - k\alpha_{11}) + \beta_{12}(\beta_{21} - k\beta_{11}) & \beta_{12}(\alpha_{21} - k\alpha_{11}) + \alpha_{12}(\beta_{21} - k\beta_{11} \end{pmatrix}$$

$$\begin{pmatrix} g_c \\ g_d \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_{c'} \\ g_{d'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Two constraints are found from above. If first matrix is shown by $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ then

Pg_c+Qg_d = ${g_c}'$ and Rg_c+Sg_d = 0 and the boundary terms change to

$$(\overline{u(c)}\ \ \overline{u(d)}\ ) \begin{pmatrix} pg'(c) \\ -pg'(d) \end{pmatrix} = (\ \overline{u_c}\ \overline{u_d}\ ) \begin{pmatrix} -\alpha_{12} & \beta_{12} \\ \beta_{12} & \alpha_{12} \end{pmatrix} \begin{pmatrix} -\alpha_{12} & \beta_{12} \\ \beta_{12} & \alpha_{12} \end{pmatrix} \begin{pmatrix} g_{c'} \\ g_{d'} \end{pmatrix}$$

$$=(\ \overline{u_c}\ \overline{u_d}\ ) \begin{pmatrix} g_{c'} \\ g_{d'} \end{pmatrix}$$

$$=\overline{u_c}{g'}_c+\overline{u_d}{g'}_d.$$

Since substitution can only be made for ${g'}_c$, it is required that u_d = 0. The boundary terms are equal to u_c(Pg_c+Qg_d). Since g_d = 0, the boundary terms are equal to u_cPg_c.

The Dirichlet formula is found as below because R-S constraint vanishes since the self-adjointness criterion dictates that $\alpha_{11}\ \alpha_{22}$ - $\alpha_{12}\,\alpha_{21}$ = $\beta_{11}\,\beta_{22}$ - $\beta_{12}\ \beta_{21}$ and kα_{12} = α_{22}, kβ_{12} = β_{22}. Since $\alpha_{12}\,(\alpha_{11}k$- $\alpha_{21})$ = β_{12} (β_{11}k-β_{21}) exists, R is found 0. If there is a parameter j such that jα_{12} = β_{11}k-β_{21} and jβ_{12} = $\alpha_{11}k$- α_{21}, S is found 0.

Then from (8) and (9) the Dirichlet formula becomes

$$\int_c^d (lg)\ \bar{u}\ \text{w dx} = \int_c^d [\ pg'\,\bar{u}' + \text{q}g\bar{u}\ ]\ dx \ +(-\ \alpha_{12}\,u(c) + \ \beta_{12}\ \text{u(d)}\ )\ (-\alpha_{11}\alpha_{12} + \beta_{11}\ \beta_{22})\ \ (-\ \alpha_{12}\ g(c) + \beta_{12}\ g(d)\ ),$$

where g and u satisfy

$$\beta_{12}\ \text{g(c)} + \ \alpha_{12}\ \text{g(d)} = 0, \qquad \beta_{12}\ \text{u(c)} + \ \alpha_{12}\ \ \text{u(d)} = 0.$$

The right-hand side of the Dirichlet formula may be used to define Sobolev inner product again.

Example 2 [19]: α_{11} = 0.6, α_{12} =0.6, β_{11}= 0.8, β_{12} =0.8, α_{21} =0.21, α_{22}=1.2, β_{21} =1, β_{22} =1.6

The matrix $\begin{pmatrix} -\alpha_{12} & \beta_{12} \\ -\alpha_{22} & \beta_{22} \end{pmatrix}$ is singular. The Dirichlet formula is

$$\int_c^d (lg)\,\bar{u}\text{ w d}x =$$

$$\int_c^d [\,pg'\,\bar{u}' + \text{q}g\bar{u}\,]\,dx + (-0.6\,u(c) + 0.8\text{u(d)})(0.28)(-0.6\text{g(c)}+0.8\text{g(d)}),$$

where g and u must satisfy the constraints

$$0.8\text{g(c)}+0.6\text{g(d)}= 0, \qquad 0.8\text{u(c)}+0.6\text{u(d)}= 0.$$

3-The zero case [19]: This case is the case when the matrix $\begin{pmatrix} -\alpha_{12} & \beta_{12} \\ -\alpha_{22} & \beta_{22} \end{pmatrix}$ is zero. The boundary conditions become

$$\begin{pmatrix} \alpha_{11} & \beta_{11} \\ \alpha_{21} & \beta_{21} \end{pmatrix} \begin{pmatrix} \text{g(c)} \\ \text{g(d)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Because the coefficient matrix possesses rank 2, the matrix above is nonsingular. Hence the boundary conditions are tantamount to g(c)= 0 and g(d)= 0. Then the Dirichlet formula becomes

$$\int_c^d (lg)\,\bar{u}\text{ w d}x = \int_c^d [\,pg'\,\bar{u}' + \text{q}g\bar{u}\,]\,dx.$$

2.4 H^1 THEORY

See the differential expression lg= (-(pg′)′ + qg) /w whose domain is constrained by boundary conditions (2).

The new inner products will be defined that depends on whether the matrix $\begin{pmatrix} -\alpha_{12} & \beta_{12} \\ -\alpha_{22} & \beta_{22} \end{pmatrix}$ is nonsingular, singular/nonzero or zero. These are respectively given by

$$1\text{-} < g,u >_{H^1} = \int_c^d [\,pg'\,\bar{u}' + \text{q}g\bar{u}\,]\,dx + (\overline{u(c)}\ \ \overline{u(d)}\,) \begin{pmatrix} \alpha_{11}\beta_{22} - \alpha_{21}\beta_{12} & \beta_{11}\beta_{22} - \beta_{12}\beta_{21} \\ \alpha_{22}\alpha_{11} - \alpha_{12}\alpha_{21} & \alpha_{22}\beta_{11} - \alpha_{12}\beta_{21} \end{pmatrix} \begin{pmatrix} \text{g(c)} \\ \text{g(d)} \end{pmatrix},$$

where α-β matrix is assumed to be positive,

$$2\text{-}< g,u >_{H^1} = \int_c^d [\, pg'\,\bar{u}' + \mathrm{q}g\bar{u}\,]\, dx \;\; +(-\,\alpha_{12}\, u(c) + \; \beta_{12}\, \mathrm{u(d)}\,)\, (-\alpha_{11}\alpha_{12} \; + \beta_{11}\, \beta_{22})(-\,\alpha_{12}\, g(c) + \beta_{12}\; g(d)\;),$$

where $-\alpha_{11}\alpha_{12} \; + \; \beta_{11}\, \beta_{22} \geq 0$ and

$$\beta_{12} g\,(\mathrm{c}) + \alpha_{12}\, g(\mathrm{d}) = 0,$$

$$3\text{-}< g,u >_{H^1} = \int_c^d [\, pg'\,\bar{u}' + \mathrm{q}g\bar{u}\,]\, dx$$

which satisfies the constraints g (c)= 0 and g (d)= 0.

Let us define the operator $\boldsymbol{\mathcal{L}}$.

Definition 2.4.1 (Operator $\boldsymbol{\mathcal{L}}$) [19]. It is denoted by Ɗ consisting of those elements g in H^1 which fulfils

1-g is absolutely continuous on every closed subinterval of [c, d].

2-pg′ is absolutely continuous on every closed subinterval of [c, d].

3-lg =(-(pg′)′ +qg)/w exists a.e and is in H^1.

The operator $\mathcal{L}$ is defined by making $\mathcal{L}g = lg$ for all g in Ɗ.

Theorem 2.4.2 [19]. L, acting on D in L_w^2 (c,d) is bounded below by ε.

Proof. For g ϵ D one obtains

$$(\mathrm{L}g,g)_{\mathrm{L}^2} = \int_c^d Lg\,\bar{g}\mathrm{wdx} = \int_c^d (-(pg'')' + \mathrm{q}g)/\mathrm{w})\,\bar{g}\mathrm{wdx} = \int_c^d [-(\,pg')'' + \mathrm{q}g]\bar{g}\,\mathrm{dx}$$

$$= -\int_c^d (pg'')'\bar{g}\; dx + \int_c^d qg\bar{g}\mathrm{dx} \; = -(\mathrm{p}g')\bar{g}\;\Big|_c^d + \int_c^d \; p|g'|^{\,2}\,\mathrm{dx} + \int_c^d \; q|g|^{\,2}\,\mathrm{dx} \; =$$

$$< g,g >_{H^1} \; \geq \int_c^d \; q|g|^{\,2}\,\mathrm{dx} \geq \; \varepsilon \int_c^d \; w|g|^{\,2}\,\mathrm{dx} = \; \varepsilon(g,g)_{\mathrm{L}^2} = \; \varepsilon\; \|g\|_{\mathrm{L}^2}^2$$

and consequently,

$$(\mathrm{L}g,g)_{\mathrm{L}^2} - \varepsilon(g,g)_{\mathrm{L}^2} = (\mathrm{L}g - \varepsilon g, g)_{\mathrm{L}^2} = \; ((\mathrm{L} - \varepsilon I)g,\; g)_{\mathrm{L}^2} > 0.$$

□

Theorem 2.4.3 [19]. L^{-1} exists and is given by a Green's function G (x,ξ) .

$$L^{-1} f(x)= \int_a^b G(x,\xi)f(\xi)w(\xi)d\xi.$$

L^{-1} is bounded by $1/\varepsilon$.

Proof. We know that

$$(Lg,g)_{L^2} = < g,g >_{H^1} \geq \varepsilon(g,g)_{L^2} .$$

Let $Lg= f$ then $L^{-1}f = g$ implies

$$(f,L^{-1}f)_{L^2} = < L^{-1}f,L^{-1}f >_{H^1} \geq \varepsilon\,(L^{-1}f,L^{-1}f)_{L^2}$$

$$\varepsilon\,(L^{-1}f,L^{-1}f)_{L^2} \leq (f,L^{-1}f)_{L^2} \leq |(f,L^{-1}f)_{L^2}| .$$

When Schwarz inequality is applied

$$\varepsilon\,\|L^{-1}f\|^2 \leq \|f\|\,\|L^{-1}f\|$$

$$\|L^{-1}f\| \leq 1/\varepsilon\|f\|.$$

□

Theorem 2.4.4 [19]. $\mathcal{L}$ is symmetric.

Proof [19]. The Dirichlet formula shows

$$(Lg,u)_{L^2} = < g,u >_{H^1}$$

for g in Ɗ, u in H^1. Suppose that u is also in Ɗ and change u by $\mathcal{L}$u.

Then

$$(Lg,Lu)_{L^2} = < g,\mathcal{L}u >_{H^1}.$$

This also shows

$$(Lg,Lu)_{L^2} = < \mathcal{L}g,u >_{H^1},$$

and symmetry is achieved.

□

Theorem 2.4.5 [19]. $\mathcal{L}^{-1}$ exists and is bounded.

Proof. $\mathcal{L}^{-1}\,g= f$ may be solved by means of Green's function. When Schwartz's inequality is applied,

$$(f,\ L^{-1}f)_{L^2} =$$

$$< \mathcal{L}^{-1}f,\ \mathcal{L}^{-1}f\ >_{H^1} = \| \mathcal{L}^{-1}f\|^{2}_{H^1} \ \leq\ \|f\|_{L^2}(\frac{1}{\varepsilon})\ \|f\|_{L^2} \leq\ (\frac{1}{\varepsilon})^{\,2}\ \|f\|^2_{H^1}.$$

Therefore

$$\| \mathcal{L}^{-1}\|_{H^1}\ \leq (\frac{1}{\varepsilon})$$

is found.

□

Theorem 2.4.6 [19]. $\mathcal{L}$ is self-adjoint in H^1.

Proof. (L-λI)Ɗ = H^1 where g ϵ Ɗ and f ϵ H^1 .

$$(L\text{-}\lambda I)g=f \Leftrightarrow g=(L-\lambda I)^{-1}\ f$$

$$Rf=g=\int_a^b G(x,t)f(t)w(t)dt.$$

g=Rf ∈ H^1. Hence the range of $\mathcal{L}$ is the whole H^1. So $\mathcal{L}$ is maximally extended and symmetric and therefore self-adjoint.

□

Theorem 2.4.7 [19]. The spectrum of $\mathcal{L}$ possesses the same eigenvalues as L, $\{\lambda_i\}_{i=1}^{\infty}$ and the same eigenfunctions $\{g_i\}_{i=1}^{\infty}$. Because $\|g_i\|^2_{H^1} = \lambda_i\ \|g_i\|^2 = \lambda_i$, i=1,2,…, however they must be normalized. These eigenfunctions create a complete orthogonal set in H^1.

Proof [19]. It is known that

$$(lg,u)_{L^2} = < g,u\ >_{H^1}.$$

Let $lg_n = \lambda_n g_n$ exist.

$$< g_n, g_n >\ _{H^1} = (lg_n, g_n)_{L^2} = (\lambda_n g_n, g_n)_{L^2} = \lambda_n (g_n, g_n)_{L^2} = \lambda_n \|g_n\|^2_{L^2},$$

$$\|g_n\|^2_{H^1} = \lambda_n \|g_n\|^2_{L^2} \Rightarrow 1/\lambda_n \|g_n\|^2_{H^1} = \|g_n\|^2_{L^2} \qquad \left\|\sqrt{1/\lambda_n}\, g_n\right\|^2_{H^1} = \|g_n\|^2_{L^2}.$$

Suppose u possesses orthogonality to the span of $\{g_i\}_{i-1}^{\infty}$. Then

$$(Lg_i, u)_{L^2}\ = < g_i, u >_{H^1} = 0$$

implies the orthogonality of u to the range of L. But this is whole $L^2_w(c,d)$ so u=0 in $L^2_w(c,d)$. Hence u=0 in H^1.

□

CHAPTER 3

REGULAR HAMILTONIAN SYSTEMS

3.1. INTRODUCTION

It can be seen from [22] that second order Sturm-Liouville equation of –(pg')' +qg =λwg + wf can be written in linear Hamiltonian format. It is defined on an interval (c,d) with p and w bigger than zero and both of them with q are continuos on (c,d). If it is permitted to write g_1= g, g_2= pg', then

$$g'_1 = (1/p)y_2,$$

$$-g'_2 = \lambda w g_1 - q g_1 + wf,$$

or

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}' = [\lambda \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -q & 0 \\ 0 & 1/p \end{pmatrix}]\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} + \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Above equation can be put in

$$JG' = [\lambda\mathcal{A} + \mathfrak{B}]G + \mathcal{A}F \quad (10),$$

where

$$J=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad G'=\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}',\quad \mathcal{A}=\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix},\ \mathfrak{B}=\begin{pmatrix} -q & 0 \\ 0 & 1/p \end{pmatrix},\ F=\begin{pmatrix} f \\ 0 \end{pmatrix}.$$

The classical linear, scalar fourth order differential equation

$$(pg'')'' - (qg')' + rg = \lambda wg + wf$$

can also be written in a linear Hamiltonian format. If it is permitted to write

$$g_1 = g,$$

$$g_2 = g',\ g_3 = -(pg'')' + qg',\ g_4 = pg''$$

then

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}$$

$$= [\ \lambda \begin{pmatrix} w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -r & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/p \end{pmatrix}] \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} + \begin{pmatrix} w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which is in the form (10).

More generally any scalar equation which is self-adjoint can be written in Hamiltonian form

$$JG' = [\lambda\mathcal{A} + \mathfrak{B}]G \quad (11),$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$\mathcal{A}$ and $\mathfrak{B}$ are equal to their adjoints and $\mathcal{A}$ is bigger than or equal to zero. They are also all real [23].

The regular second order boundary value problems in left definite context contains the Hermitian form

$$\int_c^d [pg'\,\bar{u}' + qg\bar{u}\ \ ]dx + KH,$$

where KH represents boundary terms.

In the same way, the fourth order problems involve

$$\int_c^d [pg''\,\bar{u}'' + qg'\bar{u} + rg\bar{u}\ \ ]dx + KH.$$

Higher order problems have the same form but they were not investigated in depth because of the extreme complexity of the boundary terms. All of these complexities are studied through the vector algebra in (11) in left definite context. The new variables are added to the system to formulate the boundary conditions.

The notations belong to Hinton and Shaw [24-26], Schneider and Niessen [11,12].

(10) with constraint

$$AG(c) + BG(d) = 0,$$

will considered as follows:

The equation (10) with its constraint is considered over a finite interval [c,d], where G possesses dimension 2n, J=$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, I is the n×n identity matrix, $\mathcal{A}$ = $\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$, where the components are all n×n and E=E^* ≥0 where B=B^* is 2n×2n.

If G satisfies (11) and

$$\int_c^d G^* \text{AG dx} = 0$$

then G is identically 0 which is called Atkinson definiteness condition.

The boundary coefficients A and B are 2n×2n constant matrices which satisfy the classical situation for self –adjoint boundary conditions

$$\text{AJ}A^* = \text{BJ}B^*.$$

The Hilbert space is generated by the inner product

$$(G,U)_{\mathcal{A}} = \int_c^d U^* \text{AG d}x \ (12).$$

(12) is the classical context for boundary value problem. The space of these equivalent classes for the Hilbert space is denoted by $L^2_{\mathcal{A}}$(c,d).

Now let us investigate the $L^2_{\mathcal{A}}$ theory, the Dirichlet formula, and H^1 theory respectively.

3.2 $L^2_{\mathcal{A}}$ THEORY

The problem of determining a self-adjoint operator in $L^2_{\mathcal{A}}(\text{c, d})$ is closely related with the boundary conditions appearing in the bvp.

Now let us define the operators L_M, L_m and L.

Definition 3.2.1 (Maximal Operator) [22]. Let D_M denote the set of those elements G in $L^2_{\mathcal{A}}$(c,d) which fulfils

(1) JG'-𝔅G=$\mathcal{A}$F exists a.e. and F is in $L^2_{\mathcal{A}}$(c,d).

The maximal operator L_M is defined by making $L_M G= F$ when JG'-𝔅G=$\mathcal{A}$F, G in D_M.

Definition 3.2.2 (Minimal Operator) [22]. Let D_m denote the set of those elements G in $L^2_{\mathcal{A}}$(c,d) which fulfils

(2) G is in D_M.

(3) G(c) = 0, G(d) = 0.

The minimal operator L_m is defined by making $L_m G = F$ when JG'-𝔅G=$\mathcal{A}$F, G in D_m.

Definition 3.2.3 (Operator L) [22]. Let D denote the set of those elements G in $L^2_{\mathcal{A}}$(c,d) which fulfils

(1) G is in D_M.

(2) AG(c) + BG(d)= 0 where A and B are 2n×2n matrices with rank A: B = 2n.

The operator L is defined by making LG=F when JG'-𝔅G= $\mathcal{A}$F, G in D.

Theorem 3.2.4 [22]. In $L^2_{\mathcal{A}}$(c,d), $L_m^* = L_M$, $L_M^* = L_m$.

Proof [20]. Suppose that G is in D_m and suppose that U is in the domain of L_m^*. Then

$$(L_m G, U)_{\mathcal{A}} = (G, L_m^* U)_{\mathcal{A}}.$$

This is tantamount to

$$\int_c^d U^*[JG' - \mathfrak{B}G]\, dx = \int_c^d [L_m^* U]^* \mathcal{A} G\, dx,$$

or

$$\int_c^d U^* JG'\, dx - \int_c^d [\mathfrak{B}U + \mathcal{A}(L_m^* U)]^* G\, dx = 0.$$

Integration by parts to the second integral is done, remembering that G(c) =0, G(d)= 0.

$$\int_c^d U^* JG'\, dx + \int_c^d [\ \int_c^v [\mathfrak{B}U + \mathcal{A}(L_m^* U)] d\xi]^* G'\, dx = 0,$$

or

$$\int_c^d [-JU + \int_c^v [\mathfrak{B}U + \mathcal{A}(L_m^* U)] d\xi]^* G'\, dx = 0.$$

Elements that are orthogonal to G' will be found now.

If the bracketed term is a constant R, then

$$\int_c^d R^* G'\, dx = R^* G|_c^d = 0.$$

Suppose that the bracketed term is V. By drawing upon the Gram-Schmidt procedure, it is assumed that

$$(V, R)_{\mathrm{I}} = \int_c^d R^* \mathrm{V}\, \mathrm{d}x = 0.$$

Because the constant can be arbitrary,

$$\int_c^d V \mathrm{d}x = 0.$$

Now let

$$\tilde{S} = \int_c^x V \mathrm{d}\xi.$$

Then $\tilde{S}(c) = 0$, $\tilde{S}(d) = 0$. $\tilde{S}'$ is an acceptable S' (is in the domain of L_M^*). Thus $\tilde{S}'$ is orthogonal to V. But $\tilde{S}'$ = V in L_{I}^2(c,d), and so V= 0.

It is concluded that the bracketed term must be constant, and

$$\text{-JU} + \int_c^v [\mathfrak{B}U + \mathcal{A}(L_m^* U)]\, d\xi \;\; = R.$$

This explains that U can be differentiable and

$$\mathcal{A} L_m^* U = \mathrm{JU}' - \mathfrak{B}\mathrm{U}.$$

Setting $L_m^* U$= H, we see $L_m^* U$= H if and only if

$$\mathrm{JU}' - \mathfrak{B}\mathrm{U} = \mathcal{A}\mathrm{H}.$$

The form of L_m^* was found. Because there are no constraints for U at c or d, it is found that L_m^* and L_M possess the tantamount sort of form and that the domain of L_m^*: $D_{L_m^*} \subset D_M$. Thus $L_m^* \subset L_M$.

Conversely, let G be in D_m, U be in D_M, and L_mG= F, $L_M U$ =H. Then

$$\mathrm{JG}' - \mathfrak{B}\mathrm{G} = \mathcal{A}\mathrm{F},$$

$$\mathrm{JU}' - \mathfrak{B}\mathrm{U} = \mathcal{A}\mathrm{H},$$

with F and H in $L_{\mathcal{A}}^2$(c,d). It is computed that

$$(L_m G, U)_{\mathcal{A}} = \int_c^d U^* (L_m \mathrm{G})\, \mathrm{d}x \quad (13)$$

$$= \int_c^d U^* \mathcal{A}\mathrm{F}\, \mathrm{d}x$$

$$= \int_c^d U^* [\mathrm{JG}' - \mathfrak{B}\mathrm{G}]\, \mathrm{d}x.$$

(13) is tantamount to

$$U^* \mathrm{JG}\, |_c^d + \int_c^d [JU' - \mathfrak{B}\mathrm{U}]^* \mathrm{G}\, \mathrm{d}x.$$

Because G is in D_m, G(c)= 0, G(d)=0, and

(13) becomes

$$= \int_c^d [JU' - \mathfrak{B}U]^* G \, dx$$

$$= \int_c^d H^* \mathcal{A} G \, dx$$

$$= \int_c^d (L_M U)^* \mathcal{A} G \, dx$$

$$= (G, L_M U)_{\mathcal{A}}.$$

This explains that U is in $D_{L_m^*}$, $D_M \subset D_{L_m^*}$, and $L_M U = L_m^* U$: $L_{M.} \subset L_m^*$. Hence $L_m^* = L_M$.

It is firstly noted that $L_m \subset L_M$, that is, $D_m \subset D_M$ and for G in D_m, $L_m G = L_M G$. Now suppose that U is in D_{M^*} and G is in D_m, a subset of D_M. Then $(L_M G, U)_{\mathcal{A}}$ = $(L_m G, U)_{\mathcal{A}}$. If the previous proof is again performed, it shows that U is in D_M and $L_M^* U = L_M U$. So $L_M^* \subset L_M$.

If the Green's formula is again applied, it shows that U(c) = 0, U(d)= 0, because G is in D_M, G(c) and G(d) are arbitrary. Thus $D_{M^*} \subset D_m$ and $L_M^* \subset L_m$.
Green's formula implies that $L_m \subset L_M^*$, and so $L_M^* = L_m$.

□

The operator L is self-adjoint when L is tantamount to its adjoint and these operators are determined by using boundary conditions [22].

The key way to guarantee that operator is self-adjoint is to use Green's formula. A and B are adjointed to 2n×2n matrices K and S such that the 4n×4n matrix $\begin{pmatrix} A & B \\ K & S \end{pmatrix}$ has rank 4n (is nonsingular). Then let $\tilde{A}, \tilde{B}, \tilde{K}, \tilde{S}$ be 2n×2n matrices determined by requiring the following

$$\begin{pmatrix} \tilde{A}^* & \tilde{B}^* \\ \tilde{K}^* & \tilde{S}^* \end{pmatrix} \begin{pmatrix} A & B \\ K & S \end{pmatrix} = \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}.$$

Theorem 3.2.5 [22]. (Green's Formula) Suppose that G and U are in D_M, then

$$\int_c^d U^* (L_m G) \, dx - \int_c^d (L_M U)^* \mathcal{A} G \, dx =$$

$$[\tilde{A}U(c) + \tilde{B}U(d)]^* [AG(c) + BG(d)] + [\tilde{K}U(c) + \tilde{S}U(d)]^* [KG(c) + SG(d)]. \quad (14)$$

Proof [20]. Let L_mG = AF, $L_M U$ = AH. Then the left-hand side of (14)

$$\int_c^d U^*(L_m G)\, dx - \int_c^d (L_M U)^* \mathcal{A} G\, dx$$

becomes

$$\int_c^d U^*(AF)\, dx - \int_c^d (AH)^* G\, dx$$

$$= \int_c^d [U^*(JG' - BG) - (JU' - BU)^* G]\, dx$$

$$= \int_c^d [U^* JG' + U'^* JG]\, dx$$

$$= U^* JG|_c^d .$$

Now

$$U^* JG|_c^d \;= U^*(d)JG(d) - U^*(c)JG(c)$$

$$= (U^*(c), U^*(d)) \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} G(c) \\ G(d) \end{pmatrix}.$$

If the substitution is made for the middle matrix,

$$(U^*(c), U^*(d)) \begin{pmatrix} \tilde{A}^* & \tilde{K}^* \\ \tilde{B}^* & \tilde{S}^* \end{pmatrix} \begin{pmatrix} A & B \\ K & S \end{pmatrix} \begin{pmatrix} G(c) \\ G(d) \end{pmatrix}$$

is found. This is equivalent to

$$[\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{K} & \tilde{S} \end{pmatrix} \begin{pmatrix} U(c) \\ U(d) \end{pmatrix}]^* [\begin{pmatrix} A & B \\ K & S \end{pmatrix} \begin{pmatrix} G(c) \\ G(d) \end{pmatrix}]$$

$$= \begin{pmatrix} \tilde{A}U(c) + \tilde{B}U(d) \\ \tilde{K}U(c) + \tilde{S}U(d) \end{pmatrix}^* \begin{pmatrix} AG(c) + BG(d) \\ KG(c) + SG(d) \end{pmatrix}$$

$$= [\tilde{A}U(c) + \tilde{B}U(d)]^* [AG(c) + BG(d)] + [\tilde{K}U(c) + \tilde{S}U(d)]^* [KG(c) + SG(d)].$$

This completes the proof.

□

Theorem 3.2.6 [22]. The domain of L^*, D^* possesses those elements U in $L^2_{\mathcal{A}}$(c,d) which fulfil

(1) U is in D_M,

(2) $\tilde{K}$U(c) + $\tilde{S}$U(d) = 0.

For U in D^*, L^*U = H if and only if

$$JU' - \mathfrak{B}U = \mathcal{A}H.$$

Proof [20]. Because $L_0 \subset$ L $\subset L_M$, $L_0 \subset L^* \subset L_M$ also exist . The form of L^* is tantamount to that of L. Suppose that G is in D, U is in D^* and perform (14). The left-hand side of the equation

$$\int_c^d U^*(L_m\mathrm{G})\,\mathrm{d}x - \int_c^d (L_M U)^* \mathcal{A}\mathrm{G}\,\mathrm{d}x =$$

$$[\tilde{A}\mathrm{U(c)} + \tilde{B}\mathrm{U(d)}]^* [\mathrm{AG(c) + BG(d)}] + [\tilde{K}\mathrm{U(b)} + \tilde{S}\mathrm{U(d)}]^* [\mathrm{KG(c) + SG(d)}]$$

vanishes, and the second term on the right, AG(c) + BG(d) also vanishes. But the term KG(c)+SG(d) is arbitrary, and this leads U to make $\tilde{K}$U(c)+$\tilde{S}$U(d) = 0.

Conversely, if U possesses the properties listed above, then U is in the domain of the adjoint.

□

To prove the self-adjointness condition of L, parametric conditions are needed. It is firstly noted that if G is in D and U is in D^*, then

$$\mathrm{AG(c) + BG(d)} = 0, \quad \tilde{A}\mathrm{U(c)} + \tilde{B}\mathrm{U(d)} = \psi,$$

$$\mathrm{KG(c) + SG(d)} = \varphi, \quad \tilde{K}\mathrm{U(c)} + \tilde{S}\mathrm{U(d)} = 0,$$

where φ and ψ are arbitrary.

These can be solved for G(c), G(d), U(c), U(d), leading to

$$\mathrm{G(c)} = -\mathrm{J}\tilde{K}^*\psi, \quad \mathrm{U(c)} = -\mathrm{J}A^*\varphi,$$

$$\mathrm{G(d)} = \mathrm{J}\tilde{S}^*\psi, \quad \mathrm{U(d)} = \mathrm{J}B^*\varphi,$$

by requiring that D is equal to its adjoint yields U to have the A-B boundary condition [22].

Theorem 3.2.7 [22]. The boundary value problem on the operator L, possesses self-adjointness if and only if

$$\mathrm{AJ}A^* = \mathrm{BJ}B^*.$$

Proof [20]. If L possesses self-adjointness, then Z has the D-boundary conditions. Therefore A[-J$A^*\varphi$] + B[J$B^*\varphi$] = 0, and [-AJA^*+ BJB^*] φ = 0. Since φ is arbitrary, -AJA^*+ BJB^*= 0.

Conversely, if AJA^*= BJB^*, then (-AJ BJ)$\begin{pmatrix} A^* \\ B^* \end{pmatrix}$ = 0. This explains that the columns of $\begin{pmatrix} A^* \\ B^* \end{pmatrix}$ for n independent solutions to the equation (-AJ BJ)X = 0. But from the equations computed earlier, (-AJ BJ) $\begin{pmatrix} \widetilde{K}^* \\ \tilde{S}^* \end{pmatrix}$ = 0 as well. Again full complement of n solutions exist. Thus there must take place a constant, nonsingular matrix V^* such that $\begin{pmatrix} \widetilde{K}^* \\ \tilde{S}^* \end{pmatrix} V^* = \begin{pmatrix} A^* \\ B^* \end{pmatrix}$, or (A B) = V ($\widetilde{K}$ $\tilde{S}$).

This says that the boundary conditions are of the form

$$AG(c) + BG(d) = 0$$

and

$$\widetilde{K}G(c) + \tilde{S}G(d) = 0.$$

Since the forms of L and its adjoint are equivalent, L is tantamount to its adjoint.

□

Lemma 3.2.8 [20]. Suppose that Ҡ(x, λ) is a fundamental matrix for (11) which fulfils Ҡ(a, λ) = I. (That is, Ҡ(x, λ) is an 2n×2n matrix whose columns fulfil the differential equation. With x = a, Ҡ(a, λ) = I, where I is a 2n×2n identity matrix) Then for all x,

$$Ҡ^*(x, \bar{\lambda})\, J\, Ҡ(x, \lambda) = J.$$

Proof [20]. Ҡ*(x, $\bar{\lambda}$) fulfils

$$-Ҡ^*(x, \bar{\lambda})'\, J = Ҡ^*(x, \bar{\lambda})[\lambda A + B],$$

while Ҡ (x, λ) fulfils

$$J\, Ҡ(x, \bar{\lambda})' = [\lambda A + B]\, Ҡ(x, \lambda).$$

Right multiply the first by Ҡ (x, λ), left multiply the second by Ҡ*(x, $\bar{\lambda}$), and subtract because J is constant

$$Ҡ^*(x, \bar{\lambda})'\, J\, Ҡ(x, \lambda) + Ҡ^*(x, \bar{\lambda})\, J\, Ҡ'(x, \lambda)$$

$$= [Ҡ^*(x, \bar{\lambda})\, J\, Ҡ(x, \lambda)]' = 0$$

and Ҡ*(x, $\bar{\lambda}$) J Ҡ (x, λ) = C. If x = a, Ҡ* = Ҡ = I, and so C = J.

□

Theorem 3.2.9 [21]. Let L be self-adjoint. The spectrum of L is discrete, possessing real eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ which are supplied by det(A K (c, λ) + B K (d, λ)) = 0 and which converge only at ∞. If λ is not an eigenvalue, then LG = F can be determined for G. The solution is as follows:

$$G(x) = \int_c^d R(\lambda, x, \xi)\, \mathcal{A}(\xi)\, F(\xi)\, d\xi,$$

where

$$R(\lambda, x, \xi) = -K(x, \lambda)(A K(c, \lambda)) + (B K(d, \lambda))^{-1} AK(c, \lambda)\, JK^*(\xi, \bar{\lambda}),$$

$$c < \xi < x < d,$$

$$= K(x, \lambda)(A K(c, \lambda)) + (B K(d, \lambda))^{-1} B K(c, \lambda)\, JK^*(\xi, \bar{\lambda}),$$

$$c < x < \xi < d.$$

Proof [20]. Since the solutions to the homogeneous equation are determined by G(x, λ) = P K (x, λ), for some constant P, one shall draw upon the method of the variation of parameters. There exist, with P now variable,

$$JG' = J' K P + J K P',$$

$$[\lambda A + B]G = [\lambda A + B] K P.$$

Thus

$$JG' - [\lambda A + B]G = \{J' K - [\lambda A + B] K\}P + J K P'$$

$$= J K P'$$

$$= AF.$$

Therefore

$$J K P' = AF.$$

Now from the Lemma 3.2.8

$$(JG)^{-1} = -JK^*(x, \bar{\lambda})$$

exists. So

$$P' = -J K^*(x, \bar{\lambda})\, A(x)\, F(x).$$

Thus

$$G = -K(x, \lambda)\, JK^*(\xi, \bar{\lambda})\, A(\xi)\, F(\xi) d\xi + K(x, \lambda)T.$$

But also perform the boundary condition

$$AG(c) + BG(d) = 0.$$

Here

$$G(c) = T,$$

$$G(d) = -K(d,\lambda)\int_c^d J\,K^*(\xi,\bar{y})\,A(\xi)\,F(\xi)d\xi + K(d,\lambda)T.$$

These yield

$$G(x) = -K(x,\lambda)\,[A + BK(d,\lambda)]^{-1}\,A\int_c^x J\,K^*(\xi,\bar{\lambda})\,A(\xi)\,F(\xi)d\xi$$

$$+K(x,\lambda)\,[A + BK(d,\lambda)]^{-1}\,B\int_c^d J\,K^*(\xi,\bar{\lambda})\,A(\xi)\,F(\xi)d\xi,$$

which is written as

$$G(x) = \int_c^d R(\lambda,x,\xi)\,\mathcal{A}(\xi)\,F(\xi)\,d\xi,$$

$$R(\lambda,x,\xi) = -(Kx,\lambda)\,[A + BK(d,\lambda)]^{-1}\,AJ\,K^*(\xi,\bar{\lambda}),\ c < \xi < x < d$$

$$= K(x,\lambda)\,[A + BK(d,\lambda)]^{-1}\,BJK^*(\xi,\bar{\lambda}),\ c < x < \xi < d.$$

Since L possesses self-adjointness L must take place for all complex λ. It definitely holds for all real λ apart from the zeros of det $[A + BG(d,\lambda)] = 0$. Since the determinant is analytic in λ and is not identically zero, it may possess only isolated zeros, which may accumulate only at $\mp\infty$.

□

3.3 THE DIRICHLET FORMULA

It is known from [22] that the Dirichlet formula is an important tool to construct a new inner product and a self-adjoint operator in the corresponding Sobolev space. A Sobolev inner product space will be found by the matrix $\mathfrak{B}$ with additional inner product entries determined by the boundary coefficient A and B. To obtain this new inner product Dirichlet formula will be used.

$\mathfrak{B}$ will be decomposed into $\begin{pmatrix} -B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}$ and assume that $-B_{11} \leq 0 \leq B_{22}$ and for some $\varrho > 0$, $\varrho E \leq B_{11}$. G and U will be decomposed into $\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ and $\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$.

The notation of Schneider and Niessen [11,12] will be used. Two inner products will also be used. The first,

$$(G,U)_{\mathcal{A}} = \int_c^d U_1^*\,E\,G_1\,dx$$

$$= \int_c^d U^* \mathcal{A}\, G \, \mathrm{d}x,$$

generates the classical $L^2_{\mathcal{A}}$ space where $\mathcal{A} = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$.

The second,

$$(G, U)_{\mathcal{A}} = \int_c^d [U_1^* \, B_{11} G_1 + U_2^* \, B_{22} G_2] \, \mathrm{d}x$$

$$= \int_c^d U^* \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} \mathrm{G} \, \mathrm{d}x$$

will be drawn upon as piece of the inner products in various Sobolev spaces.

A beginning variant of the Dirichlet formula is used to connect these inner products. ℓG = JG' – $\mathfrak{B}$G $= \mathcal{A}$F may be used to define LG=F. Without evaluating boundary terms, it is computed that

$$(LG, U)_{\mathcal{A}} = \int_c^d U^* (\mathrm{LG}) \, \mathrm{d}x = \int_c^d U^* (\mathcal{A}\mathrm{F}) \, \mathrm{d}x$$

$$= \int_c^d (\, U_1^* \; U_2^*) \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \mathrm{d}x = \int_c^d U_1^* E \, F_1 \mathrm{d}x.$$

Now JG' – $\mathfrak{B}$G $= \mathcal{A}$F is the same as

$$-G_2' + B_{11} G_1 - B_{12} G_2 = \mathrm{E} F_1,$$

$$G_1' - B_{12}^* G_1 - B_{22} G_2 = 0,$$

because

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} G_1' \\ G_2' \end{pmatrix} - \begin{pmatrix} -B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}.$$

Therefore,

$$\int_c^d U_1^* E \, F_1 \mathrm{d}x = \int_c^d U_1^* [\, -G_2' + B_{11} G_1 - B_{12} G_2] \, \mathrm{d}x$$

$$= -U_1^* G_2 |_c^d + \int_c^d [U_1^{*\prime} \, G_2 + U_1^* B_{11} G_1 - U_1^* B_{12} G_2] \, \mathrm{d}x$$

$$= -U_1^* G_2 |_c^d + \int_c^d [(B_{12}^* U_1 \; + B_{22} U_2)^* \, G_2 + \; U_1^* B_{11} \, G_1 - U_1^* B_{12} \, G_2] \, \mathrm{d}x$$

$$= -U_1^* G_2 |_c^d + \int_c^d [(\, U_1^* B_{12} \, G_2 + U_2^* B_{22} \, G_2 + U_1^* B_{11} \, G_1 - U_1^* B_{12} \, G_2] \, \mathrm{d}x$$

$$= -U_1^* G_2 |_c^d + \int_c^d [U_1^* B_{11} \, G_1 \; + \; U_2^* B_{22} \, G_2 \,] \, \mathrm{d}x.$$

The second component of arbitrary U will be defined as:

$$U_1' - B_{12}^* \, U_1 = B_{22} U_2.$$

One then may introduce the following Dirichlet formula:

$$(LG, U)_{\mathcal{A}} = -U_1^* G_2|_c^d + < G, U >_{H^1}. \quad (14)$$

The first term on the right-hand side of (14) will be worked on.

So see the boundary condition

$$AG(c)+BG(d) = 0,$$

which may be rewritten as

$$(A_1, A_2,\ B_1, -B_2)\begin{pmatrix} G_1(c) \\ G_2(c) \\ G_1(d) \\ -G_2(d) \end{pmatrix} = 0,$$

or

$$(A_1, B_1, A_2, -B_2)\begin{pmatrix} G_1(c) \\ G_1(d) \\ G_2(c) \\ -G_2(d) \end{pmatrix} = 0.$$

If (A_1, B_1) = M, $(-A_2, B_2)$ = N, $\begin{pmatrix} G_1(c) \\ G_1(d) \end{pmatrix} = g_1$, $\begin{pmatrix} G_2(c) \\ G_2(d) \end{pmatrix} = g_2$ are set, the below exists:

$$Mg_1 - Ng_2 = 0.$$

As it can be seen, the rank of M: N can be 2n. Suppose that the rank of N is z, $0 \leq z \leq 2n$. The sizes of $A_1, A_2,\ B_1,\ B_2$ are all 2n×n; M and N are 2n×2n. $Mg_1 - Ng_2 = 0$ will be solved for as many of the terms in g_2 as possible.

Unitary matrices B and W exist in such a way that

$$Mg_1 - Ng_2 = MBB^*g_1 - NBB^*g_2 = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} B^*g_1 - \begin{pmatrix} N_1 \\ 0 \end{pmatrix} B^*g_2 = 0,$$

and

$$W\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} B^*g_1 - W\begin{pmatrix} N_1 \\ 0 \end{pmatrix} B^*g_2 = \begin{pmatrix} M_{11} & M_{12} \\ M_{22} & M_{22} \end{pmatrix}\begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix} - \begin{pmatrix} N_{11} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g_{21} \\ g_{22} \end{pmatrix} = 0,$$

where N_{11} is nonsingular and the following exists:

$$MB = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix},\quad NB = \begin{pmatrix} N_1 \\ 0 \end{pmatrix},$$

$$W\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, W\begin{pmatrix} N_1 \\ 0 \end{pmatrix} = \begin{pmatrix} N_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

$$B^*g_1 = \begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix},\quad B^*g_2 = \begin{pmatrix} g_{21} \\ g_{22} \end{pmatrix}.$$

Therefore, the boundary condition is decomposed into two parts:

$$g_{21} = N_{11}^{-1}(M_{11}g_{11} + M_{12}g_{12}) = N_{11}^{-1}M_1B^*g_1,$$

and

$$M_{21}g_{11} + M_{22}g_{12} = M_2B^*g_1 = 0.$$

The first one will be used. The second constraint will be zero.

The boundary term in the Dirichlet formula becomes

$$-U_1^*G_2|_c^d = (U_1^*(\mathrm{c})\ U_1^*(\mathrm{d}))\begin{pmatrix} G_2(c) \\ -G_2(d) \end{pmatrix}$$

$$= u_1^*g_2 = (B^*u_1)^*(B^*g_2)$$

$$= (u_{11}^*\ u_{12}^*)\begin{pmatrix} g_{21} \\ g_{22} \end{pmatrix}$$

$$= u_{11}^*(N_{11}^{-1}(M_{11}\ M_{12})\begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix} + u_{12}^*g_{22}$$

$$= u_{11}^*(N_{11}^{-1}(M_{11}g_{11} + M_{12}g_{12})) + u_{12}^*g_{22}$$

$$= u_{11}^*(N_{11}^{-1}M_{11})\,g_{11} + u_{11}^*(N_{11}^{-1}M_{12})\,g_{12} + u_{12}^*g_{22}.$$

There exists no control over g_{22}. Hence the restriction g_{12} =0, u_{12} = 0 will take place. With this constraint, one has

$$-U_1^*G_2|_c^d = u_{11}^*(N_{11}^{-1}M_{11})\,g_{11}$$

$$= u_{11}^*\ u_{12}^*\begin{pmatrix} N_{11}^{-1}M_{11} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix}$$

$$= z_1^*\ B\begin{pmatrix} N_{11}^{-1}M_{11} & 0 \\ 0 & 0 \end{pmatrix}B^*g_1.$$

$N_{11}^{-1}M_{11} \geq 0$ is assumed and it will be showed that it is symmetric.

Next see the constraint $M_2B^*g_1 = 0$. It is easy to infer that

$$\mathrm{V}= (\mathrm{M}, -\mathrm{N})\begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \quad \mathrm{X}=(\mathrm{M}, -\mathrm{N})\begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & -I \end{pmatrix}.$$

Thus the self-adjointness condition $\mathrm{VJ}V^* = \mathrm{XJ}X^*$ is tantamount to $\mathrm{M}N^* = \mathrm{N}M^*$.

Placing the unitary matrix B, the equation

$$(\mathrm{MB})(NB)^* = (\mathrm{NB})(MB)^*$$

exists. This equivalence is the same as

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \begin{pmatrix} N_1 \\ 0 \end{pmatrix}^* = \begin{pmatrix} N_1 \\ 0 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}^* .$$

Multiply from the right by W^*, from the left by W, it is obtained

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} N_{11}^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} N_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M_{11}^* & M_{21}^* \\ M_{12}^* & M_{22}^* \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} {N_{11}}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M_{11}^* & M_{21}^* \\ M_{12}^* & M_{22}^* \end{pmatrix} \begin{pmatrix} {N_{11}}^{*-1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} {N_{11}}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} M_{11} & 0 \\ M_{21} & 0 \end{pmatrix} = \begin{pmatrix} M_{11}^* & M_{21}^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} {N_{11}}^{*-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

This leads

$$M_{11}\, N_{11}^* = \; N_{11}\, M_{11}^*$$

or

$$N_{11}^{-1} M_{11} = (N_{11}^{-1} M_{11})^*.$$

Therefore the symmetry is achieved and

$$M_{21} N_{11}^* = 0.$$

Since N_{11} is nonsingular, this makes $M_{21} = 0$.

The second boundary condition thus changes to $M_{22} g_{12} = 0$. Hence the boundary conditions change to

$$\begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix} - \begin{pmatrix} N_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_{21} \\ g_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the rank of $\begin{pmatrix} M_{11} & M_{12} & N_{11} & 0 \\ 0 & M_{22} & 0 & 0 \end{pmatrix}$ is 2n, M_{22} is nonsingular. Therefore It is concluded that, if G possesses the original boundary conditions, then $g_{12} = 0$.

In summary, the Dirichlet formula

$$(LG, U)_{\mathcal{A}} = \langle G, U \rangle_{H^1} + \; z_1^* \, B \begin{pmatrix} N_{11}^{-1} M_{11} & 0 \\ 0 & 0 \end{pmatrix} B^* g_1 \quad (15)$$

exists, where

$$g_1 = \begin{pmatrix} G_1(c) \\ G_1(d) \end{pmatrix}, \quad u_1 = \begin{pmatrix} U_1(c) \\ U_1(d) \end{pmatrix}.$$

The elements of G and U will be $g_{12} = 0$ and $u_{12} = 0$, where

$$B^* g_1 = \begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix} \text{ and } B^* u_1 = \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix}.$$

The right-hand side of (15) leads an inner product space which is known as the Sobolev space. The subspace is constrained by the linear constraint $g_{12} = u_{12} = 0$. This subspace is the context which it is sought to be.

3.4 H^1 THEORY

As the left definite problem is known from [22], it is assumed that if $\|G\|_{\mathcal{A}}= 0$, then $\|G\|_{H^1}= 0$ as well. Because $L^2_{\mathcal{A}}$(c,d) can be an collection of equivalence classes, so also can H^1 be. In addition, both norms are positive definite for scalar representations.

It must be assumed that $\|G\|_{H^1}= 0$, then $\|F\|_{H^1} = 0$ so that $\mathcal{L}$ is well defined.
There is little control over the matrices B_{11} and B_{22}. For example, in the case of a fourth order scalar problem one should have

$$B_{11} = \begin{pmatrix} r & 0 \\ 0 & q \end{pmatrix} \text{ and } B_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1/p \end{pmatrix}.$$

It may be possible that both of them or only B_{22} is singular.

Let us define operator $\mathcal{L}$.

Definition 3.4.1 (Operator $\mathcal{L}$) [22]. It is denoted by Ɗ consisting of those elements G in H^1 fulfilling

(1) ℓG = JG'-$\mathfrak{B}$G = $\mathcal{A}$F exists a.e. and F is in H^1 ($\mathcal{A} = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$).

(2) AG(c) + BG(d) = 0, where A and B are 2n×2n matrices with rank (A:B)= 2n and AJA^*= BJB^*.

The operator $\mathcal{L}$ is defined by making $\mathcal{L}G = F$ for all g in Ɗ.

Theorem 3.4.2 [22]. L, acting on D in $L^2_{\mathcal{A}}$(c,d), is bounded below by ϱ.

Proof [22]. The Dirichlet formula indicates

$$(LG, G)_{\mathcal{A}} = < G, G >_{H^1} = \varrho \ (G, G)_{\mathcal{A}}.$$

This implies that

$$((L- \varrho)G, G) \geq 0.$$

□

Corollary 3.4.3 [22]. L^{-1} holds acting on $L^2_{\mathcal{A}}$(c,d) and is bounded above by ϱ^{-1}.

Proof [22]. Let LG = F, L^{-1} F = G. Then

$$(F,\ L^{-1}\ F)_{\mathcal{A}}\ \ \geq\ \varrho\ (\ L^{-1}\ F,\ L^{-1}\ F)_{\mathcal{A}}$$

Apply the Schwartz's inequality on the left.

$$|(F,\ L^{-1}\ F)_{\mathcal{A}}| \leq \|F\|\ \|\ L^{-1}\ F\|$$

Hence,

$$\|F\|\ \|\ L^{-1}\ F\| \geq |(F,\ L^{-1}\ F)_{\mathcal{A}}| \geq \varrho\ \|\ L^{-1}\ F\|^2$$

and

$$\|\ L^{-1}\ \| = \sup \|\ L^{-1}\ F\| \ / \ \|F\| \leq 1/\varrho.$$

□

Theorem 3.4.4 [22]. $\mathcal{L}$ is symmetric.

Proof [22]. The Dirichlet formula shows

$$(LG, U)_{\mathcal{A}} = < G, U >_{H^1}\ . \quad (16)$$

Let G in Ɗ and U in H^1. Suppose that U is also in Ɗ and change U by $\mathcal{L}$U. Then

$$(LG, LU)_{\mathcal{A}} = < G, \mathcal{L}U >_{H^1}$$

exists. This implies that

$$(LG, LU)_{\mathcal{A}} = < \mathcal{L}G, U >_{H^1}$$

and symmetry is achieved.

□

Theorem 3.4.5 [22]. $\mathcal{L}^{-1}$ exists and is bounded.

Proof [22]. $\mathcal{L}G = F$ may be solved with the help of the Green's function. (16) provides

$$(F,\ L^{-1}\ F)_{\mathcal{A}}\ \ =\ <\ L^{-1}\ F,\ L^{-1}\ F >_{H^1}.$$

Applying the Schwarz inequality on the left we have

$$\| L^{-1} F\|_{H^1}^2 \leq \|F\|_{\mathcal{A}} \| L^{-1} F\|_{\mathcal{A}} \leq \|F\|_{\mathcal{A}} \|F\|_{\mathcal{A}}/ \varrho \leq \|F\|_{H^1} \|F\|_{H^1}/ \varrho^{\,2}$$

Here we take into account the inequality

$$\| L^{-1} F\|_{A} \leq \|F\|_{A}/ \varrho$$

from Corollary 3.4.3. Thus

$$\| L^{-1} F\|_{H^1} \leq \|F\|_{H^1}/ \varrho$$

$$\Rightarrow \sup \| L^{-1} F\|_{H^1}/ \|F\|_{H^1} \leq \varrho^{-1}$$

$$\Rightarrow \| L^{-1}\|_{H^1} \leq \varrho^{-1}.$$

Therefore, the proof is completed.

□

Theorem 3.4.6 [22]. $\mathcal{L}$ possesses self-adjointness in H^1.

Proof [22]. The range of $\mathcal{L}$ is the whole H^1. So $\mathcal{L}$ is maximally extended, symmetric operator. Hence it is self-adjoint.

□

Theorem 3.4.7 [22]. The spectrum of $\mathcal{L}$ possesses the same eigenvalues as L, $\{\lambda_i\}_{i=1}^{\infty}$, with the same eigenfunctions $\{G_i\}_{i=1}^{\infty}$. Because $\|G_i\|_{H^1}^{2} = \lambda_i \|G_i\|_{\mathcal{A}}^{2} = \lambda_i$, i=1,2,3,… however, they must be renormalized. These eigenfunctions make a complete orthogonal set in H^1.

Proof [22]. If $K_i = G_i/\sqrt{\lambda_i}$, renormalize the eigenfunctions. (Keep in mind that K_i has the boundary conditions. Hence g_{12}= 0 and K_i is in H^1.) To prove completeness, suppose that there takes place an element U which possesses orthogonality to the span of $\{K_i\}_{i=1}^{\infty}$. Then (16)

$$(LK_i, U)_{\mathcal{A}} = < K_i, U >_{H^1} = 0$$

leads that U possesses orthogonality to the range of L. In addition, this is whole $L_{\mathcal{A}}^2$ (c,d). Hence U = 0 in $L_{\mathcal{A}}^2$ (c,d). Therefore U= 0 in H^1.

□

3.5 EXAMPLE

These examples are shown in [22]. Suppose that

$g_1 = g$, $g_2 = g'$, $g_3 = -(pg'')' + qg'$, $g_4 = pg''$.

The fourth order problem

$$(pg'')'' - (qg')' + rg = \lambda wg + wf$$

can be shown by a four dimensional Hamiltonian system (11).

First see the boundary conditions created by the coefficients

$\alpha_{11} = cos\alpha$, $\alpha_{13} = -\sin\alpha$, $\alpha_{22} = cos\beta$, $\alpha_{24} = \sin\beta$,

$\beta_{31} = \cos\gamma$, $\beta_{33} = \sin\gamma$, $\beta_{42} = \cos\delta$, $\beta_{44} = \sin\delta$,

$0 < \alpha, \beta, \gamma, \delta < \pi/2$,

with all others α_{ij}, β_{ij} set equal to zero. g_3 and g_4 can be solved at both c and d,

$g_3(c) = \cot\alpha g_1(c)$, $g_4(c) = \cot\beta g_2(c)$,

$g_3(d) = \cot\gamma g_1(d)$, $g_4(d) = \cot\delta g_2(d)$,

and so the Dirichlet formula is

$\int_c^d [(pg'')'' - (qg')' + rg]\,\bar{u}dx = \int_c^d [pg''\bar{u}'' - qg'\bar{u}' + rg\bar{u}]dx + \cot\alpha g_1(a)\,\bar{u}_1(c)$ $+ \cot\beta g_2(c)\,\bar{u}_2(c) + \cot\gamma\, g_3(d)\,\bar{u}_3(d) + \cot\delta g_4(d)\,\bar{u}_4(d)$.

If $\alpha = 0$, then there exists a subspace constrained by g(c) =0, u(c)=0. The α term in the formula above is not included. The other cases $\beta = 0, \gamma = 0, \delta = 0$ are worked similarly.

If any of α, β, γ or δ is $\pi/2$, then again the various terms above are zero, but there does not take place subspace restriction.

Consider the boundary condition

$$\begin{pmatrix} .6I & .6I \\ .4I & 1.2I \end{pmatrix}\begin{pmatrix} G_1(c) \\ G_2(c) \end{pmatrix} + \begin{pmatrix} .8I & .8I \\ 1.0I & 1.6I \end{pmatrix}\begin{pmatrix} G_1(c) \\ G_2(c) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $G_1 = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, $G_2 = \begin{pmatrix} g_3 \\ g_4 \end{pmatrix}$.

When the above is rearranged, it is

$$\begin{pmatrix} .6I & .8I \\ .4I & 1.0I \end{pmatrix}\begin{pmatrix} G_1(c) \\ G_2(d) \end{pmatrix} - \begin{pmatrix} -.6I & .8I \\ -1.2I & 1.6I \end{pmatrix}\begin{pmatrix} G_1(c) \\ -G_2(d) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If this is multiplied by the unitary matrix W = $\begin{pmatrix} I & 0 \\ -2I & I \end{pmatrix}$, the result is

$$\begin{pmatrix} .6I & .8I \\ -.8I & -.6I \end{pmatrix}\begin{pmatrix} G_1(c) \\ G_1(d) \end{pmatrix} - \begin{pmatrix} -.6I & .8I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} G_1(c) \\ -G_2(d) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The boundary condition is broken into two pieces.

If it is permitted that

$$\begin{pmatrix} G_1(c) \\ G_2(d) \end{pmatrix} = \begin{pmatrix} .6I & -.8I \\ .8I & -.6I \end{pmatrix}\begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix}, \quad \begin{pmatrix} G_1(c) \\ -G_2(d) \end{pmatrix} = \begin{pmatrix} -.6I & -.8I \\ .8I & -.6I \end{pmatrix}\begin{pmatrix} g_{21} \\ g_{22} \end{pmatrix} \text{ exists,}$$

then

$$\begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix} = \begin{pmatrix} -.6I & .8I \\ -.8I & -.6I \end{pmatrix}\begin{pmatrix} G_1(c) \\ G_2(d) \end{pmatrix}, \quad \begin{pmatrix} g_{21} \\ g_{22} \end{pmatrix} = \begin{pmatrix} -.6I & .8I \\ -.8I & -.6I \end{pmatrix}\begin{pmatrix} G_1(c) \\ -G_2(d) \end{pmatrix} \text{ exists.}$$

Placing the first into the boundary condition supplies

$$\begin{pmatrix} .28I & -.96I \\ 0 & I \end{pmatrix}\begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g_{21} \\ g_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus g_{12}= 0, and also g_{21}= .28g_{11}-.96g_{12}. The term $U_1^* G_2|_c^d$ in the Dirichlet formula takes place as .28$z_{11}^* y_{11}$. Placing this into the Dirichlet formula while substituting for g_{11} and u_{11}, the below exists:

$$\int_c^d [\,(pg'')'' - (qg')' + rg]\,\bar{u}\,dx$$

$$= \int_c^d [\,pg''\bar{u}'' - qg'\bar{u}' + rg\bar{u}]\,dx$$

$$+(\bar{u}(c)\bar{u}'(c)\bar{u}(d)\bar{u}'(d))\begin{pmatrix} .1008 & .1008 & .1344 & .1344 \\ .1008 & .1008 & .1344 & .1344 \\ .1344 & .1344 & .1792 & .1792 \\ .1344 & .1344 & .1792 & .1792 \end{pmatrix}\begin{pmatrix} g(c) \\ g'(c) \\ g(d) \\ g'(d) \end{pmatrix}.$$

The constraint g_{12} =0 takes place as

.8g(c) + .6g(d) = 0, .8g'(c) + .6g'(d) = 0.

CHAPTER 4

WEYL'S THEORY

4.1. INTRODUCTION

In this chapter the bvp will be considered on the interval [0, ∞). The bvp will be considered as regular at zero and singular at infinity [27], [28]. Boundary condition can be given at 0 but it cannot be given directly at infinity. In 1910, Hermann Weyl introduced an extraordinary way for boundary-value problem considered on an infinite interval [9]. According to this way, differential equations can be classified at two groups:

A-All solutions are of the class $L_w^2(0,\infty)$ (limit-circle case)

B-Only one linearly independent solution is of the class $L_w^2(0,\infty)$ (limit-point case).

4.2. RIGHT DEFINITE CASE

The interval under consideration is [0, ∞). If every solution of

$$Lg := [-(p(x)g')' + q(x)g] / w(x) = \lambda g$$

satisfies

$$\int_0^\infty |g|^2 w(x)\, dx < \infty$$

for a particular complex number λ_0, then L is called the limit-circle type at infinity, otherwise L is called the limit-point type at infinity. It must be kept in mind that the classification is related to only on L and not to the particular λ_0 chosen.

Theorem 4.2.1 [27]. If every solution of $Lg = \lambda_0 g$ is of class $L_w^2(0,\infty)$ for some $\lambda_0 \in \mathbb{C}$, then for arbitrary complex λ, every solution of $Lg = \lambda g$ is of class $L_w^2(0,\infty)$.

Proof [27]. It is given that ζ, η are the linearly independent solutions of Lv = l_0v that are of the class L_w^2(0,∞). Let V be any solution of Lv = lv. Then

$$\mathrm{Lv} = l_0\mathrm{v} + (\mathrm{l}-l_0)\mathrm{v}$$

exists.

The variation of constants formula leads the following

$$\mathrm{V(t)} = c_1\zeta(\mathrm{t}) + c_2\eta(\mathrm{t}) + (\mathrm{l}-l_0)\int_c^t[\zeta(\mathrm{t})\,\eta(\mathrm{r}) - \zeta(\mathrm{r})\,\eta(\mathrm{t})]\mathrm{V(r)w(r)\,dr}$$

where c_1 and c_2 are constants.

If the notation

$$\|V\|_c = (\int_c^t (|V|^2\, wdt)^{1/2}$$

is used and if M is chosen such that $\|\zeta\|_c \leq$ M, $\|\eta\|_c \leq$ M for all t≥c, then the Schwarz inequality provides

$$|\int_c^t[\zeta(\mathrm{t})\eta(\mathrm{r}) - \zeta(\mathrm{r})\eta(\mathrm{t})]\mathrm{V(r)w(r)dr}| \leq \mathrm{M}(|\zeta(\mathrm{t})| + |\eta(\mathrm{t})|)\,\|V\|_c.$$

Using this in

$$\mathrm{V(t)} = c_1\zeta(\mathrm{t}) + c_2\eta(\mathrm{t}) + (\mathrm{l}-l_0)\int_c^t[\zeta(\mathrm{t})\eta(\mathrm{r}) - \zeta(\mathrm{r})\eta(\mathrm{t})]\mathrm{V(r)w(r)dr}$$

the Minkowski inequality gives

$$\|V\|_c \leq (|c_1| + |c_2|)\mathrm{M} + 2|\,\mathrm{l}-l_0|M^2\|V\|_c.$$

If c is thought large enough so that

$$|\,\mathrm{l}-l_0|M^2 < 1/4,$$

then

$$\|V\|_c \leq 2(|c_1| + |c_2|)\mathrm{M}.$$

Since the right-hand side of this inequality does not depend on t, V is of class L_w^2(0,∞).

□

Theorem 4.2.2 [27]. If Im λ ≠0 and ζ, η are the linearly independent solutions of Lg = lg which satisfy

$$\zeta\,(0, \lambda) = \sin\alpha, \qquad \eta\,(0, \lambda) = \cos\alpha,$$

$$\mathrm{p}(0)\,\zeta'(0, \lambda) = -\cos\alpha, \qquad \mathrm{p}(0)\,\eta'\,(0, \lambda) = \sin\alpha,$$

where 0≤α<π,

then the solution V= $\zeta + m\eta$ has the real boundary condition cosβ g(d) + sinβ p(d) g′(d) = 0, (0≤β<π) for some point d where 0<d<∞ if and only if m is on a circle C_d in the complex plane whose equation is

$$[VV](d) = 0,$$

where W[g,u] = [g,$\bar{u}$].

As d→ ∞ either $C_d \to C_\infty$, a limit circle, or $C_d \to m_\infty$, a limit point. All solutions of Lg = lg are $L_w^2(0,\infty)$ in the former case, and if Im λ ≠0 exactly one linearly independent solution is $L_w^2(0,\infty)$ in the latter case. Moreover, in the limit-circle case, a point is on the limit circle C_∞ (l) if and only if

$$[VV](\infty) = 0.$$

Proof [27]. Suppose that ζ, η are two solutions of Lg= λg which satisfy

$$\zeta(0, \lambda) = \sin\alpha, \qquad \eta(0, \lambda) = \cos\alpha,$$

$$p(0)\,\zeta'(0, \lambda) = -\cos\alpha, \qquad p(0)\,\eta'(0, \lambda) = \sin\alpha,$$

where 0≤α<π.

Then ζ, η are linearly independent solutions and $\zeta', \eta', \zeta, \eta$ are entire functions of λ and continuos in (x, λ). Besides, since

$$W[\zeta,\eta](0) = p(0)[\zeta(0)\eta'(0) - \zeta'(0)\eta(0)] = 1, \quad W[\zeta,\eta](g) = 1$$

exists for all g. These solutions are real for real λ and meet the following boundary conditions at zero:

$$\cos\alpha\, \zeta(0,\lambda) + \sin\alpha\, p(0)\, \zeta'(0, \lambda) = 0,$$

$$\sin\alpha\, \eta(0, \lambda) - \cos\alpha\, p(0)\, \eta'(0, \lambda) = 0.$$

Every solution V of Lg= λg apart from η is, for a constant multiple, in the form of

$$V = \zeta + m\eta$$

for some m which depends on λ.

Consider now a real boundary condition at d where 0< d< ∞:

$$\cos\beta\, g(d) + \sin\beta\, p(d)\, g'(d) = 0, \quad (0\le\beta<\pi).$$

$$m = -(\cot\beta\, \zeta(d,\lambda) + p(d)\, \zeta'(d, \lambda)) / (\cot\beta\, \eta(d, \lambda) + p(b)\, \eta'(d, \lambda))$$

when the solution V satisfies the boundary condition. Since m = m(λ,d,β) and ζ', η', ζ, η are entire in λ, m is meromorphic in λ and real for real λ.

If z = cotβ and if (λ, β) are held fixed, then

$$m = -(Pz + R) / (Sz + T)$$

exists.

Here P, R, S, T are fixed, z is varying over the real line and β is varying from 0 to π. The real axis of the z-plane has its image a circle C_d in the m-plane. Therefore V meets the boundary condition if and only if m lies on C_d.

From

$$z = -(R+Tm)/(P+Sm)$$

the equation of the image of the real axis (i.e. Imz=0) becomes

$$(R+Tm)/(P+Sm) = (\bar{R}+\bar{T}\bar{m})/\overline{(P}+\bar{S}\bar{m}) \Leftrightarrow$$

$$(\bar{P}+\bar{S}\bar{m})(R+Tm)-(P+Sm)(\bar{R}+\bar{T}\bar{m})=0$$

which is the equation for C_d:

$$m\bar{m}(\bar{S}T-S\bar{T}) + \bar{m}(\bar{S}R-P\bar{T}) + m(\bar{P}T-S\bar{R}) + \bar{P}R-P\bar{R} = 0.$$

$$m\bar{m}+\bar{m}(\bar{S}R-P\bar{T}/\bar{S}T-S\bar{T}) + m(\bar{P}T-S\bar{R}/\bar{S}T-S\bar{T}) + (\bar{P}R-P\bar{R}/(\bar{S}T-S\bar{T}) = 0.$$

Hence the center of C_b is

$$m_d = (\bar{P}T-S\bar{R})\ /\ (\bar{S}T-S\bar{T})$$

and the radius is

$$|PT-RS|/|\bar{S}T-S\bar{T}|.$$

Because V= $\zeta + m\eta$ and W[V,$\bar{V}$] (d)= 0, the equation of C_d is

$$W[V,\bar{V}]\ (d) = 0.$$

Since

$$P\bar{T} - \bar{S}R = W[\zeta,\bar{\eta}](d),$$

$$\bar{S}T - S\bar{T} = -W[\eta,\bar{\eta}](d),$$

$$PT-RS = W[\zeta,\eta](d) = 1.$$

$$m_d = -W[\zeta,\bar{\eta}](d)/W[\eta,\bar{\eta}](d), \quad r_d = 1/W[\eta,\bar{\eta}](d)$$

exist respectively.

It follows that the interior of C_d in the m-plane is

$$W[V,\bar{V}](b)/W[\eta,\bar{\eta}](d) < 0.$$

A direct calculation, that is (3) gives

$$W[\eta,\bar{\eta}](d) = 2iIm\lambda \int_0^d |\eta|^2\, dx$$

and

$$W[V,\bar{V}](b) = 2iIm\lambda \int_0^d |V|^2\, w(x)\, dx + W[V,\bar{V}](0).$$

Because

$$W[V,\bar{V}](0) = -2iImm,$$

$$W[V,\bar{V}](d)= 2iIm\lambda \int_0^d |G|^2\, w(x)\ dx\text{-}2iImm$$

exists.

Substitution gives

$$2iIm\lambda \int_0^d |V|^2\, w(x)\ dx\text{-}2iImm\ /2iIm\lambda \int_0^d |\eta|^2\, w(x)\ dx < 0$$

and

$$\int_0^d |V|^2 w(x) dx < \text{Im}m\ /\ \text{Im}\lambda$$

which determines the interior of C_d.

Points m are on C_d if and only if

$$\int_0^d |V|^2 w(x) dx = \text{Im}m\ /\ \text{Im}\ \lambda \quad (\text{Im}\ \lambda \neq 0)$$

The radius r_d is given for Im $\lambda > 0$ by

$$r_d = 1/(2\text{Im}\lambda \int_0^d |\psi|^2\ \text{w(x)}\ dx).$$

Now $0 < c < d < \infty$. Then if m is inside or on C_d

$$\int_0^c |V|^2\, w(x)\ dx < \int_0^d |V|^2\ w(x)\ dx \leq \text{Im}m\ /\ \text{Im}\lambda$$

so m is inside C_c.

This leads to conclude that C_c contains C_d in its interior if c<d.

As d→∞ the circles C_d converge either to a circle C_∞ or to a point m_∞ for a given λ. If the C_d converge to a circle, then its radius r_∞ = lim r_d is positive and this implies that $\eta \in L_w^2(0,\infty)$. If m_∞ is any point on C_∞, then m_∞ is inside any C_d for d>0. Therefore

$$\int_0^d |\ \zeta + m_\infty \eta\ |^2\ w(x)\ dx < Imm_\infty / \ Im\lambda.$$

If d→∞, $\zeta + m_\infty \eta \in L_w^2(0,\infty)$. In the case $C_d \to C_\infty$ all solutions are of class $L_w^2(0,\infty)$ for Im$\lambda \neq 0$. This shows the limit-circle case with the existence of the circle C_∞. Correspondingly the limit-point case is known with the existence of the point m_∞. In the case $C_d \to m_\infty$, this leads to a limr_d= 0. This situation implies that η is not of class $L_w^2(0,\infty)$.

□

4.3 LEFT DEFINITE CASE

Consider the homogeneous differential equation

$$-(pg')' + qg = \lambda wg \quad (17)$$

where p, q and w are bigger than zero and they are measurable functions over (c,d), $-\infty \leq c < d \leq \infty$; 1/p is in L^1_{loc}(c,d); q and w satisfy $\varepsilon_1 w \leq q \leq \varepsilon_2 w$ and are in L^1_{loc}(c,d) [29].

It is known that there are two linearly independent solutions of (17) for all values of the complex parameter λ.

The problem will be investigated in two contexts. The Hilbert space is defined by

$$L^2_w(c, d) = \{f: (c,d) \to C \mid \int_c^d |f(x)|^2 \, w(x)\, dx < \infty \}$$

with inner product

$$(g, u)_{L^2} = \int_c^d g(x)\, \overline{u(x)}\, w(x)\, dx.$$

The Sobolev space is defined by

$$H^1 := H^1(c,d;p,q) = \{f: (c,d) \to C \mid f \in AC_{loc}(c, d);\ p^{1/2} f',\, q^{1/2} f \in L^2(c,d) \}$$

with inner product

$$< g, u >_{H^1} = \int_c^d [pg'\bar{u}' + qg\bar{u}]\, dx .$$

Hence

$$H^1(c,d;p,q) \subset L^2(c, d; w) .$$

The equation (17) when multiplied by $\bar{g}$ and integrated from e to d', d'ϵ (e,d) leads the following:

$$\int_e^{d'} [-(pg')' + qg]\, \bar{g}\, dx \equiv \lambda \int_e^{d'} |g|^2\, w\, dx.$$

If one uses the method of integration by parts, one has the following Dirichlet formula

$$\int_e^{d'} [p|g'|^2 + q|g|^2]\, dx - (pg')\bar{g} \,\Big|_e^{d'} = \lambda \int_e^{d'} |g|^2\, w\, dx \ \epsilon\ L^2_w .$$

Let $\zeta(x, \lambda)$ be the solution which fulfils the initial conditions at e in (c,d),

$$\zeta(d, \lambda) = cos\gamma, \quad p(d)\zeta'(d, \lambda) = -sin\, \gamma,$$

for some fixed γ, real.

Let $\eta(x,\lambda)$ be the solution fulfilling

$$\eta(e,\lambda) = \sin\gamma, \ \ p(e)\eta'(e, \lambda) = \cos\gamma.$$

It is obviously seen that ζ and η are linearly independent, that is,

$$pW[\zeta,\eta] = p[\zeta\eta' - \zeta'\eta] \equiv 1$$

and η fulfils

$$\cos\gamma\eta(e,\lambda) - \sin\gamma\, p(e)\, \eta'(e,\lambda) = 0$$

while ζ fulfils

$$\sin\gamma\zeta(e,\lambda) + \cos\gamma\, p(e)\, \zeta'(c,\lambda) = 0.$$

These are two regular boundary conditions which are independent at e in (c,d).

4.3.1 The Dirichlet formula and H^1 solutions

The preliminary form of the Dirichlet formula is

$$\int_e^{d'} [-(pg')' + qg]\,\bar{g}\,dx \equiv \lambda \int_e^{d'} |g|^2\, w\, dx$$

where d' ϵ (e,d).

If one performs the integration by parts to the p term, it is found that

$$\int_e^{d'} [p|g'|^2 + q|g|^2]\,dx - (p'g)\bar{g}\ \Big|_e^{d'} = \lambda \int_e^{d'} |g|^2\, w\, dx.$$

The term $(pg')\bar{g}$ is troublesome. It is suitable to let it zero at b'. Hence it is required that

$$p(d')\, g'(d') = 0.$$

If a general boundary condition at d' is imposed,

$$\cos\delta g(d') + \sin\delta\, p(d')\, g'(d') = 0$$

for some real δ, then the solution of (17) must be in the form

$$\xi(v,\lambda) = \zeta(v,\lambda) + m(\lambda)\eta(v,\lambda).$$

Here $m(\lambda)$ is determined by

$$m(\lambda) = -(\cos\delta\zeta(d',\lambda) + \sin\delta\, p(d')\, \zeta'(d',\lambda) / \cos\delta\, \eta(d',\lambda) + \sin\delta\, p(d')\, \eta'(d',\lambda))$$

and $\tan\delta$ is determined by

$$\tan\delta = -(\zeta(d',\lambda) + m(\lambda)\eta(d',\lambda)) / (p(d')\, \zeta'(d',\lambda) + m(\lambda)\, \eta'(d',\lambda)).$$

It can be seen that as δ changes over real values from 0 to π, $m(\lambda)$ shows a circle in the complex plane. As d' approaches d, the circles approach a limit circle or a limit point. If m is on the limit circle or point, then ξ is in L_w^2(c,d).

Because

$$\xi(\mathrm{e},\lambda) = \cos\gamma + m(\lambda)\sin\gamma,$$

$$\mathrm{p(e)}\,\xi\,'(\mathrm{e},\lambda) = -\sin\gamma + m(\lambda)\cos\gamma,$$

it is also true that

$$\bar{\xi}(\mathrm{e},\lambda) = \cos\gamma + \overline{m(\lambda)}\,\sin\gamma,$$

$$\mathrm{p(e)}\,\overline{\xi\,'(\mathrm{e},\lambda)} = -\sin\gamma + \overline{m(\lambda)}\,\cos\gamma.$$

Therefore

$$\mathrm{p\,(e)}\,\xi'(\mathrm{e},\lambda)\,\bar{\xi}(\mathrm{e},\lambda) = [|m|^2-1]\sin\gamma\cos\gamma + m\cos^2\gamma - \bar{m}sin^2\gamma.$$

In addition to this,

$$\xi(\mathrm{d}',\lambda) = \mathrm{K}\sin\delta,$$

$$\mathrm{p(d')}\,\xi'(\mathrm{d}',\lambda) = -\mathrm{K}\cos\delta,$$

and

$$\overline{\xi(\mathrm{d}',\lambda)} = \bar{K}\sin\delta,$$

$$\mathrm{p(d')}\,\overline{\xi'(\mathrm{d}',\lambda)} = -\bar{K}\cos\delta,$$

so

$$\mathrm{p(d')}\,\xi'(\mathrm{d}',\lambda)\,\bar{\xi}(\mathrm{d}',\lambda) = -|K|^2\sin\delta\cos\delta,$$

where

$$|K|^2 = |\xi(\mathrm{d}',\lambda)|^2 + |\mathrm{p(d')}\,\xi\,'(\mathrm{d}',\lambda)\,|^2.$$

If g is changed by ξ in

$$\int_e^{d'}[p|g'|^2 + \mathrm{q}|g|^2]\,\mathrm{dx} - (\mathrm{p}g')\bar{g}\,\Big|_e^{d'}$$

and boundary values

$$\bar{\xi}(\mathrm{e},\lambda) = \cos\gamma + \overline{m(\lambda)}\,\sin\gamma$$

$$\mathrm{p(e)}\,\overline{\xi'(\mathrm{e},\lambda)} = -\sin\gamma + \overline{m(\lambda)}\,\cos\gamma$$

$$\overline{\xi(\mathrm{d}',\lambda)} = \bar{K}\sin\delta$$

$$\mathrm{p(d')}\,\overline{\xi'(\mathrm{d}',\lambda)} = -\bar{K}\cos\delta$$

are inserted, the following is found:

$$\int_e^{d'}[p\,|\xi'|^2 + \mathrm{q}|\xi|^2]\,\mathrm{dx} + |K|^2\sin\delta\cos\delta + [|m|^2-1]\sin\gamma\cos\gamma + m\cos^2\gamma - \bar{m}sin^2\gamma \quad (18)$$

$$= (\mu + \mathrm{iv})\int_e^{d'}|\xi|^2\,w\,dx, \qquad \lambda = \mu + \mathrm{iz}.$$

The imaginary part of (18) is

$$\mathrm{Im}(m) = \mathrm{v}\int_e^{d'}|\xi|^2\,w\,dx.$$

The real part of (18) is

$$\int_e^{d'}[p\,|\xi'|^2 + q|\xi|^2]\,dx + |K|^2 \sin\delta\,\cos\delta =$$

$$[1-|m|^2]\sin\gamma\,\cos\gamma - \mathrm{Re}(m)\cos^2\gamma + \mathrm{Re}(m)\,sin^2\gamma + \mu\,\mathrm{Im}(m)/z.$$

If d' approaches d, m approaches the limit point or a point on the limit circle. Nonetheless, only if δ is in $[0, \pi/2]$, it is seen that ξ is in H^1 (e,d; p,q), since otherwise the two terms on the left in the above formula become infinite.

If δ is fixed at $\pi/2$ with

$$m(\lambda) = -\zeta'(d', \lambda)/\eta'(d', \lambda),$$

it is encountered that

$$\int_e^{d'}[p\,|\xi'|^2 + q|\xi|^2]\,dx = [1-|m|^2]\sin\gamma\,\cos\gamma - \mathrm{Re}(m)\cos^2\gamma + \mathrm{Re}(m)\,sin^2\gamma + \mu\,\mathrm{Im}(m)/z.$$

If d' is fixed in the upper limit in the integral and suppose that all other d''s approach d, the following is found:

$$\int_e^{d'}[p\,|\xi'|^2 + q|\xi|^2]\,dx \le [1-|m|^2]\sin\gamma\,\cos\gamma - \mathrm{Re}(m)\cos^2\gamma + \mathrm{Re}(m)\,sin^2\gamma + \mu\,\mathrm{Im}(m)/z.$$

where, now, all of the m's are on the limit point or limit circle.

Therefore, one may pass to the following theorem.

Theorem 4.3.1.1 [29]. For all λ, $\mathrm{Im}(\lambda) \neq 0$, there takes place a solution

$$\xi_d(x,\lambda) = \zeta(x,\lambda) + m_d(\lambda)\,\eta(x,\lambda)$$

of (17) which is in H^1 (e,d; p,q), the Sobolev space with inner product

$$< g,u >_{H^1} = \int_c^d [pg'\bar{u}' + qg\bar{u}]\,dx.$$

Even if (17) is in the limit circle case with two solutions in L_w^2(e,d), there need not be more than one solution $\xi_d(x,\lambda)$ in H^1 (e,d;p,q). Even if the L_w^2(e,d) theory is limit-circle with two L_w^2(e,d) solutions for (17), there is one solution $\xi_d(x,\lambda)$ in H^1 (e,d;p,q).

Theorem 4.3.1.2 [29]. Let $\xi_d(x,\lambda)$ be the solution of (17) in H^1 (e,d; p,q) generated by approaching solutions $\xi_d'(x,\lambda)$ satisfying

$$p(d')\,\xi_{d'}'(d',\lambda) = 0.$$

Then

$$\lim_{x \to d} p(x)\ \xi_d'(x,\lambda) = 0.$$

Proof [29]. $p(d')\ \xi_d'(d',\lambda) - p(d')\ \xi_{d'}'(d',\lambda) = (b_d - b_{d'})\ p(d')\ \zeta'(d',\lambda)$ exists. Now

$$p(d')\ \zeta'(d',\lambda) = p(e)\ \zeta'(e,\lambda) + \int_e^{d'} (p'\zeta')'\, dx =$$

$$p(e)\ \zeta'(e,\lambda) + \int_e^{d'} [q(x) - \lambda\, w(x)]\zeta(x,\lambda)dx.$$

Thus by the triangle inequality

$$|\ p(d')\ \zeta'(d',\lambda)\ | \le |\ p(e)\ \zeta'(e,\lambda)\ | + K \int_e^{d'} w\ |\ \zeta\ |\ dx$$

and by Cauchy-Schwartz inequality

$$\le |\ p(e)\ \zeta'(e,\lambda)\ | + K[\int_e^{d'} w\, dx]^{1/2}\ [\int_e^{d'} w\ |\zeta|^2 dx]^{1/2}$$

In the limit point case

$$|m_d - m_{d'}| < 2/\ |z|\ \int_e^{d'} w\ |\zeta|^2 dx.$$

Thus

$$|m_d - m_{d'}|\ |p(d')\zeta'(d',\lambda)\ | \le (A + B[\int_e^{d'} w\ |\zeta|^2 dx]^{1/2})\ /\ (\int_e^{d'} w\ |\zeta|^2\ dx)$$

which approaches 0 as d' approaches d.

In the limit circle case one has

$$|p(d')\zeta'(d',\lambda)\ | < K.$$

Since

$$m_{d'} \to m_d$$

one obtains

$$\lim_{d' \to d} p(d')\zeta'(d',\lambda)\ (m_{d'} \to m_d) = 0.$$

□

Theorem 4.3.1.3 [29]. Suppose that g and u are in H^1 (e,d;p,q) and

$$\lim_{x \to d} p(x) g'(x) = 0$$

If

$$\lim_{x \to d} p(x) g'(x) \overline{u(x)}$$

exists, then

$$\lim_{x \to d} \mathrm{p(x)g'(x)}\overline{\mathrm{u}(x)} = 0.$$

Proof [29]. Keep in mind that $p^{1/2}g'$ and $p^{1/2}$u' are in L^2(e,d). Let

$$\alpha = \lim_{x \to d} \mathrm{p(x)}g'\mathrm{(x)u}\overline{(x)},$$

and let $\alpha \neq 0$. Then near b,

$$p^{1/2}g' \sim \alpha \,/\, (p^{1/2}\bar{u}).$$

However $p^{1/2}g'$ is in L^2(e,d), so it will be deduced that $(p^{1/2}\,\bar{u})^{-1}$ is in $L^2(e_0$,d), for some e_0 in (e,d).

Now $p^{1/2}$u′ is in L^2(e,d), so

$$\mathrm{u'/u} = (p^{1/2}\mathrm{u'})\ (p^{1/2}\mathrm{u})^{-1}$$

is in $L^1(e_0$,d). However, then

$$(\ln |\mathrm{u}|)' = (\mathrm{Re\ u'}\bar{u}\ ) \,/\, (\mathrm{u}\bar{\mathrm{u}})$$

is in $L^1(e_0$,d). Thus $\lim_{x \to d} \ln|\mathrm{u}|$ exists and is finite. Hence u is bounded as x approaches d. This makes a contradiction, since u is bounded and

$$\lim_{x \to d} \mathrm{p(x)g'(x)} = 0.$$

implies $\alpha = 0$.

□

As a corollary the following can be stated:

Theorem 4.3.1.4 [29]. Suppose that g and u be are H^1 (e,d;p,q), let pg' be in $\mathrm{AC_{loc}}$ (c,e), and

$$\lim_{\mathrm{x \to c}} \mathrm{p(x)g'(x)} = 0$$

If

$$\lim_{\mathrm{x \to c}} \mathrm{p(x)g'(x)}\,\overline{\mathrm{u(x)}} \text{ exists,}$$

then

$$\lim_{\mathrm{x \to c}} \mathrm{p(x)g'(x)}\,\overline{\mathrm{u(x)}} = 0.$$

CHAPTER 5

CONCLUSION AND DISCUSSION

In this thesis, we have collected some results on regular and singular Sturm-Liouville equations and operators. As is known that real-world problems can be identified by second/fourth/… order Sturm-Liouville equations with appropriate boundary conditions. A useful method to get some information about the spectrum of such problems is to pass to the associated operators defined on suitable spaces. Positiveness condition of the function appearing at the right-hand side of the equation

$$-(p(x)g')' + q(x)g = \lambda w(x)g, \ x \in (c,d),$$

gives rise the well-known Lebesgue space L_W^2 (c,d) with the usual inner product

$$(g,u) = \int_c^d g\, \bar{u} w \, dx.$$

On the other side, if one considers Lg instead of g in the inner product, where

$$Lg = [\ -(p(x)g')' + q(x)g] \ / \ w(x)$$

then a nice formula (9) arises as follows

$$(Lg,u) = -pg'\bar{u}\ |_c^d + \int_c^d pg'\, \bar{u}' dx + \int_c^d qg\, \bar{u}\, dx.$$

In the literature, some authors have tried to make the right-hand side of the equation an inner product. This aim now depends on p, q and the remaining part

$$-pg'\bar{u}\ |_c^d = -p(d)\, g'(d)\, \bar{u}(d) + p(c)\, g'(c)\, \bar{u}(c).$$

Krall achieved to construct an inner product by choosing appropriate boundary conditions and p,q>0. Then he investigated some properties of the operators in this new inner product space. Moreover, the similar results have been obtained for the left-definite Hamiltonian systems.

Another important problem in the literature is to determine the number of the linearly independent solutions of the equation

$$-(p(x)g')' + q(x)g = \lambda w(x)g, \ x \in [c,\infty)$$

belonging to the Lebesgue space L_w^2 (c,∞). This problem was solved by H. Weyl in 1910. He showed that at least one of the linearly independent solutions must lie in L_w^2 (c,∞). Furthermore, both of them may lie in L_w^2 (c,∞). These cases are known as limit-point and limit-circle cases, respectively. However, such a result has not been introduced for left-definite case until 1992. Krall and Race have showed that at least one solution must lie in the Sobolev space H^1(c,∞;p,q) but there is not more information about the other linearly independent solution.

REFERENCES

1. **NAIMARK, M.A.** (1968), *Linear Differential Operators Part II*, Frederick Ungar Publishing Co., London.

2. **MÖLLER, M.**, **PIVOVARCHIK V.** (2015), *Spectral Theory of Operator Pencils, Hermite –Biehler Functions, and their Applications, Operator theory: Advances and applications Volume 246,* Birkhäuser, New York.

3. **KELLEY, W.G., PETERSON A.C.** (2010), *The Theory of Differential Equations,* Springer, New York.

4. **PLEJEL, A.** (1969), Some remarks about the limit point and limit circle theory, *Arkiv Mat.,* 543-550, vol.7.

5. **PLEJEL, A.** (1971), Complementary remarks about the limit point and limit circle theory, *Arkiv Mat.,* 45-47, vol.8.

6. **EVERITT, W.N.** (1981), On Certain Regular Ordinary Differential Expressions and Related Differential Operators, *Spect. Th. Ord. Diff. Ops.* Eds. I.W. Knowles and R.T. Lewis, North Holland, Amsterdam 115-167.

7. **BENNEWITZ, C., EVERITT, W.N.** (1983), On second order left definite boundary value problems, *Lect. Notes Math.* 31-67, vol. 1032.

8. **KRALL, A.M.** (1992), Regular left definite boundary value problems of even order, *Quaes. Math.*, 105-118, vol.15.

9. **WEYL, H.** (1910), Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen, *Mathematische Annalen,* 220-269, vol.68

10. **ATKINSON, F.V., EVERITT, W.N., AND ONG, K.S.** (1974), On the m-coefficient of Weyl for a differential equation with an indefinite weight function, *Proc.London Math. Soc.* 368-384, vol.29.

11. **SCHNEIDER, A., NIESSEN, H.D.** (1976), Linksdefinite singuläre kanonische Eigenwertprobleme, *I.J.f.d. reine ang. Math.* 13-52, vol.281.

12. **SCHNEIDER, A., NIESSEN, H.D.** (1977), Linksdefinite singuläre kanonische Eigenwertprobleme, *II.J.f.d. reine ang. Math.* 62-84, vol.289.

13. **ONYANGO-OTIENO, V.P.** (1980), *The application of ordinary differential operators to the study of the classical orthogonal polynomials,* Ph.D. Dissertation, University of Dundee.

14. **EVERITT, W.N., LITTLEJOHN, L.L.** (1988), Differential operators and the Legendre type polynomials, *Diff. And Int. Eq.* 97-116, vol. 1.

15. **KRALL A.M., LITTLEJOHN L.L.** (1993), The Legendre polynomials under a left definite energy norm, *Quaes. Math.* 393-403, vol.16.

16. **HAJMIRZAAHMAD, M.** (1990), *The spectral resolution of Laguerre operators in right definite and left definite spaces,* Ph.D. Dissertation, The Pennsylvania State University, University Park.

17. **HAJMIRZAAHMAD, M.** (1995), Laguerre polynomial expansions, *J.Comp.Appl. Math.*, 25-37, vol.59.

18. **HAJMIRZAAHMAD, M.** (1994), Jacobi polynomial expansions, *J. Math. Anal. Appl.,* 35-61, vol.181.

19. **KRALL, A.M.** (1995), Left Definite Theory for Second Order Differential Operators with Mixed Boundary Conditions, *Journal of Differential Equations,* 153-165, vol. 118.

20. **KRALL, A.M.** (2002)**,** *Hilbert Space*, *Boundary Value Problem and Orthogonal Polynomials*, Birkhauser Verlag, Basel.

21. **ZETTL, A.** (2005), *Sturm-Liouville Theory,* American Mathematical Society, USA

22. **KRALL, A.M.** (1995), Left–Definite Regular Hamiltonian Systems, *Math. Nachr.* 203-217, vol. 174.

23. **WALKER, P.W.** (1974), A Vector-Matrix Formulation for Formally Symmetric Ordinary Differential Equations with Applications to Solutions of Integrable Square, *J. London Math. Soc.*, 151-159, vol. 9.

24. **HINTON, D.B., SHAW J.K.** (1981), Titchmarsh-Weyl Theory for Hamiltonian Systems*, Spec. Th. Diff. Op.*, Eds. I. W. Knowles and R. T. Lewis, North Holland, N.Y.

25. **HINTON, D.B., SHAW J.K.** (1981), On Titchmarsh-Weyl M (λ) functions for Linear Hamiltonian Systems, *J.Diff. Eq.* 316-342, vol.40.

26. **HINTON, D.B., SHAW J.K.** (1981), Hamiltonian Systems of Limit Point or Limit Circle Type with Both End Points Singular, *J.Diff.Eq.,* 444-464, vol.50.

27. **CODDINGTON, E.A., LEVINSON N.** (1955), *Theory of Ordinary Differential Equations,* McGraw-Hill, New York.

28. **UĞURLU, E.** (2009), *Sturm-Liouville Denklemlerinin Weyl Teorisi*, Yüksek Lisans Semineri, Ankara Üniversitesi, Fen Bilimleri Enstitüsü, Ankara.

29. **KRALL, A.M., RACE, D.** (1995), Self-adjointness for the Weyl problem under an energy norm, *Quaes. Math.*, 407-426, vol. 18