mathematics

MDPI

Article

Solving the Lane–Emden Equation within a Reproducing Kernel Method and Group Preserving Scheme

Mir Sajjad Hashemi [1], Ali Akgül [2,*], Mustafa Inc [3], Idrees Sedeeq Mustafa [2] and Dumitru Baleanu [4,5]

1. Department of Mathematics, Basic Science Faculty, University of Bonab, Bonab 55517-61167, Iran; hashemi@bonabu.ac.ir
2. Department of Mathematics, Art and Science Faculty, Siirt University, TR-56100 Siirt, Turkey; idrisaland@gmail.com
3. Department of Mathematics, Science Faculty, Fırat University, 23119 Elazığ, Turkey; minc@firat.edu.tr
4. Department of Mathematics, Art and Science Faculty, Çankaya University, TR-06300 Ankara, Turkey; dumitru@cankaya.edu.tr
5. Department of Mathematics Bucharest, Institute of Space Sciences, Bucharest 179141, Romania

* Correspondence: aliakgul00727@gmail.com

Received: 17 October 2017; Accepted: 5 December 2017; Published: 12 December 2017

Abstract: We apply the reproducing kernel method and group preserving scheme for investigating the Lane–Emden equation. The reproducing kernel method is implemented by the useful reproducing kernel functions and the numerical approximations are given. These approximations demonstrate the preciseness of the investigated techniques.

Keywords: Lane–Emden equation; group preserving scheme; reproducing kernel functions; approximate solutions

JEL Classification: 47B32; 46E22; 74S30

1. Introduction

The work of singular initial value problems modeled by second order nonlinear ordinary differential equations (ODEs) have captivated many mathematicians and physicists. One of the equations in this class is the Lane–Emden equation [1]. We use the reproducing kernel method (RKM) and the group preserving scheme (GPS) to investigate this equation in this paper. We have investigated solutions of the following problem:

$$\varsigma'' + \frac{2}{\eta}\varsigma' + \varsigma^3 = 0, \quad 0 < \eta \leq 10, \tag{1}$$

with the initial conditions

$$\varsigma(0) = 1, \quad \varsigma'(0) = 0, \tag{2}$$

where $\varsigma(\eta)$ is a sufficiently smooth function. We recall that this problem is in the class of Astrophysics equations [2–5].

We recall that there are many papers on the solution of the nonlinear problems with a reproducing kernel method. The notion of the reproducing kernel can be traced back to the paper of Zaremba in 1908. It was presented to discuss the boundary value problems of the harmonic functions. In the early development stage of the reproducing kernel theory, most of the works were implemented by Bergman. This researcher obtained the corresponding kernels of the harmonic functions with one or several variables, and the corresponding kernel of the analytic function in squared metric, and implemented them in the research of the boundary value problem of the elliptic partial differential equation. This is the first stage in the development history of reproducing kernel. The second stage of the reproducing kernel theory was started by Mercer who discovered that the continuous kernel of the positive definite integral equation has the positive definite property as [6]:

$$\sum_{i,j=1}^{n} k(x_i, y_j)\xi_i\xi_j \geq 0.$$

He named the kernel with this property positive definite Hermite matrix. He presented a Hilbert space with inner product $\langle f, g\rangle$, and showed the reproducibility of the kernel as:

$$v(s) = \langle v(t), k(t,s)\rangle .$$

In 1950, Aronszajn collected the works of the formers and studied a systematic reproducing kernel theory including the Bergman kernel function.

Reproducing kernel theory has valuable implementations in integral equations, differential equations, probability and statistics. This theory has been implemented for many model problems in recent years. The RKM, which accurately calculates the series solution, is of efficient interest to applied sciences. Recently, a lot of research work has been devoted to the application of RKM [6–11]. For more details, see [12–22].

The GPS in the present paper is based on the group invariant schemes, introduced by Liu [23]. The most important difference between GPS andthe conventional techniques, such as the Runge–Kutta method, is that these techniques are all formulated directly in the usual Euclidean $\mathbb{R}^k$. Furthermore, none of the methods above are considered in Minkowski space $\mathcal{M}^{k+1}$. One straight advantage of the formulation in $\mathcal{M}^{k+1}$ is that the new techniques can avoid the lacking of spurious solutions and ghost fixed points. Some interesting papers in GPS are [24–33].

This work is prepared as follows. Section 2 presents some useful reproducing kernel functions. The approximate solutions of Lane–Emden equations are presented in this section. In addition, some numerical experiments are shown. We explained the GPS and apply it to our investigated equation in Section 3. Conclusions are discussed in the final section.

2. Reproducing Kernel Functions

We define some useful reproducing kernel spaces and find some reproducing kernel functions in this section.

Definition 1. $W_2^1[0,1]$ *is given as:*

$$W_2^1[0,1] = \{u \in AC[0,1] : \ u' \in L^2[0,1]\},$$

where AC defines the space of absolutely continuous functions.

$$\langle u, g\rangle_{W_2^1} = \int_0^1 \left(u(\eta)g(\eta) + u'(\eta)g'(\eta)\right) \mathrm{d}\eta, \quad u, g \in W_2^1[0,1] \tag{3}$$

and

$$\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}} \quad u \in W_2^1[0,1] \tag{4}$$

are the inner product and the norm in $W_2^1[0,1]$, respectively. Reproducing kernel function $T_\eta(\varsigma)$ of $W_2^1[0,1]$ is given by [6]

$$T_\eta(\varsigma) = \frac{1}{2\sinh(1)}\left[\cosh(\eta + \varsigma - 1) + \cosh(|\eta - \varsigma| - 1)\right]. \tag{5}$$

Definition 2. *We describe the space* ${}^{o}W_2^3[0,1]$ *by*

$$ {}^{o}W_2^3[0,1] = \{u \in AC[0,1] : \; u', u'' \in AC[0,1], \; u^{(3)} \in L^2[0,1], \; u(0) = 0 = u'(0)\}. $$

$$ \langle u, v \rangle_{{}^{o}W_2^3} = \sum_{i=0}^{2} u^{(i)}(0) v^{(i)}(0) + \int_0^1 u^{(3)}(\eta) v^{(3)}(\eta) \mathrm{d}\eta, \quad u, v \in {}^{o}W_2^3[0,1] $$

and

$$ \|u\|_{{}^{o}W_2^3} = \sqrt{\langle u, u \rangle_{{}^{o}W_2^3}}, \quad u \in {}^{o}W_2^3[0,1], $$

are the inner product and the norm in ${}^{o}W_2^3[0,1]$ *respectively.*

Theorem 1. *The reproducing kernel function* r_ς *of* ${}^{o}W_2^3[0,1]$ *is given as*

$$ r_\varsigma(\eta) = \begin{cases} \sum_{k=0}^{5} c_{k+1}(\varsigma)\eta^k, & 0 \le \eta < \varsigma \le 1, \\ \\ \sum_{k=0}^{5} d_{k+1}(\varsigma)\eta^k, & 0 \le \varsigma < \eta \le 1, \end{cases} \tag{6} $$

where

$$ \begin{aligned} c_1(\varsigma) &= 0, \quad c_2(\varsigma) = 0, \quad c_3(\varsigma) = \frac{1}{4}\varsigma^2, \quad c_4(\varsigma) = \frac{1}{12}\varsigma^2, \\ c_5(\varsigma) &= -\frac{1}{24}\varsigma, \quad c_6(\varsigma) = \frac{1}{120}, \\ d_1(\varsigma) &= \frac{1}{120}\varsigma^5, \quad d_2(\varsigma) = -\frac{1}{24}\varsigma^4, \\ d_3(\varsigma) &= \frac{1}{12}\varsigma^3 + \frac{1}{4}\varsigma^2, \\ d_4(\varsigma) &= 0, \quad d_5(\varsigma) = 0, \quad d_6(\varsigma) = 0. \end{aligned} $$

Proof. Let $u \in {}^{o}W_2^3[0,1]$ and $0 \le \varsigma \le 1$. Define r_ς by Equation (6). We have

$$ r'_\varsigma(\eta) = \begin{cases} \sum_{k=0}^{4} (k+1) c_{k+1}(\varsigma)\eta^k, & 0 \le \eta < \varsigma \le 1, \\ \\ \sum_{k=0}^{4} (k+1) d_{k+1}(\varsigma)\eta^k, & 0 \le \varsigma < \eta \le 1, \end{cases} $$

$$ r''_\varsigma(\eta) = \begin{cases} \sum_{k=0}^{3} (k+1)(k+2) c_{k+2}(\varsigma)\eta^k, & 0 \le \eta < \varsigma \le 1, \\ \\ \sum_{k=0}^{3} (k+1)(k+2) d_{k+2}(\varsigma)\eta^k, & 0 \le \varsigma < \eta \le 1, \end{cases} $$

$$ r^{(3)}_\varsigma(\eta) = \begin{cases} \sum_{k=0}^{2} (k+1)(k+2)(k+3) c_{k+3}(\varsigma)\eta^k, & 0 \le \eta < \varsigma \le 1, \\ \\ \sum_{k=0}^{2} (k+1)(k+2)(k+3) d_{k+3}(\varsigma)\eta^k, & 0 \le \varsigma < \eta \le 1, \end{cases} $$

$$ r^{(4)}_\varsigma(\eta) = \begin{cases} \sum_{k=0}^{1} (k+1)(k+2)(k+3)(k+4) c_{k+4}(\varsigma)\eta^k, & 0 \le \eta < \varsigma \le 1, \\ \\ \sum_{k=0}^{1} (k+1)(k+2)(k+3)(k+4) d_{k+4}(\varsigma)\eta^k, & 0 \le \varsigma < \eta \le 1, \end{cases} $$

and

$$r_\varsigma^{(5)}(\eta)=\begin{cases}120c_5(\varsigma), & 0\le\eta<\varsigma\le1,\\ \\ 120d_5(\varsigma), & 0\le\varsigma<\eta\le1.\end{cases}$$

We get

$$\begin{aligned}\langle u,r_\varsigma\rangle_{{}^oW_2^3} &= \sum_{i=0}^{2}u^{(i)}(0)r_\varsigma^{(i)}(0)+\int_0^1 u^{(3)}(\eta)r_\varsigma^{(3)}(\eta)\mathrm{d}\eta\\
&= u'(0)r'_\varsigma(0)+u''(0)r''_\varsigma(0)+u''(1)r_\varsigma^{(3)}(1)-u''(0)r_\varsigma^{(3)}(0)\\
&\quad -u'(1)r_\varsigma^{(4)}(1)+u'(0)r_\varsigma^{(4)}(0)+\int_0^1 u'(\eta)r_\varsigma^{(5)}(\eta)\mathrm{d}\eta\\
&= c_1(\varsigma)u'(0)+2c_2(\varsigma)u''(0)\\
&\quad +6(d_3(\varsigma)+4d_4(\varsigma)+10d_5(\varsigma))u''(1)-6c_3(\varsigma)u''(0)\\
&\quad -24(d_4(\varsigma)+5d_5(\varsigma))u'(1)+24c_4(\varsigma)u'(0)\\
&\quad +\int_0^\varsigma 120c_5(\varsigma)u'(\eta)\mathrm{d}\eta+\int_\varsigma^1 120d_5(\varsigma)u'(\eta)\mathrm{d}\eta\\
&= (c_1(\varsigma)+24c_4(\varsigma))u'(0)+2(c_2(\varsigma)-3c_3(\varsigma))u''(0)\\
&\quad +6(d_3(\varsigma)+4d_4(\varsigma)+10d_5(\varsigma))u''(1)-24(d_4(\varsigma)+5d_5(\varsigma))u'(1)\\
&\quad +120(c_5(\varsigma)-d_5(\varsigma))u(\varsigma)\\
&= u(\varsigma).\end{aligned}$$

□

2.1. Solutions in ${}^oW_2^3[0,1]$

The solution of Equation (1) is considered in the reproducing kernel space ${}^oW_2^3[0,1]$. On describing the operator

$$L:\ {}^oW_2^3[0,1]\to W_2^1[0,1]$$

as

$$Lv(\eta)=v''(\eta)+\frac{2}{\eta}v'(\eta), \tag{7}$$

model problems (1) and (2) convert to the following problem:

$$\begin{cases}Lv=M(\eta,v), & \eta\in[0,1],\\ v(0)=0=v'(0).\end{cases} \tag{8}$$

Theorem 2. *L defined by Equation (7) is a bounded linear operator.*

Proof. We need to prove $\|Lv\|^2_{W_2^1}\le M\,\|v\|^2_{{}^oW_2^3}$, where $P>0$. By Equations (3) and (4), we obtain

$$\|Lv\|^2_{W_2^1}=\langle Lv,Lv\rangle_{W_2^1}=\int_0^1\left(Lv(\eta)^2+Lv'(\eta)^2\right)\mathrm{d}\eta.$$

We get

$$v(\eta)=\langle v(\cdot),r_\eta(\cdot)\rangle_{{}^oW_2^3}$$

by reproducing property and

$$Lv(\eta)=\langle v(\cdot),Lr_\eta(\cdot)\rangle_{{}^oW_2^3},$$

so

$$|Lv(\eta)| \leq \|v\|_{{}^oW_2^3} \left\|Lr_\eta\right\|_{{}^oW_2^3} = P_1 \|v\|_{{}^oW_2^3},$$

where $P_1 > 0$. Therefore, we get

$$\int_0^1 [(Lv)(\eta)]^2 \, d\eta \leq P_1^2 \|v\|^2_{{}^oW_2^3}.$$

Since

$$(Lv)'(\eta) = \langle v(\cdot), (Lr_\eta)'(\cdot) \rangle_{{}^oW_2^3},$$

then

$$\left|(Lv)'(\eta)\right| \leq \|v\|_{{}^oW_2^3} \left\|(Lr_\eta)'\right\|_{{}^oW_2^3} = P_2 \|v\|_{{}^oW_2^3},$$

where $P_2 > 0$. Thus, we acquire

$$\left[(Lv)'(\tau)\right]^2 \leq P_2^2 \|v\|^2_{{}^oW_2^3}$$

and

$$\int_0^1 \left[(Lv)'(\eta)\right]^2 d\eta \leq P_2^2 \|v\|^2_{{}^oW_2^3},$$

that is

$$\|Lv\|^2_{W_2^1} \leq \int_0^1 \left([(Lv)(\eta)]^2 + \left[(Lv)'(\eta)\right]^2\right) d\eta \leq \left(P_1^2 + P_2^2\right) \|v\|^2_{{}^oW_2^3} = P \|v\|^2_{{}^oW_2^3},$$

where $P = P_1^2 + P_2^2 > 0$ is a positive constant. □

2.2. The Main Results

Let $\varphi_i(\eta) = T_{\eta_i}(\eta)$ and $\psi_i(\eta) = L^*\varphi_i(x)$; L^* is a conjugate operator of L. The orthonormal system $\{\overline{\Psi}_i(\eta)\}_{i=1}^\infty$ of ${}^oW_2^3[0,1]$ can be achieved from Gram–Schmidt orthogonalization operation of $\{\psi_i(\eta)\}_{i=1}^\infty$,

$$\overline{\psi}_i(\eta) = \sum_{k=1}^{i} \beta_{ik}\psi_k(\eta), \quad (\beta_{ii} > 0, \quad i = 1, 2, \ldots). \tag{9}$$

Theorem 3. *Let $\{\eta_i\}_{i=1}^\infty$ be dense in $[0,1]$ and $\psi_i(\eta) = L_\varsigma r_\eta(\varsigma)\big|_{\varsigma=\eta_i}$. Then, the sequence $\{\psi_i(\eta)\}_{i=1}^\infty$ is a complete system in ${}^oW_2^3[0,1]$.*

Proof. By reproducing property and property of the operator, we get

$$\psi_i(\eta) = (L^*\varphi_i)(\eta) = \left\langle (L^*\varphi_i)(\varsigma), r_\eta(\varsigma) \right\rangle = \left\langle \varphi_i)(\varsigma), L\varsigma r_\eta(\varsigma) \right\rangle = L_\varsigma r_\eta(\varsigma)\big|_{\varsigma=\eta_i}.$$

It is clear that $\psi_i(\eta) \in {}^oW_2^3[0,1]$. For each fixed $u(\eta) \in {}^oW_2^3[0,1]$, let $\langle u(\eta), \psi_i(\eta) \rangle = 0$, $(i = 1, 2, \ldots)$,

$$\langle u(\eta), (L^*\varphi_i)(\eta) \rangle = \langle Lu(\cdot), \varphi_i(\cdot) \rangle = (Lu)(\eta_i) = 0.$$

$\{\eta_i\}_{i=1}^\infty$ is dense in $[0,1]$. Therefore, $(Lu)(\eta) = 0$. $u \equiv 0$ by the L^{-1}. □

Theorem 4. *If $u(\eta)$ is the exact solution of Equation (8), then*

$$u = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} M(\eta_k, u_k) \widehat{\Psi}_i(\eta), \tag{10}$$

where $\{(\eta_i)\}_{i=1}^{\infty}$ *is dense in* $[0,1]$.

Proof. We obtain

$$\begin{aligned} u(\eta) &= \sum_{i=1}^{\infty} \left\langle u(\eta), \widehat{\Psi}_i(\eta) \right\rangle_{{}^oW_2^3} \widehat{\Psi}_i(\eta) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle u(\eta), \Psi_k(\eta) \right\rangle_{{}^oW_2^3} \widehat{\Psi}_i(\eta) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle u(\eta), L^*\varphi_k(\eta) \right\rangle_{{}^oW_2^3} \widehat{\Psi}_i(\eta) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left\langle Lu(\eta), \varphi_k(\eta) \right\rangle_{W_2^1} \widehat{\Psi}_i(\eta) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} Lu(\eta_k) \widehat{\Psi}_i(\eta) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} M(\eta_k, u_k) \widehat{\Psi}_i(\eta), \end{aligned}$$

from Equation (9) and the uniqueness of solution Equation (8). □

The approximate solution u_n can be achieved as:

$$u_n = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} M(\eta_k, u_k) \widehat{\Psi}_i(\eta). \tag{11}$$

Lemma 1. *If* $\|u_n - u\|_{{}^oW_2^3} \to 0$, $\eta_n \to \eta$, $(n \to \infty)$ *and* $M(\eta, u)$ *is continuous for* $\eta \in [0,1]$, *then* [6]

$$M(\eta_n, u_{n-1}(\eta_n)) \to M(\eta, u(\eta)) \quad \text{as } n \to \infty.$$

Theorem 5. *For any fixed* $u_0(\eta) \in {}^oW_2^3[0,1]$, *assume the following conditions are satisfied:*

(i)

$$u_n(\eta) = \sum_{i=1}^{n} A_i \overline{\psi}_i(\eta), \tag{12}$$

$$A_i = \sum_{k=1}^{i} \beta_{ik} M(\eta_k, u_{k-1}(\eta_k)), \tag{13}$$

(ii) $\|u_n\|_{{}^oW_2^3}$ *is bounded;*
(iii) $\{\eta_i\}_{i=1}^{\infty}$ *is dense in* $[0,1]$;
(iv) $M(\eta, u) \in W_2^1[0,1]$ *for any* $u(\eta) \in {}^oW_2^3[0,1]$.

Then, $u_n(\eta)$ *in Equation* (13) *converges to the exact solution of Equation* (10) *in* ${}^oW_2^3[0,1]$ *and*

$$u(\eta) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(\eta),$$

where A_i *is given by (13).*

Proof. Let us demonstrate the convergence of $u_n(\eta)$ firstly. By Equation (12), we obtain

$$u_{n+1}(\eta) = u_n(\eta) + A_{n+1} \widehat{\Psi}_{n+1}(x). \tag{14}$$

From the orthonormality of $\{\widehat{\Psi}_i\}_{i=1}^{\infty}$, we acquire

$$\|u_{n+1}\|^2 = \|u_n\|^2 + A_{n+1}^2 = \|u_{n-1}\|^2 + A_n^2 + A_{n+1}^2 = \ldots = \sum_{i=1}^{n+1} A_i^2. \tag{15}$$

From boundedness of $\|u_n\|_{{}^oW_2^3}$, we get

$$\sum_{i=1}^{\infty} A_i^2 < \infty,$$

i.e.,

$$\{A_i\} \in l^2, \quad (i = 1, 2, \ldots).$$

Let $m > n$, by $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \ldots \perp (u_{n+1} - u_n)$, we acquire

$$\begin{aligned} \|u_m - u_n\|_{{}^oW_2^3}^2 &= \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \ldots + u_{n+1} - u_n\|_{{}^oW_2^3}^2 \\ &\leq \|u_m - u_{m-1}\|^2 + \ldots + \|u_{n+1} - u_n\|_{{}^oW_2^3}^2 \\ &= \sum_{i=n+1}^{m} A_i^2 \to 0, \quad m, n \to \infty, \end{aligned}$$

where $\perp$ denotes the orthogonality. Taking into consideration the completeness of ${}^oW_2^3[0,1]$, there exists $u(\eta) \in {}^oW_2^3[0,1]$, such that

$$u_n(\eta) \to u(\eta) \quad as\ n \to \infty.$$

Taking limits in Equation (9) gives

$$u(\eta) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(\eta).$$

Since

$$(Lu)\left(\eta_j\right) = \sum_{i=1}^{\infty} A_i \left\langle L\overline{\psi}_i(\eta), \varphi_j(\eta) \right\rangle_{W_2^1} = \sum_{i=1}^{\infty} A_i \left\langle \overline{\psi}_i(\eta), L^*\varphi_j(\eta) \right\rangle_{{}^oW_2^3} = \sum_{i=1}^{\infty} A_i \left\langle \overline{\psi}_i(\eta), \overline{\psi}_j(\eta) \right\rangle_{{}^oW_2^3},$$

it follows that

$$\sum_{j=1}^{n} \beta_{nj}(Lu)(\eta_j) = \sum_{i=1}^{\infty} A_i \left\langle \overline{\psi}_i(\eta), \sum_{j=1}^{n} \beta_{nj} \overline{\psi}_j(\eta) \right\rangle_{{}^oW_2^3} = \sum_{i=1}^{\infty} A_i \left\langle \overline{\psi}_i(\eta), \overline{\psi}_n(\eta) \right\rangle_{{}^oW_2^3} = A_n.$$

If $n = 1$, then

$$Lu(\eta_1) = M(\eta_1, u_0(\eta_1)). \tag{16}$$

If $n = 2$, then

$$\beta_{21}(Lu)(\eta_1) + \beta_{22}(Lu)(\eta_2) = \beta_{21} M(\eta_1, u_0(\eta_1)) + \beta_{22} M(\eta_2, u_1(\eta_2)). \tag{17}$$

From Equations (16) and (17),

$$(Lu)(\eta_2) = M(\eta_2, u_1(\eta_2)).$$

Additionally, it is simple to show by induction that

$$(Lu)(\eta_j) = M(\eta_j, u_{j-1}(\eta_j)). \tag{18}$$

Therefore, we get

$$(Lu)(\varsigma) = M(\varsigma, u(\varsigma)),$$

that is, $u(\eta)$ is the solution of Equation (8) and

$$u(\eta) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i,$$

where A_i are given by Equation (13). This completes the proof. □

Theorem 6. *If $u \in {}^{o}W_2^3[0,1]$, then*

$$\|u_n - u\|_{{}^{o}W_2^3} \to 0, \quad n \to \infty.$$

Moreover, a sequence $\|u_n - u\|_{{}^{o}W_2^3}$ is monotonically decreasing in n.

Proof. We acquire

$$\|u_n - u\|_{{}^{o}W_2^3} = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(\eta_k, u_k,) \widehat{\Psi}_i \right\|_{{}^{o}W_2^3},$$

by Equations (10) and (11). Thus, we get

$$\|u_n - u\|_{{}^{o}W_2^3} \to 0, \quad n \to \infty.$$

$$\begin{aligned} \|u_n - u\|^2_{{}^{o}W_2^3} &= \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(\eta_k, u_k) \widehat{\Psi}_i \right\|^2_{{}^{o}W_2^3} \\ &= \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} M(\eta_k, u_k) \widehat{\Psi}_i \right)^2. \end{aligned}$$

Clearly, $\|u_n - u\|_{{}^{o}W_2^3}$ is monotonically decreasing in n. □

3. Group Preserving Scheme

Internal symmetry group of a system, especially dynamical systems obtained from Equation (1), preserves using the GPS and when we do not have the symmetry group of nonlinear Lane–Emden equation, it is possible to embed them into the augmented dynamical systems. Consider a dynamical system corresponding to a differential equation as follows:

$$\mathbf{y}' = \Psi(\eta, \mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^k, \eta \in \mathbb{R}. \tag{19}$$

Then, by using a definition for a unit vector of the orientation of the state vector y for Equation (19), we have:

$$\mathbf{n} := \frac{\mathbf{y}}{\|\mathbf{y}\|}, \tag{20}$$

where $\|\mathbf{y}\| = \sqrt{\mathbf{y} \cdot \mathbf{y}} > 0$ is the Euclidean norm of $\mathbf{y}$. Equations (19) and (20) conclude:

$$\dot{\mathbf{n}} := \frac{\Psi(\eta, \mathbf{y})}{\|\mathbf{y}\|} - \left(\frac{\Psi(\eta, \mathbf{y})}{\|\mathbf{y}\|} \cdot \mathbf{n} \right) \mathbf{n}. \tag{21}$$

In addition, upon utilizing Equations (19) and (20), we can write:

$$\frac{d}{d\eta} \|\mathbf{y}\| = \frac{d}{d\eta} \sqrt{\mathbf{y} \cdot \mathbf{y}} = \dot{\mathbf{y}} \cdot \mathbf{n} = \Psi(\eta, \mathbf{y}) \cdot \mathbf{n}. \tag{22}$$

From Equations (21) and (22), it follows:

$$\frac{d}{d\eta}\begin{bmatrix} \mathbf{y} \\ \|\mathbf{y}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{k\times k} & \frac{\Psi(\eta,\mathbf{y})}{\|\mathbf{y}\|} \\ \frac{\Psi^T(\eta,\mathbf{y})}{\|\mathbf{y}\|} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{y} \\ \|\mathbf{y}\| \end{bmatrix}. \tag{23}$$

Obviously, the first equation in Equation (23) is the same as the original Equation (19), but the addition of the second equation presents us a Minkowskian structure of the augmented state variables of $\mathbf{Y} := (\mathbf{y}^T, \|\mathbf{y}\|)^T \in \mathcal{M}^{k+1}(\mathbb{R})$, which describes an inner product on $\mathbb{R}^{k+1}$ given by:

$$\langle U, V\rangle = U^T \Lambda V = u_1 v_1 + \cdots + u_k v_k - u_{k+1} v_{k+1}, \tag{24}$$

where

$$\Lambda = \begin{bmatrix} I_k & \mathbf{0}_{k\times 1} \\ \mathbf{0}_{1\times k} & -1 \end{bmatrix}, \tag{25}$$

and

$$U^T = (u_1, \ldots, u_k, u_{k+1})^T, \;\; V^T = (v_1, \ldots, v_k, v_{k+1})^T.$$

This is the Lorentz inner product on $\mathbb{R}^{k+1}$.

Actually, the null vector in $\mathcal{M}^{k+1}(\mathbb{R})$ lies in the set

$$\mathcal{H}_{k,1}(0) = \{\mathbf{X} \in \mathbb{R}^{k+1} : \mathbf{X} \neq 0, \; \langle \mathbf{X}, \mathbf{X}\rangle = \mathbf{x}\cdot\mathbf{x} - t^2 = 0\}.$$

It is easy to investigate that, in the Minkowskian structure, the augmented variable $\mathbf{Y} := (\mathbf{y}^T, \|\mathbf{y}\|)^T$ is a null vector and, from the Lorentz inner product, satisfy the cone condition:

$$\langle \mathbf{Y}, \mathbf{Y}\rangle = \mathbf{Y}^T \Lambda \mathbf{Y} = 0. \tag{26}$$

Equation (23) can be written in the abstract form:

$$\mathbf{Y}' = \Omega \mathbf{Y}, \;\; \mathbf{Y} \in \mathcal{H}_{k,1}(0), \tag{27}$$

where

$$\Omega := \begin{bmatrix} \mathbf{0}_{k\times k} & \frac{\Psi(\eta,\mathbf{y})}{\|\mathbf{y}\|} \\ \frac{\Psi^T(\eta,\mathbf{y})}{\|\mathbf{y}\|} & 0 \end{bmatrix}. \tag{28}$$

Definition 3. *Let A be a real square matrix. Then,*

$$Sk\_Sym_k(\mathcal{M}^k(\mathbb{R})) = \{A : A^T\Lambda + \Lambda A = 0\}$$

is the space of skew symmetric matrices in Minkowskian structure.

We have to note that, in Equation (27), $\Omega \in Sk\_Sym_{k+1}(\mathcal{M}^{k+1}(\mathbb{R}))$.

There is a group of real square matrices that is well-known as a global linear group, defined by:

$$GL_k(\mathbb{R}) = \{G \in M_{k,k} : det(G) \neq 0\}.$$

Moreover, we can consider the closed subgroup

$$O(k, 1) = \{G \in GL_{k+1}(\mathbb{R}) : G^T \Lambda G = \Lambda\}.$$

We have to note that $G \in O(k,1)$ if and only if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{k+1}$, $\langle G\mathbf{x}, G\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$. Thus, $O(k,1)$ consist of all the Lorentzian isometries of $\mathbb{R}^{k+1}$. Notice that, for $G \in O(k,1)$, we have $det(G) = \pm 1$. Another useful subgroup of $O(k,1)$ is

$$SO_0(k,1) = \{G \in O(k,1) : det(G) = 1\},$$

which is well-known as the proper Orthochronous Lorentz group. Connections between the Lie groups and Lie algebras are specified by the exponential map. That is, if $so(k,1)$ is the Lie algebra of $SO_0(k,1)$, then

$$exp : so(k,1) \to SO_0(k,1). \tag{29}$$

Moreover, we know that $so(k,1) = Sk\_Sym_{k+1}(\mathcal{M}^{k+1}(\mathbb{R}))$ (See reference [34], p. 82). Therefore, in Equation (27), $\Omega \in so(k,1)$ and the corresponding discretized $G \in SO_0(k,1)$, obtained from the exponential map (29), have the following properties:

$$G^T \Lambda G = \Lambda, \;\; det(G) = 1. \tag{30}$$

Now, we are ready to develop our desired numerical scheme in the form:

$$\mathbf{Y}_{n+1} = G(n)\mathbf{Y}_n, \tag{31}$$

where $\mathbf{Y}_n$ interprets the numerical value of $\mathbf{Y}$ at a discrete t_n, and the discretized group element $G(n)$ is obtained through a Cayley transform as follows:

$$\begin{aligned} G(n) &= [I_k - \Delta\eta\ \Omega(n)]^{-1}[I_k + \Delta\eta\ \Omega(n)] \\ &= \begin{bmatrix} I_k + \frac{2\Delta\eta^2 \Psi_n \Psi_n^T}{\|\mathbf{y}_n\|^2 - \Delta\eta^2 \|\Psi_n\|^2} & \frac{2\Delta\eta\ \|\mathbf{y}_n\| \Psi_n}{\|\mathbf{y}_n\|^2 - \Delta\eta^2 \|\Psi_n\|^2} \\ \frac{2\Delta\eta\ \|\mathbf{y}_n\| \Psi_n^T}{\|\mathbf{y}_n\|^2 - \Delta\eta^2 \|\Psi_n\|^2} & \frac{\|\mathbf{y}_n\|^2 + \Delta\eta^2 \|\Psi_n\|^2}{\|\mathbf{y}_n\|^2 - \Delta\eta^2 \|\Psi_n\|^2} \end{bmatrix}. \end{aligned} \tag{32}$$

Substituting Equation (32) into Equation (31) and taking its first row, we get

$$\mathbf{y}_{n+1} = \mathbf{y}_n + 2\Delta\eta \frac{\|\mathbf{y}_n\|^2 + \Delta\eta\ \Psi_n \cdot \mathbf{y}_n}{\|\mathbf{y}_n\|^2 - \Delta\eta^2 \|\Psi_n\|^2} \Psi_n = \mathbf{y}_n + \sigma_n \Psi_n. \tag{33}$$

Now, we are ready to use the GPS for solving Equation (1) with initial conditions (2). According to Equation (19), we have:

$$\Psi(\eta, \mathbf{y}) := \begin{pmatrix} y_2(\eta) \\ -\frac{2}{\eta} y_2(\eta) - y_1^3(\eta) \end{pmatrix},$$

$$\mathbf{y} = \begin{pmatrix} y_1(\eta) \\ y_2(\eta) \end{pmatrix} = \begin{pmatrix} \varsigma(\eta) \\ \varsigma'(\eta) \end{pmatrix}.$$

Results of this example are obtained by fixing $\Delta\eta = 10^{-7}$. Figure 1 shows the graph of the approximate solution obtained by GPS. Moreover, the approximate solutions of Equation (1) obtained by the reproducing kernel method and group preserving scheme are reported in Table 1. Results of this paper show that two investigated methods are in good agreement and approximate solutions are reliable. We calculated all our results with Maple 13 (Siirt University). We used

$$\eta_i = \frac{i}{m}, \quad i = 1, 2, 3, \cdots, m$$

for our numerical results. Using the reproducing kernel method, we choose 100 points. It is possible to improve the results by increasing the points.

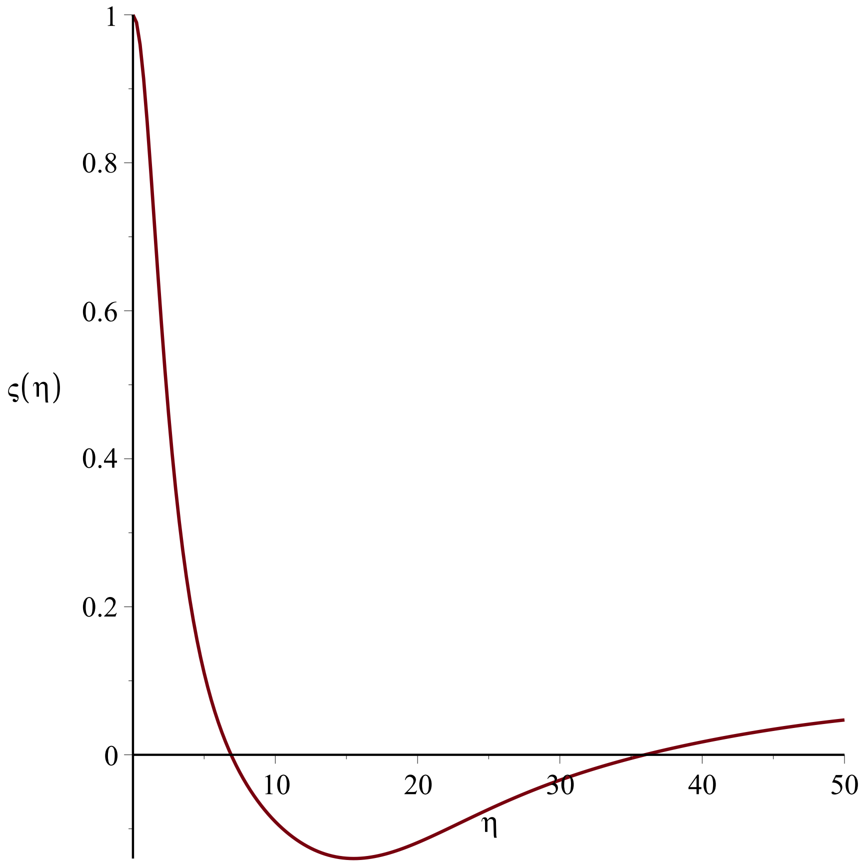

Figure 1. Numerical solution of Equation (1) obtained by the group preserving scheme (GPS).

Table 1. Comparison of approximate solutions obtained by reproducing kernel method (RKM) and GPS for Equation (1).

η	RKM	GPS
0.1	0.9983360948	0.998335829543602
0.5	0.9598395393	0.959839062543164
1.0	0.8550592570	0.855057541543122
2.0	0.5829639252	0.582850462212463
3.0	0.3592354020	0.359226444051538
4.0	0.2091578370	0.209281565659890
5.0	0.1106289100	0.110819798197543
6.0	0.0435212480	0.043737947433237
7.0	−0.004536310	−0.00431221951973
8.0	−0.040571182	−0.04034773735436
9.0	−0.068517970	−0.06829954400156
10.0	−0.090565560	−0.09035595601487

4. Conclusions

We discussed the RKM and the GPS for solving the Lane–Emden equation with initial conditions expressed given by Equation (1). An example depicted in Equation (1) was presented and the computational accuracy was illustrated. We found the approximate solutions for different values of η by using RKM and GPS, respectively. As it is shown in Table 1, these two investigated methods are very accurate. In addition, we reported very useful reproducing kernel functions and a geometric approach in this work.

Acknowledgments: This research was supported by 2017-SİÜFED-39 and 2017-SİÜFEB-40.

Author Contributions: All authors have contributed equally in this paper. All authors read and approved the final manuscript. Ali Akgül and Mir Sajjad Hashemi conceived and designed the experiments; Mustafa Inc performed the experiments; Idrees Sedeeq Mustafa and Dumitru Baleanu analyzed the data; Idrees Sedeeq Mustafa contributed reagents/materials/analysis tools; Ali Akgül wrote the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Singh, O.P.; Pandey, R.; Singh, V.K. An analytic algorithm of Lane-Emden type equations arising in astrophysics using modified homotopy analysis method. *Comput. Phys. Commun.* **2009**, *180*, 1116–1124.
2. Kaur, H.; Mittal, R.C.; Mishra, V. Haar wavelet approximate solutions for the generalized lane–emden equations arising in astrophysics. *Comput. Phys. Commun.* **2013**, *184*, 2169–2177.
3. Nasab, A.K.; Kılıçman, A.; Atabakan, Z.P.; Leong, W.J. A numerical approach for solving singular nonlinear lane-emden type equations arising in astrophysics. *New Astron.* **2015**, *34*, 178–186.
4. Pandey, R.K.; Kumar, N. Solution of lane–emden type equations using bernstein operational matrix of differentiation. *New Astron.* **2012**, *17*, 303–308.
5. Pandey, R.K.; Kumar, N.; Bhardwaj, A.; Dutta, G. Solution of lane–emden type equations using legendre operational matrix of differentiation. *Appl. Math. Comput.* **2012**, *218*, 7629–7637.
6. Cui, M.; Lin, Y. *Nonlinear Numerical Analysis in the Reproducing Kernel Space*; Nova Science Publishers Inc.: New York, NY, USA, 2009.
7. Akgül, A. A new method for approximate solutions of fractional order boundary value problems. *Neural Parallel Sci. Comput.* **2014**, *22*, 223–237.
8. Aronszajn, N. Theory of reproducing kernels. *Trans. Am. Math. Soc.* **1950**, *68*, 337–404.
9. Chen, Z. The exact solution of system of linear operator equations in reproducing kernel spaces. *Appl. Math. Comput.* **2008**, *203*, 56–61.
10. Geng, F.; Cui, M.; Zhan, B. Method for solving nonlinear initial value problems by combining homotopy perturbation and reproducing kernel Hilbert space methods. *Nonlinear Anal. Real World Appl.* **2010**, *11*, 637–644.
11. Turkyilmazoglu, M. Effective computation of exact and analytic approximate solutions to singular nonlinear equations of Lane-Emden-Fowler type. *Appl. Math. Model.* **2013**, *37*, 7539—7548.
12. Adem, A.; Khalique, C.; Biswas, A. Solutions of Kadomtsev-Petviashvili equation with power law nonlinearity in $1+3$ dimensions. *Math. Methods Appl. Sci.* **2011**, *34*, 532–543.
13. Biswas, A.; Zerrad, E. Higher order Gabitov-Turitsyn equation for dispersion-managed solitons in multiple channels. *Int. J. Math. Anal.* **2007**, *1*, 565–582.
14. Ebadi, G.; Biswas, A. The $\frac{G'}{G}$ method and topological soliton solution of the $K(m,n)$ equation. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 2377–2382.
15. Ebadi, G.; Biswas, A. Application of G'/G-expansion method to Kuramoto-Sivashinsky equation. *Acta Math. Appl. Sin. Engl. Ser.* **2016**, *32*, 623–630.
16. Jafari, H.; Tajadodi, H.; Biswas, A. Homotopy analysis method for solving a couple of evolution equations and comparison with Adomian's decomposition method. *Waves Random Complex Media* **2011**, *21*, 657–667.
17. Jafari, H.; Sooraki, A.; Talebi, Y.; Biswas, A. The first integral method and traveling wave solutions to Davey-Stewartson equation. *Nonlinear Anal. Model. Control* **2012**, *17*, 182–193.
18. Johnpillai, A.G.; Kara, A.H.; Biswas, A. Symmetry reduction, exact group-invariant solutions and conservation laws of the Benjamin-Bona-Mahoney equation. *Appl. Math. Lett.* **2013**, *26*, 376–381.
19. Khalique, C.M.; Biswas, A. A Lie symmetry approach to nonlinear Schrödinger's equation with non-Kerr law nonlinearity. *Commun. Nonlinear Sci. Numer. Simul.* **2009**, *14*, 4033–4040.
20. Kumar, S.; Hama, A.; Biswas, A. Solutions of Konopelchenko-Dubrovsky equation by traveling wave hypothesis and Lie symmetry approach. *Appl. Math. Inf. Sci.* **2014**, *8*, 1533–1539.
21. Milovic, D.; Biswas, A. Doubly periodic solution for nonlinear Schrödinger's equation with triple power law nonlinearity. *Int. J. Nonlinear Sci.* **2009**, *7*, 420–425.
22. Morris, R.M.; Kara, A.H.; Biswas, A. An analysis of the Zhiber-Shabat equation including Lie point symmetries and conservation laws. *Collect. Math.* **2016**, *67*, 55–62.

23. Liu, C.-S. Cone of nonlinear dynamical system and group preserving schemes. *Int. J. Non-Linear Mech.* **2001**, *36*, 1047–1068.
24. Abbasbandy, S.; Hashemi, M.S. Group preserving scheme for the cauchy problem of the laplace equation. *Eng. Anal. Bound. Elem.* **2011**, *35*, 1003–1009.
25. Akgül, A.; Hashemi, M.S.; Raheem, S.A. Constructing two powerful methods to solve the thomas—Fermi equation. *Nonlinear Dyn.* **2016**, *87*, 1435–1444.
26. Akgül, A.; Hashemi, M.; Inc, M. Group preserving scheme and reproducing kernel method for the poisson–boltzmann equation for semiconductor devices. *Nonlinear Dyn.* **2017**, *88*, 2817–2829.
27. Hashemi, M.S. Constructing a new geometric numerical integration method to the nonlinear heat transfer equations. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *22*, 990–1001.
28. Hashemi, M.S.; Abbasband, S. A geometric approach for solving troesch's problem. *Bull. Malays. Math. Sci. Soc.* **2017**, *40*, 97–116.
29. Hashemi, M.S.; Baleanu, D.; Parto-Haghigh, M. A lie group approach to solve the fractional poisson equation. *Rom. J. Phys.* **2015**, *60*, 289–1297.
30. Hashemi, M.S.; Darvishi, E.; Baleanu, D. A geometric approach for solving the density dependent diffusion nagumo equation. *Adv. Differ. Equ.* **2016**, *2016*, doi:10.1186/s13662-016-0818-2.
31. Hashemi, M.S.; Inc, M.; Karatas, E.; Akgül, A. A numerical investigation on burgers equation by mol-gps method. *J. Adv. Phys.* **2017**, *6*, 413–417.
32. Hashemi, M.S.; Nucci, M.C.; Abbasbandy, S. Group analysis of the modified generalized vakhnenko equation. *Commun. Nonlinear Sci. Numer. Simul.* **2013**, *18*, 867–877.
33. Liu, C.-S. Group preserving scheme for backward heat conduction problems. *Int. J. Heat Mass Transf.* **2004**, *47*, 2567–2576.
34. Baker, A. *Matrix Groups: An Introduction to Lie Group Theory*; Springer: London, UK, 2002.