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# A method for solving nonlinear Volterra's population growth model of noninteger order

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## Abstract

Many numerical methods have been developed for nonlinear fractional integro-differential Volterra's population model (FVPG). In these methods, to approximate a function on a particular interval, only a restricted number of points have been employed. In this research, we show that it is possible to use the fuzzy transform method (F-transform) to tackle with FVPG. It makes the F-transform preferable to other methods since it can make full use of all points on this interval. We also make a comparison showing that this method is less computational and is more convenient to be utilized for coping with nonlinear integro-differential equation (IDEs), fractional nonlinear integro-differential equation (FIDEs), and fractional ordinary differential equations (FODEs).

**MSC:** 45J05; 26A33; 26A33

**Keywords:** population growth; fuzzy transform; Caputo derivative; integro-differential equation

## 1 Introduction

Fractional arithmetic and fractional differential equations appear in many sciences, including medicine [1], economics [2], dynamical problems [3, 4], chemistry [5], mathematical physics [6], traffic model [7], fluid flow [8], and so on. Scholars and researchers are invited to check books that have been written to take advantage of fractional arithmetic [9–11].

In this research work, we have for the first time shown that it is possible to use the  $F$ -transform method (FTM) to tackle with the nonlinear Volterra population growth model of noninteger order (FVPG). We have investigated the following FVPG:

$$D_t^\beta P(\tilde{t}) = AP(\tilde{t}) - BP^2(\tilde{t}) - CP(\tilde{t})I_0^\alpha P(s), \quad 0 < \beta \leq 1, P(0) = P_0, \quad (1.1)$$

where

- $D^\beta$  denotes the fractional differential operator of order  $\beta$  defined by

$$D^\beta P(t) = \frac{1}{\Gamma(k-\beta)} \int_0^t (t-s)^{\beta-1} P^{(k)}(s) ds, \quad k-1 < \beta \leq k, k \in \mathbb{N}, \quad (1.2)$$

- the positive parameter  $A$  stands for the coefficient of birth rate,
- the positive parameter  $B$  is the coefficient of crowding,
- the positive parameter  $C$  is the coefficient of toxicity, which demonstrates the important treatment of the population evolution as long as its level falls to zero in the long run,
- $P_0$  is the primary population,
- $P = P(\tilde{t})$  denotes the population at time  $\tilde{t}$ , and
- the integral  $I_0^{\tilde{t}}P(s) = \int_0^{\tilde{t}} P(s) ds$  indicates the effect of toxin gathering.

We utilize the scale population and time by putting  $u = \frac{BP}{A}$  and  $t = \frac{C\tilde{t}}{B}$  to gain the following problem:

$$\begin{cases} \lambda D^\beta u(t) = u(t) - u^2(t) - u(t)I_0^t u(s), & 0 < \beta \leq 1, \\ u(0) = u_0. \end{cases} \tag{1.3}$$

In the nondimensional problem (1.3):

- $u(t)$  at time  $t$  is the scaled population of similar individuals,
- $\lambda = \frac{C}{AB}$  is a prescribed nondimensional parameter.

The analytical solution for  $\beta = 1$  of (1.3) is [12]

$$u(t) = u_0 \exp\left(\frac{1}{\lambda} \int_0^t \left(1 - u(s) - \int_0^s u(x) dx\right) ds\right), \tag{1.4}$$

which shows that if  $u_0 > 0$ , then  $u(t)$  is positive for all  $t$ .

Various approximate methods for approximating the VPG and FVPG have been investigated by scholars: Hicdurmaz and Can [13], the pseudospectral method of the Legendre functions of noninteger order; Parand and Delkhosh [14], the generalized Chebyshev orthogonal functions of non-integer order of the first kind and the collocation method; Maleki and Maleki [15], the multidomain Legendre-Gauss pseudospectral approach; Suat Erturk et al. [16], the Padé approximations and differential transform method; Khan et al. [17], a new homotopy perturbation method; Krishnaveni et al. [18], the shifted Legendre polynomial method, Dehghan and Shahini [19], the rational pseudospectral approximation, Ghasemi [20], a new homotopy analysis method; Yildirim and Gulkanat [21], the homotopy-Padé technique; Fathizadeh [22], the hybrid rational Haar wavelets; and so on.

The  $F$ -transform has been implemented for dealing with ordinary differential equations as compared with many other classical procedures [23]. In approximate methods, for the purpose of approximating a function on particular interval, only a restricted number of points are used. The  $F$ -transform is preferable to other methods because it uses all points in this interval.

The FTM has recently been utilized in [24–26] to find an approximate solution of the first-order fuzzy differential equations and two-point boundary value problems.

Along the same line of research, Chen and Shen [27] have established an algorithm to gain the numerical solutions of second-order primary amount problems.

## 2 Fuzzy partition and fuzzy transform

In this section, we outline the main definitions of the  $F$ -transform to be utilized in the subsequent sections of numerical implementations.

**Definition 2.1** ([23]) Presuming that for  $n \geq 2, t_1 = a < t_2 < \dots < t_{n-1} < t_n = b$  are specified nodes, we express that fuzzy sets  $B_1, \dots, B_n$  defined on  $[a, b]$  with their membership functions  $B_1(t), \dots, B_n(t)$  form a fuzzy partition of  $[a, b]$  if they satisfy the following conditions:

- (1) the functions  $B_k$  on  $[a, b]$  taking values in  $[0, 1]$  are continuous,  $\sum_{k=1}^n B_k(t) = 1$  for  $t \in [a, b]$ , and  $B_k(t_k) = 1$ ,
- (2)  $B_k(t) = 0$  if  $t \notin (t_{k-1}, t_{k+1})$  with  $t_0 = a$  and  $t_{n+1} = b$ ,
- (3)  $B_k(t)$  increases on  $[t_{k-1}, t_k]$  for  $k = 2, \dots, n$  and decreases on  $[t_k, t_{k+1}]$ ,  $k = 1, \dots, n - 1$ .

The membership functions  $B_1, B_2, \dots, B_n$  are called basic functions (BFs).

The following formulas give standard triangular membership functions:

$$\begin{aligned}
 B_1(t) &= \begin{cases} 1 - \frac{t-t_1}{h_1}, & t_1 \leq t \leq t_2, \\ 0 & \text{otherwise,} \end{cases} \\
 B_k(t) &= \begin{cases} \frac{t-t_{k-1}}{h_{k-1}}, & t_{k-1} \leq t \leq t_k, \\ 1 - \frac{t-t_k}{h_k}, & t_k \leq t \leq t_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \\
 B_n(t) &= \begin{cases} \frac{t-t_{n-1}}{h_{n-1}}, & t_{n-1} \leq t \leq t_n, \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned} \tag{2.1}$$

The following formulas for  $k = 2, \dots, n - 1$  give standard sinusoidal membership functions:

$$\begin{aligned}
 B_1(t) &= \begin{cases} 0.5(1 + \cos \frac{\pi}{h}(t - t_1)), & t_1 \leq t \leq t_2, \\ 0 & \text{otherwise,} \end{cases} \\
 B_k(t) &= \begin{cases} 0.5(1 + \cos \frac{\pi}{h}(t - t_k)), & t_{k-1} \leq t \leq t_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \\
 B_n(t) &= \begin{cases} 0.5(1 + \cos \frac{\pi}{h}(t - t_n)), & t_{n-1} \leq t \leq t_n, \\ 0 & \text{otherwise,} \end{cases}
 \end{aligned} \tag{2.2}$$

where  $h_k = t_{k+1} - t_k, k = 1, \dots, n - 1$ . A fuzzy partition of  $[a, b]$  for  $k = 1, 2, \dots, n - 1$  is uniform if  $t_{k+1} - t_k = h = \frac{b-a}{n-1}$  and two additional properties are satisfied:

- (4)  $B_k(t_k - t) = B_k(t_k + t)$  for all  $t \in [0, h], k = 2, \dots, n - 1$ ,
- (5)  $B_k(t) = B_{k-1}(t - h)$  and  $B_{k+1}(t) = B_k(t - h)$  for all  $k = 2, \dots, n - 1$  and  $t \in [t_k, t_{k+1}]$ .

**Lemma 2.2** ([23]) Let  $n \geq 3$ , and let  $B_1, B_2, \dots, B_n$  be the BF's forming a uniform partition of  $[a, b]$ . Then

$$\int_{t_1}^{t_2} B_1(t) dt = \int_{t_{n-1}}^{t_n} B_n(t) dt = \frac{h}{2} \tag{2.3}$$

and

$$\int_{t_{k-1}}^{t_{k+1}} B_k(t) dt = h \tag{2.4}$$

for  $k = 2, \dots, n - 1$ , where  $h$  is the distance between two neighboring nodes.

**Lemma 2.3** Let  $B_1, B_2, \dots, B_n, n \geq 3$ , be the sinusoidal membership functions for BFs forming a uniform partition of  $[a, b]$ . Then

$$\int_0^{t_k} B_i(t) dt = \begin{cases} h, & k \geq i + 1, \\ \frac{h}{2}, & k = i, \\ 0, & k \leq i - 1. \end{cases}$$

*Proof* Regarding Definition 2.1, if  $t \notin (t_{i-1}, t_{i+1})$ , then  $B_i(t) = 0$ . Otherwise, if  $k \leq i - 1$ , then  $t \leq t_k$ , so  $B_i(t) = 0$ , and we have  $\int_0^{t_k} B_i(t) dt = 0$ .

By Lemma 2.2 and Definition 2.1, if  $k = i$ , then

$$\int_0^{t_k} B_i(t) dt = \int_0^{t_{k-1}} B_i(t) dt + \int_{t_{k-1}}^{t_k} B_i(t) dt = 0 + \frac{h}{2} = \frac{h}{2}.$$

If  $k \geq i + 1$ , then, by Lemma 2.2 and Definition 2.1,

$$\int_0^{t_k} B_i(t) dt = \int_0^{t_{k-1}} B_i(t) dt + \int_{t_{k-1}}^{t_{k+1}} B_i(t) dt + \int_{t_{k+1}}^{t_k} B_i(t) dt = h. \quad \square$$

**Definition 2.4** ([23]) Let  $f$  be any function from  $C([a, b])$ , and let  $B_1, B_2, \dots, B_n$  be BFs that form a fuzzy partition of  $[a, b]$ . We claim that the  $n$ -tuple  $[F_1, F_2, \dots, F_n]$  of real numbers given by

$$F_k = \frac{\int_a^b f(t)B_k(t) dt}{\int_a^b B_k(t) dt} \tag{2.5}$$

for  $k = 1, 2, \dots, n$  is the  $F$ -transform of  $f$  in relation to  $B_1, B_2, \dots, B_n$ .

**Definition 2.5** ([23]) Let  $[F_1, F_2, \dots, F_n]$  be the  $F$ -transform of a function  $f$  featuring BFs  $B_1, B_2, \dots, B_n$ . Then

$$f_n(t) = \sum_{k=1}^n F_k B_k(t)$$

is called the inverse  $F$ -transform of a function  $f$  on  $[a, b]$ .

**Theorem 2.6** ([23]) Let  $B_1, B_2, \dots, B_n$  be the BFs forming a fuzzy partition of  $[a, b]$ , and let  $f$  be a continuous function on  $[a, b]$ . Thus, the  $k$ th component of the integral  $F$ -transform over  $[f(a), f(b)]$  gives the minimum of the function

$$\phi(y) = \int_a^b (f(t) - y)^2 B_k(t) dt.$$

**Lemma 2.7** ([23] (Convergence)) Let  $f$  be a continuous function on  $[a, b]$ . Thus, for every  $\epsilon > 0$ , there exist  $n_\epsilon$  and a fuzzy partition  $B_1, \dots, B_{n_\epsilon}$  of  $[a, b]$  such that

$$|f(t) - f_{n_\epsilon}(t)| \leq \epsilon \tag{2.6}$$

for all  $t \in [a, b]$ .

**Theorem 2.8** *Let  $B_1, B_2, \dots, B_n$  be BFs, and let*

$$FT(B_1, B_2, \dots, B_n) = \left\{ g(t) \mid g(t) = \sum_{i=1}^n c_i B_i(t) \right\}, \tag{2.7}$$

where  $c_1, c_2, \dots, c_n$  are optional real numbers, and  $f \in L_2([a, b], B_1, B_2, \dots, B_n)$ . Considering

$$\|f\|_k = \sqrt{\int_{t_{k-1}}^{t_{k+1}} f^2(t) B_k(t) dt}, \quad k = 1, \dots, n,$$

and the metric distance  $d_k(f, g) = \|f - g\|_k$ , the components  $F_1, F_2, \dots, F_n$  of the  $F$ -transform of  $f$  minimize the sum  $\sum_{k=1}^n d_k^2(f, c_k)$  with respect to the parameters  $c_1, c_2, \dots, c_n$ .

*Proof*  $L_2([a, b], B_1, B_2, \dots, B_n)$  is a normed space (see Lemma 5 in [23]). With the metric distance  $d_k(f, g)$  and Theorem from [23], the proof is complete.  $\square$

### 3 Description of the new approach

By Theorem 2.8, if  $u \in L_2([0, T], B_0, B_1, \dots, B_n)$ , then we get an approximation of the function  $u$  for (1.2) with basic functions BFs of the form

$$u(t) \approx u_{\text{BFT}}(t) = \sum_{k=0}^n u_k B_k(t), \quad t \in [0, T], \tag{3.1}$$

where  $u_k = u(t_k)$  for  $k = 0, 1, 2, \dots, n$  with  $t_k = kh$ , and  $h = \frac{T}{n}$  is the time step length (in which  $n \geq 1$  is integer). It is evident that  $u(t_k) = u_k = u_{\text{BFT}}(t_k)$ .

Using the approximation for the Caputo derivative [28] of equation (1.3), we have:

$$D^\beta u(t_{k+1}) \approx \frac{1}{\Gamma(2-\beta)h^\beta} \sum_{j=0}^k (u(t_{j+1}) - u(t_j))((k-j+1)^{1-\beta} - (k-j)^{1-\beta}). \tag{3.2}$$

**Remark 3.1** Substituting  $u_{\text{BFT}}(t)$  into (3.2), we get the equation

$$D^\beta u_{\text{BFT}}(t_{k+1}) \approx \frac{1}{\Gamma(2-\beta)h^\beta} \sum_{j=0}^k (u_{j+1} - u_j)((k-j+1)^{1-\beta} - (k-j)^{1-\beta}). \tag{3.3}$$

Now, to gain the approximate solution of problem (1.2), we use  $u_{\text{BFT}}(t)$ , and then

$$D_t^\beta u_{\text{BFT}}(t) - \frac{1}{\lambda} \left( u_{\text{BFT}}(t) - u_{\text{BFT}}^2(t) - u_{\text{BFT}}(t) \int_0^t u_{\text{BFT}}(x) dx \right) = 0, \tag{3.4}$$

$$0 < \beta \leq 1, 0 < t \leq T,$$

and also

$$D^\beta u_{\text{BFT}}(t_k) - \frac{1}{\lambda} \left( u_{\text{BFT}}(t_k) - u_{\text{BFT}}^2(t_k) - u_{\text{BFT}}(t_k) \int_0^{t_k} u_{\text{BFT}}(x) dx \right), \tag{3.5}$$

$$k = 0, 1, \dots, n.$$

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**Algorithm 1** An approximation algorithm for the  $n$  partition of  $[0, t_n]$  with sinusoidal  $F$ -transform.

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**Step 1.** Input  $u_0 = u(0)$ ,  $n$ , and  $h$ .

**Step 2.** Locate  $t_k \leftarrow kh, k = 0, 1, 2, \dots, n$ .

**Step 3.** Choose sinusoidal  $BFs B_k(t)$  for  $k = 0, 1, 2, \dots, n$ .

**Step 4.** Set

$$\frac{1}{h^\beta \Gamma(2 - \beta)} \sum_{j=0}^k (u_{j+1} - u_j) ((k - j + 1)^{1-\beta} - (k - j)^{1-\beta}) - \frac{1}{\lambda} \left( u_{k+1} - u_{k+1}^2 - \sum_{i=1}^{k+1} a_i u_i u_{k+1} \right) = 0.$$

**Step 5.** Calculate every  $u_k, k = 1, 2, \dots, n$ , of an equation of degree two.

**Step 6.** The approximate solution is

$$u_{\text{BFT}}(t) \approx \sum_{i=0}^n u_i B_i(t).$$


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Regarding (1.3), (3.1), and (3.3), we obtain the following result:

$$\frac{1}{h^\beta \Gamma(2 - \beta)} \sum_{j=0}^k (u_{j+1} - u_j) \times ((k - j + 1)^{1-\beta} - (k - j)^{1-\beta}) \approx \frac{1}{\lambda} \left( u_{k+1} - u_{k+1}^2 - \sum_{i=1}^{k+1} a_i u_i u_{k+1} \right), \tag{3.6}$$

where  $k = 0, 1, 2, \dots, n - 1, a_1 = a_k = \frac{h}{2}$ , and  $a_2 = a_3 = \dots = a_{k-1} = h$ .

Using the boundary condition  $u_0 = u(0)$ , we can calculate  $u_1, u_2, \dots, u_n$  and then get the approximate solution  $u(t) \approx u_{\text{BFT}}(t)$  for (1.3).

An approximation for this method stated in Algorithm 1.

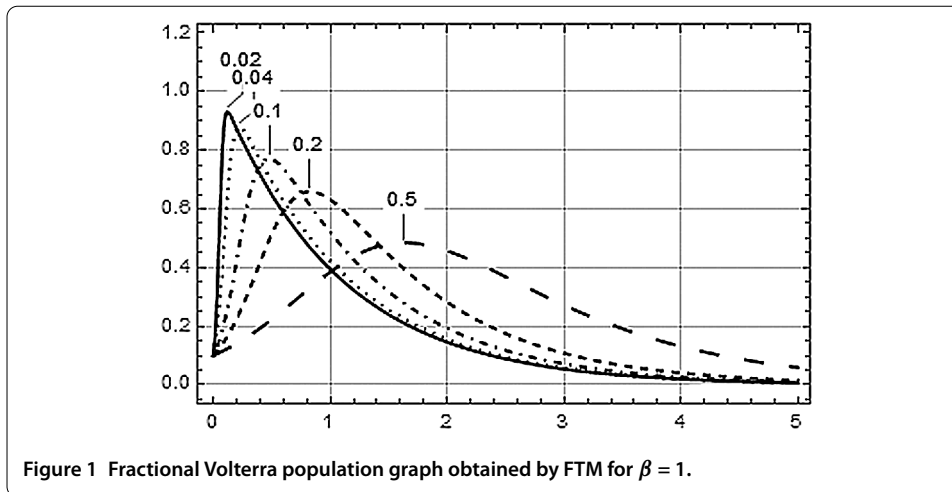
Figure 1 shows fractional Volterra population graph obtained by FTM with  $n = 500, \beta = 1$ , different values of  $\lambda = (0.02, 0.04, 0.1, 0.2, 0.5), h = 0.01$ , and  $u(0) = 0.1$ .

#### 4 Numerical results

Table 1 for  $\beta = 1$  illustrates a comparison of the FTM and approximate methods: new homotopic perturbation (NHPM), second derivative multistep approaches (SDMM), rational Chebyshev collocation method (RCC), Hermite functions collocation (HFC), homotopy perturbation method (HPM), with the exact values of [17]

$$u_{\text{maximum}} = 1 + \lambda \ln \left( \frac{\lambda}{1 + \lambda - u(0)} \right). \tag{4.1}$$

In the last two columns of Table 1, we can see that by increasing the amount  $n$  a more accurate answer can be achieved.



**Table 1** A comparison for  $\beta = 1$  of the FTM and approximate methods

$\lambda$	NHPM	HPM	HFC	RCC	SDMM	$u_{max}$	Fuzzy transform method	
							$h = 0.01, n = 500$	$h = 0.001, n = 5000$
0.02	0.922942037	0.90383805	0.92342704	0.92342715	0.92342714	0.92342717	0.910786	0.923059
0.04	0.873725344	0.86124017	0.87371998	0.87371998	0.87371998	0.87371998	0.863645	0.873381
0.10	0.765113089	0.76511308	0.76974149	0.76974149	0.76974140	0.76974149	0.760485	0.769751
0.20	0.659050432	0.65791230	0.65905038	0.65905038	0.65905037	0.65905038	0.651225	0.659057
0.50	0.485190290	0.48528234	0.48519030	0.48519030	0.48519029	0.48519030	0.480624	0.485189

### 5 Conclusion

In the present paper, we have applied the sinusoidal BFs for an approximate solution of FVPG. The advantage of this method can be its simple way with approximate accuracy indicated through a given example. As it has been illustrated in the previous section, the approximate solution resulted in this paper is consistent with either exact solution or with other counterparts. Eventually, it is recommended to utilize the proposed approach to solve differential equations, partial integro-differential equations, and equation systems of arbitrary order.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors read and approved the final manuscript.

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