Advances in Difference Equations
a SpringerOpen Journal

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Local existence for an impulsive fractional neutral integro-differential system with Riemann–Liouville fractional derivatives in a Banach space

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Abstract

In this manuscript, we investigate a sort of fractional neutral integro-differential equations with impulsive outcomes and extend the formula of general solutions for the impulsive fractional neutral integro-differential system in a Banach space. By using the analysis of the limit case and the operator generating compact semigroup, we derive the main results. Finally, an example is discussed to illustrate the efficiency of the results.

MSC: Primary 34A08; 35R12; secondary 45J05; 34A37

Keywords: Fractional differential equations; Riemann–Liouville fractional derivatives; Impulsive

1 Introduction

Fractional calculus is a field of mathematics study that grows out of traditional definitions of calculus integral and derivative operators in much the same way fractional exponents are an outgrowth of exponents with integer value. The concept of fractional (fractional derivatives and integrals) is popularly believed to have stemmed from a question raised in the year 1695 by Marquis de L'Hopital (1661–1704) to Gottfried Wilhelm Leibniz (1646–1716), which sought the meaning of Leibniz's (currently popular) notation $\frac{d^n y}{dx^n}$ for the derivative of order $n \in \mathbb{N}$ when $n = \frac{1}{2}$; that is, "What if n is fractional?". In his reply, dated 30 September 1695, Leibniz wrote to L' Hopital as follows:

"This is an apparent paradox from which, one day, useful consequences will be drawn." That is, "$d^{\frac{1}{2}}x$ are going to be adequate $x\sqrt{dx : x}$."

It is typically acknowledged that integer-order derivatives and integrals have clear physical and geometric interpretations. However, just in case of fractional-order integration and differentiation, that represent an apace growing field each in theory and in applications to planet issues, it is not thus. Since the looks of the thought on differentiation and integration of arbitrary (not necessary integer) order, there was not any acceptable geometric and physical interpretation of those operations for some three hundred years. In [22], it is shown that geometric interpretation of fractional integration is "Shadows on the walls" and its physical interpretation is "Shadows of the past". Fractional differential equations (abbreviated, FDEs) and integro-differential equations have gained wide importance as a result of their applications in numerous fields like physics, mechanics, control theory, and engineering, one will create relevance to the books [2, 11, 22, 29] and also the papers [5, 10, 20, 25, 26, 30–32].

The definitions of Riemann–Liouville (abbreviated RL) FDEs or integral initial conditions play a very important role in some fractional problems within the world. Heymans and Podlubny [9] verified that it had been attainable to attribute physical desiring to initial conditions expressed in terms of RL fractional derivatives or integrals on the sector of the viscoelasticity. For more details, one can see [1, 15, 16, 19, 27].

In addition, the speculation of impulsive differential equations seems to be a natural description of many real processes subject to sure perturbations whose length is negligible as compared with the overall length of the method, such changes are going to be fairly well approximated as being quick changes of state, or inside the design of impulses. This methodology tends to be extra fittingly sculptured by impulsive differential equations, which allow for discontinuities inside the evolution of the state, considered in such fields as drugs, biology, engineering science, chemical technology, etc. Therefore, it appears fascinating to check the fractional impulsive differential and integro-differential equations.

Furthermore, impulsive fractional evolution systems with the Caputo fractional derivative with completely different conditions were studied by several authors, one can see [4, 7, 8, 12–14, 23, 24]. However, abundant less is thought regarding the impulsive fractional evolution systems with RL fractional derivative, see [17, 18, 28].

In specific, the following fractional order integro-differential equation in a Banach space using the Caputo fractional derivative:

$$\begin{cases} {}^{C}D_t^{\alpha} u(t) + Au(t) = f(t,u(t)) + \int_0^t q(t-s)g(t,u(s))\,ds, & t \geq 0, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), & k = 1,\dots,m, \\ u(0) = u_0 \in X, \end{cases}$$

was mentioned by Gou and Li [8], and they established the local and global existence of mild solution to an impulsive fractional semilinear integro-differential equation with noncompact semigroup. In [18], Liu and Bin gave the approximate controllability of an impulsive RL fractional system with the help of the Banach contraction principle in a Banach space. Liu et al. [17] established the approximate controllability of impulsive fractional neutral evolution equations with RL fractional derivatives by using the Banach contraction principle. Later, Zhang et al. [28] analyzed the general solution of impulsive systems with RL fractional derivatives by using a limit case (as impulse tends to zero).

However, local existence for impulsive fractional neutral integro-differential equations with RL has not been fully investigated in the literature.

Inspired by the above-mentioned works, we investigate the following impulsive fractional integro-differential equation with RL fractional derivative of the form

$$\begin{aligned} {}^{L}D_t^{\gamma}\big[w(t) - \mathscr{D}\big(t,w(t)\big)\big] &= \mathscr{A}w(t) + \mathscr{L}\big(t,w(t)\big) \\ &\quad + \int_0^t q(t-s)\mathscr{P}\big(t,w(s)\big)\,ds, \qquad (1.1) \\ &\quad \mathscr{I}' = t \in (0,T], \quad t \neq t_k, k = 1,\dots,m, \end{aligned}$$

$$\Delta I_{0^+}^{1-\gamma} w|_{t=t_k} = I_{0^+}^{1-\gamma} w\bigl(t_k^+\bigr) - I_{0^+}^{1-\gamma} w\bigl(t_k^-\bigr) = I_k\bigl(w\bigl(t_k^-\bigr)\bigr), \tag{1.2}$$

$$I_{0^+}^{1-\gamma}\bigl[w(t) - \mathscr{D}\bigl(t, w(t)\bigr)\bigr]|_{t=0} = w_0 \in \mathbb{H}, \tag{1.3}$$

where ${}^L D_t^\gamma$ $(0 < \gamma < 1)$ represents the RL fractional derivative of order γ and $I_{0+}^{1-\gamma}$ denotes the RL integral of order $1-\gamma$. Throughout this paper, we take $\mathscr{I} = [0, T]$ is an operational interval. Let $\mathscr{A} : D(\mathscr{A}) \subseteq \mathbb{H} \to \mathbb{H}$ be the infinitesimal generator of a c_0 semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ in a Banach space $\mathbb{H}$. There exists a constant $\mathcal{M} \geq 1$ such that $\|\mathcal{T}(t)\| \leq \mathcal{M}$, $\mathscr{D}, \mathscr{L}, \mathscr{P} : \mathscr{I} \times \mathbb{H} \to \mathbb{H}$, $q : \mathscr{I} \to \mathbb{H}$ and $I_k : \mathbb{H} \to \mathbb{H}$ are apposite continuous functions. $0 = t_0 < t_1 <, \ldots, < t_m = T$. Here $I_{0+}^{1-\gamma} w(t_k^+) = \lim_{\epsilon\to 0^+} I_{0+}^{1-\gamma} w(t_k + \epsilon)$ and $I_{0+}^{1-\gamma} w(t_k^-) = \lim_{\epsilon\to 0^-} I_{0+}^{1-\gamma} w(t_k + \epsilon)$ denote the right and left limits of $I_{0+}^{1-\gamma} w(t)$ at $t = t_k$, respectively.

For impulsive system (1.1)–(1.3), we have

$$\lim_{I_1\to 0,\ldots,I_m\to 0}\bigl\{(1.1)\text{–}(1.3)\bigr\} = \begin{cases} {}^L D_t^\gamma [w(t) - \mathscr{D}(t, w(t))] \\ \quad = \mathscr{A} w(t) + \mathscr{L}(t, w(t)) \\ \qquad + \int_0^t q(t-s)\mathscr{P}(t, w(s))\, ds, \\ \qquad t \in (0, T], \\ I_{0+}^{1-\gamma} [w(t) - \mathscr{D}(t, w(t))]|_{t=0} \\ \quad = w_0 \in \mathbb{H}. \end{cases} \tag{1.4}$$

As a result, it implies that there exists a hidden condition

$$\begin{aligned} &\lim_{I_1\to 0,\ldots,I_m\to 0}\bigl\{\text{the solution of impulsive system (1.1)–(1.3)}\bigr\} \\ &\quad = \bigl\{\text{the solution of system (1.4)}\bigr\}. \end{aligned} \tag{1.5}$$

Consequently, the definition of solution for impulsive framework (1.1)–(1.3) is given below.

Definition 1.1 Let $w(t) : [0, T] \to \mathbb{H}$ be the solution of the fractional structure (1.1)–(1.3) if $I_{0+}^{1-\gamma} [w(t) - \mathscr{D}(t, w(t))]|_{t=0} = w_0$, the problem ${}^L D_t^\gamma [w(t) - \mathscr{D}(t, w(t))] = \mathscr{A} w(t) + \mathscr{L}(t, w(t)) + \int_0^t q(t-s)\mathscr{P}(t, w(s))\, ds$ for each $t \in (0, T]$ is proved, the impulsive conditions $\Delta I_{0^+}^{1-\gamma} w|_{t=t_k} = I_k(w(t_k^-))$ (here $k = 1, \ldots, m$) are fulfilled, the confinement of $w(t)$ to the interval $(t_k, t_{k+1}]$ (here $k = 1, \ldots, m$) is continuous, and therefore condition (1.5) holds.

The rest of this paper is composed as follows. In Sect. 2, we present some preliminaries which will be used to prove our necessary and sufficient conditions. In Sect. 3, the existence of solutions for problem (1.1)–(1.3) is analyzed under appropriate fixed point techniques. In Sect. 4, as a final point, examples are given to illustrate our results.

2 Preliminaries

In this preliminary, we tend to recall some basic definitions, lemmas, and theorem which will be used throughout this paper. The norm of a Banach space $\mathbb{H}$ is denoted by $\|\cdot\|_{\mathbb{H}}$. Let $C(\mathscr{I}, \mathbb{H})$ represent the Banach space of all $\mathbb{H}$-valued continuous functions from $\mathscr{I}$ to $\mathbb{H}$ with the norm $\|w\|_C = \sup_{t\in\mathscr{I}} \|w(t)\|_{\mathbb{H}}$. So as to outline the mild solution of problem

(1.1)–(1.3), we tend to additionally take the Banach space $C_{1-\gamma}(\mathscr{I},\mathbb{H}) = \{w \in C(\mathscr{I},\mathbb{H}) : t^{1-\gamma}w(t) \in C(\mathscr{I},\mathbb{H})\}$ with the norm

$$\|w\|_{C_{1-\gamma}} = \sup\{t^{1-\gamma}\|w(t)\|_{\mathbb{H}}, t \in \mathscr{I}\}.$$

Obviously, the space $C_{1-\gamma}$ is a Banach space.

In order to outline the mild solutions of system (1.1)–(1.3), we also consider the Banach space $PC_{1-\gamma}(\mathscr{I},\mathbb{H}) = \{w : (t-t_k)^{1-\gamma}w(t) \in PC_{1-\gamma}(\mathscr{I},\mathbb{H})$ is continuous from left and has right limits at $t \in \{t_1, t_2, \dots, t_m\}\}$

$$\|w\|_{PC_{1-\gamma}} = \max\Big\{\sup_{t\in(t_k,t_{k+1}]}(t-t_k)^{1-\gamma}\|w(t)\|_{\mathbb{H}} : k = 0,1,\dots,m\Big\}.$$

Definition 2.1 ([30]) Let $\mathscr{A}$ be the infinitesimal generator of an analytic semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ of uniformly bounded linear operators on $\mathbb{H}$. If $0 \in \rho(-\mathscr{A})$, where $\rho(-\mathscr{A})$ is the resolvent set of $\mathscr{A}$, then for $0 < \eta \leq 1$, it is possible to define the fractional power $\mathscr{A}^{\eta}$ as a closed linear operator on its domain $D(\mathscr{A}^{\eta})$. For an analytic semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, the following properties will be used.

1. There is $\mathcal{M} \geq 1$ such that

$$\mathcal{M} := \sup_{t\in[0,+\infty)}|\mathcal{T}(t)| < \infty.$$

2. For any $\eta \in (0,1]$, there exists $\mathcal{M}_{\eta} > 0$ ensuring that

$$\|\mathscr{A}^{\eta}\mathcal{T}(t)\| \leq \frac{\mathcal{M}_{\eta}}{t^{\eta}}, \quad 0 < t \leq T.$$

For additional details regarding the semigroup theory and fractional powers of operators, we advise the reader to refer to [21].

Currently, we offer a few basic definitions and results of the fractional calculus theory that happen to be used in addition as a chunk of this manuscript.

Definition 2.2 ([11]) The fractional integral of order γ with the lower limit 0 for a function f is determined as

$$I_t^{\gamma}f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{f(s)}{(t-s)^{1-\gamma}}\,ds, \quad t > 0, \gamma > 0,$$

given the right part is point-wise described on $[0,+\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.3 ([11]) The RL derivative of order γ with the lower limit 0 for a function $f \in L^1(\mathscr{I},\mathbb{H})$ is characterized as

$${}^{L}D_t^{\gamma}f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{1-n+\gamma}}\,ds, \quad t > 0, n-1 < \gamma < n.$$

Consider the initial value problem

$$\begin{cases} D_{a^+}^{\gamma}w(t) = \mathscr{A}w(t) + f(t,w(t)), & \gamma \in \mathbb{C} \text{ and } \mathcal{R}(\gamma) \in (0,1), t \in (a,T], \\ I_{a^+}^{1-\gamma}w(a) = w_a, & w_a \in \mathbb{C} \end{cases}$$

is equivalent to the following nonlinear Volterra integral equation of the second kind:

$$w(t) = \frac{w_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}\big[\mathscr{A}w(s) + f\big(s,w(s)\big)\big]\,ds.$$

The piecewise function for (1.1)–(1.3) is given by

$$\begin{aligned}\overline{w}(t) &= \frac{1}{\Gamma(\gamma)} I_{a^+}^{1-\gamma} w\big(t_k^+\big)(t-t_k)^{\gamma-1} + \mathscr{D}\big(t,w(t)\big)\\ &\quad + \frac{1}{\Gamma(\gamma)}\int_{t_k}^t (t-s)^{\gamma-1}\big[\mathscr{A}w(s) + \Lambda_1(s)\big]\,ds,\end{aligned}$$

where $\Lambda_1(s) = \mathscr{L}(s,w(s)) + \int_0^s q(s-\tau)\mathscr{P}(\tau,w(\tau))\,d\tau$, $t \in (t_k,t_{k+1}]$, $k = 0,1,\dots,m$, with

$$I_{a^+}^{1-\gamma} w\big(t_k^+\big) = I_{a^+}^{1-\gamma} w\big(t_k^-\big) + \Delta_k\big(w\big(t_k^-\big)\big).$$

By Definition 2.3, we have

$$\begin{aligned}&D_{a^+}^{\gamma}\big[\overline{w}(t) - \mathscr{D}\big(t,w(t)\big)\big]\\
&\quad = D_{a^+}^{\gamma}\left(\frac{1}{\Gamma(\gamma)} I_{a^+}^{1-\gamma} w\big(t_k^+\big)(t-t_k)^{\gamma-1}\right)\\
&\qquad + D_{a^+}^{\gamma}\left(\frac{1}{\Gamma(\gamma)}\int_{t_k}^t (t-s)^{\gamma-1}\big[\mathscr{A}w(s) + \Lambda_1(s)\big]\,ds\right)\\
&\quad = \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)}\frac{d}{dt}\int_{t_k}^t (t-\eta)^{1-\gamma-1} I_{a^+}^{1-\gamma} w\big(t_k^+\big)(\eta-t_k)^{\gamma-1}\,d\eta + \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)}\\
&\qquad (\times)\frac{d}{dt}\left(\int_a^t (t-\eta)^{1-\gamma-1}\int_{t_k}^{\eta} (\eta-s)^{\gamma-1}\big[\mathscr{A}w(s) + \Lambda_1(s)\big]\,ds\,d\eta\right)\\
&\quad = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_{t_k}^t (t-\eta)^{1-\gamma-1}\left(\frac{1}{\Gamma(\gamma)}\int_{t_k}^{\eta} (\eta-s)^{\gamma-1}\big[\mathscr{A}w(s) + \Lambda_1(s)\big]\,ds\right)d\eta\\
&\quad = D_{a^+}^{\gamma}\left(\frac{1}{\Gamma(\gamma)}\int_{t_k}^{\eta} (\eta-s)^{\gamma-1}\big[\mathscr{A}w(s) + \Lambda_1(s)\big]\,ds\right)\\
&\quad = D_{a^+}^{\gamma}\left(I_{a^+}^{\gamma}\left[\mathscr{A}w(t) + \mathscr{L}\big(t,w(t)\big) + \int_0^t q(t-s)\mathscr{P}\big(s,w(s)\big)\,ds\right]\right)\\
&\quad = \mathscr{A}w(t) + \mathscr{L}\big(t,w(t)\big) + \int_0^t q(t-s)\mathscr{P}\big(s,w(s)\big)\,ds,\end{aligned}$$

where $a = 0$ and $t \in (t_k,t_{k+1}]$.

So, $\overline{w}(t)$ fulfills the condition of fractional differential framework (1.1)–(1.3), and it does not fulfill condition (1.5). In this way, we accept that $\overline{w}(t)$ is an approximate solution for the exact solution of impulsive framework (1.1)–(1.3).

Theorem 2.1 *Suppose that ξ is a constant and $a = 0$. $w(t)$ is a general solution of model (1.1)–(1.3) if and only if $w(t)$ fulfills the fractional integral equation*

$$w(t) = \begin{cases} \frac{w_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1} \\ \quad (\times)[\mathscr{A}w(s) + \Lambda_1(s)]\,ds, \quad (a,t_1], \\ \frac{w_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}[\mathscr{A}w(s) + \Lambda_1(s)]\,ds \\ \quad + \sum_{i=1}^k \frac{\Delta_i(w(t_i^-))}{\Gamma(\gamma)}(t-t_i)^{\gamma-1} - \sum_{i=1}^k \frac{\xi\Delta_i(w(t_i^-))}{\Gamma(\gamma)} \\ \quad (\times)\{w_a(t-a)^{\gamma-1} + \int_a^t (t-s)^{\gamma-1}[\mathscr{A}w(s) + \Lambda_1(s)]\,ds \\ \quad - (w_a + \int_a^{t_i}[\mathscr{A}w(s) + \Lambda_1(s)]\,ds)(t-t_i)^{\gamma-1} - \int_{t_i}^t (t-s)^{\gamma-1} \\ \quad (\times)[\mathscr{A}w(s) + \Lambda_1(s)]\,ds\}, \end{cases} \tag{2.1}$$

where $a = 0$ and $t \in (t_k, t_{k+1}]$, given that the integral in (2.1) exists.

Proof "Necessity": First we are able to simply verify that equation (2.1) fulfills the shrouded condition (1.5). Next, taking the RL fractional derivative to equation (2.1) for every $t \in (t_k, t_{k+1}]$, $k = 0, 1, \ldots, m$, we get

$$\begin{aligned} &D_{a^+}^{\gamma}\bigl[w(t) - \mathscr{D}\bigl(t, w(t)\bigr)\bigr] \\ &= D_{a^+}^{\gamma}\left(\frac{w_a}{\Gamma(\gamma)}(t-a)^{\gamma-1}\right) + D_{a^+}^{\gamma}\left(\frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}\bigl[\mathscr{A}w(s) + \Lambda_1(s)\bigr]\,ds\right) \\ &\quad + D_{a^+}^{\gamma}\left(\sum_{i=1}^k \frac{\Delta_i(w(t_i^-))}{\Gamma(\gamma)}(t-t_i)^{\gamma-1}\right) - D_{a^+}^{\gamma}\left(\sum_{i=1}^k \frac{\xi\Delta_i(w(t_i^-))}{\Gamma(\gamma)}\right. \\ &\quad (\times)\left\{w_a(t-a)^{\gamma-1} + \int_a^t (t-s)^{\gamma-1}\bigl[\mathscr{A}w(s) + \Lambda_1(s)\bigr]\,ds - \left(w_a + \int_a^{t_i}\bigl[\mathscr{A}w(s)\right.\right. \\ &\quad \left.\left.\left. + \Lambda_1(s)\bigr]\,ds\right)(t-t_i)^{\gamma-1} - \int_{t_i}^t (t-s)^{\gamma-1}\bigl[\mathscr{A}w(s) + \Lambda_1(s)\bigr]\,ds\right\}\right) \\ &= \bigl(\mathscr{A}w(t) + \Lambda_1(t)\bigr)_{t\geq a} - \xi\sum_{i=1}^k \Delta_i\bigl(w(t_i^-)\bigr)\bigl[\bigl(\mathscr{A}w(t) + \Lambda_1(t)\bigr)_{t\geq a} - \bigl(\mathscr{A}w(t) \\ &\quad + \Lambda_1(t)\bigr)_{t\geq t_i}\bigr]_{t\in(t_k,t_{k+1}]} \\ &= \left(\mathscr{A}w(t) + \mathscr{L}\bigl(t, w(t)\bigr) + \int_0^t q(t-s)\mathscr{P}\bigl(s, w(s)\bigr)\,ds\right)_{t\in(t_k,t_{k+1}]}. \end{aligned}$$

Therefore, equation (2.1) fulfills the RL fractional derivative of model (1.1)–(1.3). Using (2.1) for every t_k, $k = 1, 2, \ldots, m$, we have

$$\begin{aligned} &I_{a^+}^{1-\gamma}w\bigl(t_k^+\bigr) - I_{a^+}^{1-\gamma}w\bigl(t_k^-\bigr) \\ &= \left\{\frac{1}{\Gamma(1-\gamma)}\int_a^t (t-\eta)^{1-\gamma-1}w(\eta)\,d\eta\right\}_{t\to t_k^+} \\ &\quad - \left\{\frac{1}{\Gamma(1-\gamma)}\int_a^t (t-\eta)^{1-\gamma-1}w(\eta)\,d\eta\right\}_{t=t_k} \\ &= \Delta_k\bigl(w\bigl(t_k^-\bigr)\bigr) - \xi\Delta_k\bigl(w\bigl(t_k^-\bigr)\bigr)\left\{w_a + \int_a^t \bigl(\mathscr{A}w(s) + \Lambda_1(s)\bigr)\,ds\right. \end{aligned}$$

$$
\begin{aligned}
&- \left(w_a + \int_a^{t_k} \left[\mathscr{A} w(s) + \Lambda_1(s) \right] ds \right) - \int_{t_k}^{t} \left(\mathscr{A} w(s) + \Lambda_1(s) \right) ds \Bigg\}_{t \to t_k} \\
&= \Delta_k \left(w\left(t_k^-\right) \right).
\end{aligned}
$$

Therefore, equation (2.1) fulfills the impulsive conditions of model (1.1)–(1.3). Then equation (2.1) satisfies the conditions of system (1.1)–(1.3) with $a = 0$.

"Sufficiency": We demonstrate that the solutions of framework (1.1)–(1.3) fulfill condition (2.1) by scientific induction. By Definition 2.2, the solution of model (1.1)–(1.3) fulfills

$$
w(t) = \frac{w_a}{\Gamma(\gamma)} (t-a)^{\gamma-1} + \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} \left[\mathscr{A} w(s) + \Lambda_1(s) \right] ds, (a, t_1]. \tag{2.2}
$$

By (2.2), we have

$$
\begin{aligned}
I_{a^+}^{1-\gamma} w\left(t_1^+\right) &= I_{a^+}^{1-\gamma} w\left(t_1^-\right) + \Delta_1\left(w\left(t_1^-\right) \right) \\
&= w_a + \Delta_1\left(w\left(t_1^-\right) \right) + \int_a^{t_1} \left[\mathscr{A} w(s) + \Lambda_1(s) \right] ds,
\end{aligned}
$$

and the approximate solution $\overline{w}(t)$, $t \in (t_1, t_2]$ is defined by

$$
\begin{aligned}
\overline{w}(t) &= \frac{1}{\Gamma(\gamma)} I_{a^+}^{1-\gamma} w\left(t_1^+\right)(t-t_1)^{\gamma-1} + \mathscr{D}\left(t, w(t)\right) \\
&\quad + \frac{1}{\Gamma(\gamma)} \int_{t_1}^t (t-s)^{\gamma-1} \left[\mathscr{A} w(s) \Lambda_1(s) \right] ds \\
&= \frac{1}{\Gamma(\gamma)} \left(w_a + \int_a^{t_1} \left[\mathscr{A} w(s) + \Lambda_1(s) \right] ds + \Delta_1\left(w\left(t_1^-\right) \right) \right) (t-t_1)^{\gamma-1} \\
&\quad + \mathscr{D}\left(t, w(t)\right) + \frac{1}{\Gamma(\gamma)} \int_{t_1}^t (t-s)^{\gamma-1} \left[\mathscr{A} w(s) + \Lambda_1(s) \right] ds, \quad t \in (t_1, t_2],
\end{aligned}
$$

with $e_1(t) = w(t) - \overline{w}(t)$, $t \in (t_1, t_2]$. By

$$
\begin{aligned}
\lim_{\Delta_1(w(t_1^-)) \to 0} w(t) &= \lim_{\Delta_1(w(t_1^-)) \to 0} \left\{ \frac{w_a}{\Gamma(\gamma)} (t-a)^{\gamma-1} + \mathscr{D}\left(t, w(t)\right) \right. \\
&\quad \left. + \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} \left[\mathscr{A} w(s) + \Lambda_1(s) \right] ds + \Delta_1\left(w\left(t_1^-\right) \right) \right\} \\
&= \frac{w_a}{\Gamma(\gamma)} (t-a)^{\gamma-1} + \mathscr{D}\left(t, w(t)\right) + \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} \left[\mathscr{A} w(s) \right. \\
&\quad \left. + \Lambda_1(s) \right] ds, \quad t \in (t_1, t_2].
\end{aligned}
$$

We get

$$
\begin{aligned}
\lim_{\Delta_1(w(t_1^-)) \to 0} e_1(t) &= \lim_{\Delta_1(w(t_1^-)) \to 0} \left\{ w(t) - \overline{w}(t) \right\} \\
&= \frac{w_a}{\Gamma(\gamma)} (t-a)^{\gamma-1} + \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} \left[\mathscr{A} w(s) + \Lambda_1(s) \right] ds
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\gamma)}(t-t_1)^{\gamma-1}\left(w_a + \int_a^{t_1}\left[\mathscr{A}w(s) + \Lambda_1(s)\right]ds\right) \\
&- \frac{1}{\Gamma(\gamma)}\int_{t_1}^{t}(t-s)^{\gamma-1}\left[\mathscr{A}w(s) + \Lambda_1(s)\right]ds.
\end{aligned}
$$

Then we assume

$$
\begin{aligned}
e_1(t) &= \sigma\left(\Delta_1\left(w\left(t_1^-\right)\right)\right)\lim_{\Delta_1(w(t_1^-))\to 0} e_1(t) \\
&= \frac{\sigma(\Delta_1(w(t_1^-)))}{\Gamma(\gamma)}\left\{w_a(t-a)^{\gamma-1} + \int_a^t (t-s)^{\gamma-1}\left[\mathscr{A}w(s) + \Lambda_1(s)\right]ds\right. \\
&\quad - \left(w_a + \int_a^{t_1}\left[\mathscr{A}w(s) + \Lambda_1(s)\right]ds\right)(t-t_1)^{\gamma-1} - \int_{t_1}^t (t-s)^{\gamma-1}\left[\mathscr{A}w(s)\right. \\
&\quad \left.\left. + \Lambda_1(s)\right]ds\right\},
\end{aligned}
$$

where the function $\sigma(\cdot)$ is an undetermined function with $\sigma(0) = 1$. Thus,

$$
\begin{aligned}
w(t) &= \overline{w}(t) + e_1(t) \\
&= \frac{1}{\Gamma(\gamma)}\left\{\sigma\left(\Delta_1\left(w\left(t_1^-\right)\right)\right)\left(w_a(t-a)^{\gamma-1} + \int_a^t (t-s)^{\gamma-1}\left[\mathscr{A}w(s)\right.\right.\right. \\
&\quad \left.\left. + \Lambda_1(s)\right]ds\right) + \Delta_1\left(w\left(t_1^-\right)\right)(t-t_1)^{\gamma-1} + \left[1 - \sigma\left(\Delta_1\left(w\left(t_1^-\right)\right)\right)\right]\left(w_a\right. \\
&\quad \left. + \int_a^{t_1}\left[\mathscr{A}w(s) + \Lambda_1(s)\right]ds\right)(t-t_1)^{\gamma-1} + \left[1 - \sigma\left(\Delta_1\left(w\left(t_1^-\right)\right)\right)\right] \\
&\quad \left.(\times)\int_{t_1}^t (t-s)^{\gamma-1}\left[\mathscr{A}w(s) + \Lambda_1(s)\right]ds\right\} + \mathscr{D}\left(t, w(t)\right), \quad t \in (t_1, t_2].
\end{aligned}
\tag{2.3}
$$

Using (2.3), we get

$$
\begin{aligned}
I_{a^+}^{1-\gamma}w\left(t_2^+\right) &= I_{a^+}^{1-\gamma}w\left(t_2^-\right) + \Delta_2\left(w\left(t_2^-\right)\right) \\
&= w_a + \Delta_1\left(w\left(t_1^-\right)\right) + \Delta_2\left(w\left(t_2^-\right)\right) + \int_a^{t_2}\left[\mathscr{A}w(s) + \Lambda_1(s)\right]ds,
\end{aligned}
$$

and the approximate solution $\overline{w}(t)$, $t \in (t_2, t_3]$ is given by

$$
\begin{aligned}
\overline{w}(t) &= \frac{1}{\Gamma(\gamma)}I_{a^+}^{1-\gamma}w\left(t_2^+\right)(t-t_2)^{\gamma-1} + \mathscr{D}\left(t, w(t)\right) \\
&\quad + \frac{1}{\Gamma(\gamma)}\int_{t_2}^t (t-s)^{\gamma-1}\left[\mathscr{A}w(s) + \Lambda_1(s)\right]ds \\
&= \frac{1}{\Gamma(\gamma)}\left(w_a + \int_a^{t_2}\left[\mathscr{A}w(s) + \Lambda_1(s)\right]ds + \Delta_1\left(w\left(t_1^-\right)\right) + \Delta_2\left(w\left(t_2^-\right)\right)\right) \\
&\quad (\times)(t-t_2)^{\gamma-1} + \mathscr{D}\left(t, w(t)\right) + \frac{1}{\Gamma(\gamma)}\int_{t_2}^t (t-s)^{\gamma-1}\left[\mathscr{A}w(s) + \Lambda_1(s)\right]ds
\end{aligned}
$$

for $t \in (t_2, t_3]$. Similarly, we get

$$\begin{aligned}
w(t) &= \overline{w}(t) + e_2(t) \\
&= \frac{1}{\Gamma(\gamma)}\Big\{\big[\sigma\big(\Delta_1\big(w(t_1^-)\big)\big) + \sigma\big(\Delta_2\big(w(t_2^-)\big)\big) - 1\big]\Big(w_a(t-a)^{\gamma-1} \\
&\quad + \int_a^t (t-s)^{\gamma-1}\big[\mathscr{A}w(s) + \Lambda_1(s)\big]\,ds\Big) + \Delta_1\big(w(t_1^-)\big)(t-t_1)^{\gamma-1} \\
&\quad + \Delta_2\big(w(t_2^-)\big)(t-t_2)^{\gamma-1} \\
&\quad + \big[1 - \sigma\big(\Delta_1\big(w(t_1^-)\big)\big)\big]\Big(w_a + \int_a^{t_1}\big[\mathscr{A}w(s) + \Lambda_1(s)\big]\,ds\Big)(t-t_1)^{\gamma-1} \\
&\quad + \big[1 - \sigma\big(\Delta_1\big(w(t_1^-)\big)\big)\big]\int_{t_1}^t (t-s)^{\gamma-1}\big[\mathscr{A}w(s) + \Lambda_1(s)\big]\,ds \\
&\quad + \big[1 - \sigma\big(\Delta_2\big(w(t_2^-)\big)\big)\big]\Big(w_a + \int_a^{t_2}\big[\mathscr{A}w(s) + \Lambda_1(s)\big]\,ds\Big)(t-t_2)^{\gamma-1} \\
&\quad + \big[1 - \sigma\big(\Delta_2\big(w(t_2^-)\big)\big)\big]\int_{t_2}^t (t-s)^{\gamma-1}\big[\mathscr{A}w(s) + \Lambda_1(s)\big]\,ds\Big\} \\
&\quad + \mathscr{D}\big(t, w(t)\big), \quad t \in (t_2, t_3].
\end{aligned} \tag{2.4}$$

Moreover, letting $t_2 \to t_1$, we have

$$\begin{aligned}
&\lim_{t_2 \to t_1}\begin{cases}
D_{a^+}^{\gamma}[w(t) - \mathscr{D}(t, w(t))] \\
\quad = \mathscr{A}w(t) + \mathscr{L}(t, w(t)) \\
\qquad + \int_0^t q(t-s)\mathscr{P}(t, w(s))\,ds, \quad \gamma \in (0,1), t \in (a, t_3], t \neq t_1 \text{ and } t \neq t_2, \\
\Delta I_{a^+}^{1-\gamma} w|_{t=t_k} = I_{a^+}^{1-\gamma} w(t_k^+) - I_{a^+}^{1-\gamma} w(t_k^-) = \Delta_k(w(t_k^-)), \quad k = 1, 2, \\
I_{a+}^{1-\gamma}[w(t) - \mathscr{D}(t, w(t))]|_{t=a} = w_a \in \mathbb{H}
\end{cases} \\
&= \begin{cases}
D_{a^+}^{\gamma}[w(t) - \mathscr{D}(t, w(t))] \\
\quad = \mathscr{A}w(t) + \mathscr{L}(t, w(t)) \\
\qquad + \int_0^t q(t-s)\mathscr{P}(t, w(s))\,ds, \quad \gamma \in (0,1), t \in (a, t_3] \text{ and } t \neq t_1, \\
\Delta I_{a^+}^{1-\gamma} w|_{t=t_1} = I_{a^+}^{1-\gamma} w(t_1^+) - I_{a^+}^{1-\gamma} w(t_1^-) + I_{a^+}^{1-\gamma} w(t_2^+) - I_{a^+}^{1-\gamma} w(t_2^-) \\
\quad = \Delta_1(w(t_1^-)) + \Delta_2(w(t_1^-)), \\
I_{a+}^{1-\gamma}[w(t) - \mathscr{D}(t, w(t))]|_{t=0} = w_0 \in \mathbb{H}.
\end{cases}
\end{aligned}$$

Using (2.3) and (2.4), we have $1 - \sigma(\Delta_1 + \Delta_2) = 1 - \sigma(\Delta_1) + 1 - \sigma(\Delta_2)$. Taking $\rho(z) = 1 - \sigma(z)$, we have $\rho(z + \omega) = \rho(z) + \rho(\omega)$ for $\forall z, \omega \in \mathbb{C}$. So, $\rho(z) = \xi z$. Thus,

$$\begin{aligned}
w(t) &= \frac{w_a}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}\big[\mathscr{A}w(s) + \Lambda_1(s)\big]\,ds \\
&\quad + \frac{\Delta_1(w(t_1^-))}{\Gamma(\gamma)}(t-t_1)^{\gamma-1} - \frac{\xi\,\Delta_1(w(t_1^-))}{\Gamma(\gamma)}\Big\{w_a(t-a)^{\gamma-1} \\
&\quad + \int_a^t (t-s)^{\gamma-1}\big[\mathscr{A}w(s) + \Lambda_1(s)\big]\,ds
\end{aligned}$$

$$\begin{aligned}
&-\left(w_a+\int_a^{t_1}\bigl[\mathscr{A}w(s)+\Lambda_1(s)\bigr]\,ds\right)(t-t_1)^{\gamma-1}\\
&-\int_{t_1}^{t}(t-s)^{\gamma-1}\bigl[\mathscr{A}w(s)+\Lambda_1(s)\bigr]\,ds\Bigr\}+\mathscr{D}\bigl(t,w(t)\bigr),\quad t\in(t_1,t_2].
\end{aligned}\tag{2.5}$$

Continuing in this way, we obtain, for $t \in (t_k, t_{k+1}]$,

$$\begin{aligned}
w(t)={}&\frac{w_a}{\Gamma(\gamma)}(t-a)^{\gamma-1}+\frac{1}{\Gamma(\gamma)}\int_a^t(t-s)^{\gamma-1}\bigl[\mathscr{A}w(s)+\Lambda_1(s)\bigr]\,ds\\
&+\sum_{i=1}^{k}\frac{\Delta_i(w(t_i^-))}{\Gamma(\gamma)}(t-t_i)^{\gamma-1}-\sum_{i=1}^{k}\frac{\xi\,\Delta_i(w(t_i^-))}{\Gamma(\gamma)}\Bigl\{w_a(t-a)^{\gamma-1}\\
&+\int_a^t(t-s)^{\gamma-1}\bigl[\mathscr{A}w(s)+\Lambda_1(s)\bigr]\,ds-\biggl(w_a+\int_a^{t_i}\bigl[\mathscr{A}w(s)\\
&+\Lambda_1(s)\bigr]\,ds\biggr)(t-t_i)^{\gamma-1}-\int_{t_i}^{t}(t-s)^{\gamma-1}\bigl[\mathscr{A}w(s)+\Lambda_1(s)\bigr]\,ds\Bigr\}\\
&+\mathscr{D}\bigl(t,w(t)\bigr),\quad t\in(t_k,t_{k+1}].
\end{aligned}\tag{2.6}$$

So, the solution of system (1.1)–(1.3) satisfies equation (2.1) with $a = 0$. Therefore, the impulsive system (1.1)–(1.3) is equivalent to the integral equation (2.1) with $a = 0$. □

Lemma 2.1 *If $a = 0$ and $\xi = 0$ in (2.6) are given by*

$$w(t)=\begin{cases}
\frac{w_0}{\Gamma(\gamma)}t^{\gamma-1}+\mathscr{D}(t,w(t))\\
\quad+\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}[\mathscr{A}w(s)+\Lambda_1(s)]\,ds, & t\in(0,t_1],\\
\frac{w_0}{\Gamma(\gamma)}t^{\gamma-1}+\frac{1}{\Gamma(\gamma)}\int_0^t(t-s)^{\gamma-1}[\mathscr{A}w(s)+\Lambda_1(s)]\,ds\\
\quad+\sum_{i=1}^{k}\frac{\Delta_i(w(t_i^-))}{\Gamma(\gamma)}(t-t_i)^{\gamma-1}, & t\in(t_k,t_{k+1}],
\end{cases}\tag{2.7}$$

where $\Lambda_1(s)=\mathscr{L}(s,w(s))+\int_0^s q(s-\tau)\mathscr{P}(\tau,w(\tau))\,d\tau$. Then, for $t > 0$, we have

$$w(t)=\begin{cases}
t^{\gamma-1}\mathbb{T}_\gamma(t)w_0+\mathscr{D}(t,w(t))+\int_0^t(t-s)^{\gamma-1}\mathscr{A}\mathbb{T}_\gamma(t-s)\\
\quad(\times)\mathscr{D}(s,w(s))\,ds+\int_0^t(t-s)^{\gamma-1}\mathbb{T}_\gamma(t-s)[\mathscr{L}(s,w(s))\\
\quad+\int_0^s q(s-\tau)\mathscr{P}(\tau,w(\tau))\,d\tau]\,ds, & t\in(0,t_1],\\
t^{\gamma-1}\mathbb{T}_\gamma(t)w_0+\mathscr{D}(t,w(t))+\int_0^t(t-s)^{\gamma-1}\mathscr{A}\mathbb{T}_\gamma(t-s)\\
\quad(\times)\mathscr{D}(s,w(s))\,ds+\int_0^t(t-s)^{\gamma-1}\mathbb{T}_\gamma(t-s)[\mathscr{L}(s,w(s))\\
\quad+\int_0^s q(s-\tau)\mathscr{P}(\tau,w(\tau))\,d\tau]\,ds\\
\quad+\sum_{i=1}^{k}\mathbb{T}_\gamma(t-t_i)(t-t_i)^{\gamma-1}I_i(w(t_i^-)), & t\in(t_k,t_{k+1}],
\end{cases}$$

where

$$\begin{aligned}
&\mathbb{T}_\gamma(t)=\gamma\int_0^\infty\theta\xi_\gamma(\theta)\mathbb{T}\bigl(t^\gamma\theta\bigr)\,d\theta,\\
&\xi_\gamma(\theta)=\frac{1}{\gamma}\theta^{-1-\frac{1}{\gamma}}\phi_\gamma\bigl(\theta^{\frac{-1}{\gamma}}\bigr),
\end{aligned}$$

$\phi_\gamma(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\gamma n-1}\frac{\Gamma(n\gamma+1)}{n!}\sin(n\pi\gamma), \theta \in (0,\infty)$. *Here* ξ_γ *is the probability density function in* $(0,\infty)$, *that is*, $\xi_\gamma(\theta) \geq 0$ *and* $\int_0^\infty \xi_\gamma(\theta)\,d\theta = 1$.

Proof The proof is very similar to the proof of [30, 32, Lemma 3.1, Lemma 3.3], hence here we omit it. □

Lemma 2.2 ([30, Lemma 3.2]) *If* $t \geq 0$, $\mathbb{T}_\gamma(t)$ *is a linear and bounded operator. That is, for any* $w \in \mathbb{H}$,

$$\left\|\mathbb{T}_\gamma(t)w\right\| \leq \frac{\mathcal{M}}{\Gamma(\gamma)}\|w\|.$$

Lemma 2.3 ([30, Lemma 3.5]) *For any* $\beta \in (0,1)$, $\eta \in (0,1]$, *and for all* $w \in D(\mathscr{A})$, *there exists a positive constant* $\mathcal{M}_\eta$ *in a way that*

$$\mathscr{A}\mathbb{T}_\gamma(t)w = \mathscr{A}^{1-\beta}\mathbb{T}_\gamma(t)\mathscr{A}^\beta w, \quad 0 \leq t \leq T,$$

and

$$\left\|\mathscr{A}^\eta\mathbb{T}_\gamma(t)\right\| \leq \frac{\gamma\mathcal{M}_\eta\Gamma(2-\eta)}{t^{\gamma\eta}\Gamma(1+\gamma(1-\eta))}, \quad 0 < t \leq T.$$

Lemma 2.4 ([27, Lemma 2.7]) *Assume* $\xi, \eta \in \mathbb{R}$, $\eta > -1$ *and* $n \in \mathbb{N}^+$, *and then when* $t > 0$, *we have*

$$\left(I_{0+}^{\xi}\frac{s^\eta}{\Gamma(\eta+1)}\right)(t) = \begin{cases} \frac{t^{\xi+\eta}}{\Gamma(\xi+\eta+1)}, & \xi+\eta \neq -n, \\ 0, & \xi+\eta = -n. \end{cases}$$

Next, we have a tendency to recall some properties of the measure of noncompactness which will be employed in the proof of our main results. We tend to denote by $\omega(\cdot)$ the Kuratowski measure of noncompactness on both the finite sets of $\mathbb{H}$ and $C(\mathscr{I},\mathbb{H})$. For more points of interest of the measure of noncompactness, see [3, 6]. For any $D \subset C(\mathscr{I},\mathbb{H})$ and $t \in \mathscr{I}$, let $D(t) = \{u(t)|u \in D\} \subset \mathbb{H}$. If $D \subset C(\mathscr{I},\mathbb{H})$ is bounded, then $D(t)$ is bounded in $\mathbb{H}$ and $\omega(D(t)) \leq \omega(D)$.

3 Existence results

In this area, we display and demonstrate the existence results for problem (1.1)–(1.3). In view of Lemma 2.1, first, we define the mild solution for model (1.1)–(1.3) with the help of a probability density function and the Laplace transform.

Definition 3.1 ([27, Definition 3.1]) A function $w \in PC_{1-\gamma}(\mathscr{I},\mathbb{H})$ is said to be a mild solution of model (1.1)–(1.3) if the following hold:

(i) $I_{0^+}^{1-\gamma}[w(t) - \mathscr{D}(t,w(t))]|_{t=0} = w_0 \in \mathbb{H}$;

(ii)

$$w(t)=\begin{cases} t^{\gamma-1}\mathbb{T}_\gamma(t)w_0+\int_0^t(t-s)^{\gamma-1}\mathscr{A}\mathbb{T}_\gamma(t-s)\mathscr{D}(s,w(s))\,ds \\ \quad +\mathscr{D}(t,w(t))+\int_0^t(t-s)^{\gamma-1}\mathbb{T}_\gamma(t-s)[\mathscr{L}(s,w(s)) \\ \quad +\int_0^s q(s-\tau)\mathscr{P}(\tau,w(\tau))\,d\tau]\,ds, & t\in(0,t_1], \\ t^{\gamma-1}\mathbb{T}_\gamma(t)w_0+\int_0^t(t-s)^{\gamma-1}\mathscr{A}\mathbb{T}_\gamma(t-s)\mathscr{D}(s,w(s))\,ds \\ \quad +\mathscr{D}(t,w(t))+\int_0^t(t-s)^{\gamma-1}\mathbb{T}_\gamma(t-s)[\mathscr{L}(s,w(s)) \\ \quad +\int_0^s q(s-\tau)\mathscr{P}(\tau,w(\tau))\,d\tau]\,ds \\ \quad +\sum_{i=1}^k\mathbb{T}_\gamma(t-t_i)(t-t_i)^{\gamma-1}I_i(w(t_i^-)), & t\in(t_k,t_{k+1}]. \end{cases}$$

Now, we are in a position to introduce the hypotheses on framework (1.1)–(1.3) as follows.

(H1) $\mathcal{T}(t)$, $t>0$ is a strongly continuous semigroup and continuous in the uniform operator topology.

(H2) $\mathscr{L}:\mathscr{I}\times\mathbb{H}\to\mathbb{H}$ is continuous, and we can discover constants $N_{\mathscr{L}}>0$ in a way that

$$\big\|\mathscr{L}(t,w)-\mathscr{L}(t,v)\big\|_{\mathbb{H}}\le N_{\mathscr{L}}\|w-v\|_{\mathbb{H}},\quad t\in\mathscr{I},w,v\in\mathbb{H}.$$

(H3) $\mathscr{P}:\mathscr{I}\times\mathbb{H}\to\mathbb{H}$ is continuous, and we can discover constants $N_{\mathscr{P}}>0$ such that

$$\big\|\mathscr{P}(t,w)-\mathscr{P}(t,v)\big\|_{\mathbb{H}}\le N_{\mathscr{P}}\|w-v\|_{\mathbb{H}},\quad t\in\mathscr{I},w,v\in\mathbb{H}.$$

(H4) The function $\mathscr{D}:\mathscr{I}\times\mathbb{H}\to\mathbb{H}$, we can discover constants $\beta\in(0,1)$ and $N_{\mathscr{D}}>0$ in ways that $\mathscr{D}\in D(\mathscr{A}^\beta)$, and for any $w,v\in\mathbb{H}$, $t\in\mathscr{I}$, the function $\mathscr{A}^\beta\mathscr{D}(\cdot,w)$ is strongly measurable and $\mathscr{A}^\beta\mathscr{D}(t,w)$ satisfies

$$\big\|\mathscr{A}^\beta\mathscr{D}(t,w)-\mathscr{A}^\beta\mathscr{D}(t,v)\big\|_{\mathbb{H}}\le N_{\mathscr{D}}\|w-v\|_{\mathbb{H}}.$$

(H5) There exist constants $0<d_k<\Gamma(\gamma)/[\mathcal{M}\sum_{i=1}^k(t_i-t_{i-1})^{\gamma-1}]$, $k=1,\dots,m+1$, ensuring that

$$\big\|I_i(w)-I_i(v)\big\|_{\mathbb{H}}\le d_i\|w-v\|_{\mathbb{H}}\quad\text{and}\quad\big\|I_i(w)\big\|\le N_I,\quad(w,v)\in\mathbb{H}^2.$$

(H6) For any $R>0$ and $a>0$, there exists $L_i(R,a)>0$, $i=1,2,3$, ensuring that for any equicontinuous and countable set $D\subset B_R=\{w\in\mathbb{H}:\|w\|\le R\}$,

$$\omega\big(\mathscr{L}(t,D)\big)\le L_2\omega(D),\qquad\omega\big(\mathscr{D}(t,D)\big)\le L_1\omega(D),$$
$$\omega\big(\mathscr{P}(t,D)\big)\le L_3\omega(D),\quad t\in[0,a].$$

Remark 3.1 Throughout this paper, we define a few notations:

$$\big\|\mathscr{A}^{-\beta}\big\|=\mathcal{M}_0\quad\text{and}\quad K(\gamma,\beta)=\frac{\mathcal{M}_{1-\beta}\Gamma(1+\beta)\tau^{\gamma\beta}}{\beta\Gamma(1+\gamma\beta)}.$$

Theorem 3.1 *Assume that hypotheses* (H1)–(H6) *are satisfied. Then, for every* $w_0 \in PC_{1-\gamma}(\mathscr{I},\mathbb{H})$, *there exists* $\tau_1 = \tau_1(w_0)$, $0 < \tau_1 < T$, *ensuring that model* (1.1)–(1.3) *has a solution* $w \in PC_{1-\gamma}((0,\tau_1],\mathbb{H})$.

Proof Since we are interested here just in local solutions, we may assume that $T < \infty$. By using our conditions (H1)–(H5), $t' > 0$ and $r > 0$ are such that $B_r(w_0) = \{w : (t-t_k)^{1-\gamma}\|w - w_0\| \le r\}$ and $(t-t_k)^{1-\gamma}\|\mathscr{L}(t,w(t))\| \le N_{\mathscr{L}}$, $(t-t_k)^{1-\gamma}\|\mathscr{P}(t,w(t))\| \le N_{\mathscr{P}}$, $(t-t_k)^{1-\gamma}\|\mathscr{A}^{\beta}\mathscr{D}(t,w(t))\| \le N_{\mathscr{D}}$ for $0 \le t \le t'$ and $w \in B_r(w_0)$ and select

$$\begin{aligned}\tau_0 = \min\biggl\{ & t', T, \biggl[\frac{\Gamma(\gamma+1)}{\mathcal{M}(N_{\mathscr{L}} + q^* N_{\mathscr{P}})}\bigl(1 - \bigl(\mathcal{M}_0 + K(\gamma,\beta)L_{\mathscr{D}}\bigr)\biggr]^{\frac{1}{\gamma}}, \\ & \biggl[\frac{\Gamma(\gamma+1)}{\mathcal{M}(N_{\mathscr{L}} + q^* N_{\mathscr{P}})}\bigl(1 - \bigl(\mathcal{M}_0 + K(\gamma,\beta)\bigr)N_{\mathscr{D}} + \overline{u}\bigr)\biggr]^{\frac{1}{\gamma}}\biggr\},\end{aligned} \tag{3.1}$$

where $q^* = \sup_{0\le t\le T}\int_0^t \|q(t-s)\|\,ds$, $\overline{u} = \frac{\mathcal{M}}{\Gamma(\gamma)}[\|w_0\| - \sum_{i=1}^k (t_i - t_{i-1})^{\gamma-1}]$. Set $\Omega = \{w \in PC_{1-\gamma}([0,\tau_1],\mathbb{H}) : t^{1-\gamma}\|w(t)\| \le r, t \in [0,\tau_1]\}$, then Ω is a closed ball in $PC_{1-\gamma}([0,\tau_1],\mathbb{H})$ with center θ and radius r. Consider the operator $\Upsilon : \Omega \to PC_{1-\gamma}([0,\tau_1],\mathbb{H})$ defined by

$$\begin{aligned}(\Upsilon w)(t) = {} & t^{\gamma-1}\mathbb{T}_\gamma(t)w_0 + \mathscr{D}\bigl(t,w(t)\bigr) \\ & + \int_0^t (t-s)^{\gamma-1}\mathscr{A}\mathbb{T}_\gamma(t-s)\mathscr{D}\bigl(s,w(s)\bigr)\,ds \\ & + \int_0^t (t-s)^{\gamma-1}\mathbb{T}_\gamma(t-s)\biggl[\mathscr{L}\bigl(s,w(s)\bigr) + \int_0^s q(s-\tau)\mathscr{P}\bigl(\tau,w(\tau)\bigr)\,d\tau\biggr]ds \\ & + \sum_{0<t_k<t}\mathbb{T}_\gamma(t-t_k)(t-t_k)^{\gamma-1}I_k\bigl(w(t_k^-)\bigr).\end{aligned}$$

For any $w \in \Omega$ and $t \in [0,\tau_1]$, by Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}&(t-t_k)^{1-\gamma}\bigl\|(\Upsilon w)(t)\bigr\| \\ &\quad = (t-t_k)^{1-\gamma}t^{\gamma-1}\bigl\|\mathbb{T}_\gamma(t)w_0\bigr\| + (t-t_k)^{1-\gamma}\bigl\|\mathscr{D}\bigl(t,w(t)\bigr)\bigr\| \\ &\quad\quad + (t-t_k)^{1-\gamma}\int_0^t (t-s)^{\gamma-1}\bigl\|\mathscr{A}\mathbb{T}_\gamma(t-s)\mathscr{D}\bigl(s,w(s)\bigr)\bigr\|\,ds \\ &\quad\quad + (t-t_k)^{1-\gamma}\int_0^t (t-s)^{\gamma-1}\biggl\|\mathbb{T}_\gamma(t-s)\biggl[\mathscr{L}\bigl(s,w(s)\bigr) \\ &\quad\quad + \int_0^s q(s-\tau)\mathscr{P}\bigl(\tau,w(\tau)\bigr)\,d\tau\biggr]\biggr\|\,ds \\ &\quad\quad + (t-t_k)^{1-\gamma}\biggl\|\sum_{0<t_k<t}\mathbb{T}_\gamma(t-t_k)(t-t_k)^{\gamma-1}I_k\bigl(w(t_k^-)\bigr)\biggr\| \\ &\quad = \sum_{j=1}^4 J_j,\end{aligned} \tag{3.2}$$

where

$$
\begin{aligned}
J_1 &= (t-t_k)^{1-\gamma} t^{\gamma-1} \left\| \mathbb{T}_\gamma(t) w_0 \right\| + (t-t_k)^{1-\gamma} \left\| \mathscr{D}\big(t, w(t)\big) \right\| \\
&\leq \left\| \mathbb{T}_\gamma(t) w_0 \right\| + \left\| \mathscr{A}^{-\beta} \right\| (t-t_k)^{1-\gamma} \left\| \mathscr{A}^\beta \mathscr{D}\big(t, w(t)\big) \right\| \\
&\leq \frac{\mathcal{M}\|w_0\|}{\Gamma(\gamma)} + \mathcal{M}_0 N_{\mathscr{D}}, \\
J_2 &= (t-t_k)^{1-\gamma} \int_0^t (t-s)^{\gamma-1} \left\| \mathscr{A} \mathbb{T}_\gamma(t-s) \mathscr{D}\big(s, w(s)\big) \right\| ds \\
&\leq (t-t_k)^{1-\gamma} \int_0^t (t-s)^{\gamma-1} \left\| \mathscr{A}^{1-\beta} \mathbb{T}_\gamma(t-s) \right\| \left\| \mathscr{A}^\beta \mathscr{D}\big(s, w(s)\big) \right\| ds \\
&\leq (t-t_k)^{1-\gamma} \frac{\gamma \mathcal{M}_{1-\beta} \Gamma(1+\beta)}{\Gamma(1+\gamma\beta)} \int_0^t (t-s)^{\gamma\beta-1} \left\| \mathscr{A}^\beta \mathscr{D}\big(s, w(s)\big) \right\| ds \\
&\leq K(\gamma, \beta) N_{\mathscr{D}}, \\
J_3 &= (t-t_k)^{1-\gamma} \int_0^t (t-s)^{\gamma-1} \left\| \mathbb{T}_\gamma(t-s) \left[\mathscr{L}\big(s, w(s)\big) \right.\right. \\
&\quad \left.\left. + \int_0^s q(s-\tau) \mathscr{P}\big(\tau, w(\tau)\big)\, d\tau \right] \right\| ds \\
&\leq (t-t_k)^{1-\gamma} \frac{\mathcal{M}}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \left[\left\| \mathscr{L}\big(s, w(s)\big) \right\| + q^* \left\| \mathscr{P}\big(s, w(s)\big) \right\| \right] ds \\
&\leq \frac{\mathcal{M}}{\Gamma(\gamma)} \frac{t^\gamma}{\gamma} \left[N_{\mathscr{L}} + q^* N_{\mathscr{P}} \right] \\
&\leq \frac{\mathcal{M}}{\Gamma(\gamma+1)} \left[N_{\mathscr{L}} + q^* N_{\mathscr{P}} \right] \tau^\gamma, \\
J_4 &= (t-t_k)^{1-\gamma} \left\| \sum_{0<t_k<t} \mathbb{T}_\gamma(t-t_k)(t-t_k)^{\gamma-1} I_k\big(w(t_k^-)\big) \right\| \\
&\leq (t-t_k)^{1-\gamma} \frac{\mathcal{M}}{\Gamma(\gamma)} \sum_{i=1}^k (t-t_i)^{\gamma-1} \left\| I_i\big(w(t_i^-)\big) \right\| \\
&\leq \frac{\mathcal{M}}{\Gamma(\gamma)} \sum_{i=1}^k (t_i-t_{i-1})^{\gamma-1} (t_i-t_{i-1})^{1-\gamma} \left\| I_i\big(w(t_i^-)\big) \right\| \\
&\leq \frac{\mathcal{M}}{\Gamma(\gamma)} \sum_{i=1}^k (t_i-t_{i-1})^{\gamma-1} N_I.
\end{aligned}
$$

Using $J_1 - J_4$ in equation (3.2), we get

$$
\begin{aligned}
(t-t_k)^{1-\gamma} \left\| (\Upsilon w)(t) \right\| &\leq \frac{\mathcal{M}}{\Gamma(\gamma)} \left[\|w_0\| + \sum_{i=1}^k (t_i-t_{i-1})^{\gamma-1} N_I \right] \\
&\quad + N_{\mathscr{D}}\big(\mathcal{M}_0 + K(\gamma, \beta)\big) + \frac{\mathcal{M}}{\Gamma(\gamma+1)} \left[N_{\mathscr{L}} + q^* N_{\mathscr{P}} \right] \tau^\gamma \\
&\leq R.
\end{aligned}
$$

Therefore, $\Upsilon w \in \Omega$. Now we show that Υ is continuous from Ω into Ω. To show this, we first observe that since $\mathscr{L}$, $\mathscr{D}$, and $\mathscr{P}$ are continuous in $\mathscr{I} \times \mathbb{H}$, for any $\epsilon > 0$ and for fixed $w \in B_R(w_0)$, there exists $\delta_1(w), \delta_2(w) > 0$ ensuring that for any $v \in B_R(w_0)$ and let $\delta(w) = \min\{\delta_1(w), \delta_2(w)\}$. Then, for any $v \in \Omega$, $(t-t_k)^{1-\gamma}\|w(t) - v(t)\| < \delta(w)$ and choose

$$\left[\left(\left\|\mathscr{A}^{-\beta}\right\| + K(\gamma,\beta)\right)N_{\mathscr{D}} + \frac{\mathcal{M}}{\Gamma(\gamma+1)}\tau^{\gamma}\left[N_{\mathscr{L}} + q^{*}N_{\mathscr{P}}\right] + \frac{\mathcal{M}}{\Gamma(\gamma)}md_k\tau^{\gamma-1}\right] < \frac{\epsilon}{\delta(w)}.$$

Then we have

$$\begin{aligned}
&(t-t_k)^{1-\gamma}\left\|(\Upsilon w)(t) - (\Upsilon v)(t)\right\| \\
&\quad\leq (t-t_k)^{1-\gamma}\left\|\mathscr{D}\big(t,w(t)\big) - \mathscr{D}\big(t,v(t)\big)\right\| \\
&\qquad + (t-t_k)^{1-\gamma}\int_0^t (t-s)^{\gamma-1}\left\|\mathscr{A}\mathbb{T}_\gamma(t-s)\left[\mathscr{D}\big(s,w(s)\big) - \mathscr{D}\big(s,v(s)\big)\right]\right\| ds \\
&\qquad + (t-t_k)^{1-\gamma}\int_0^t (t-s)^{\gamma-1}\left\|\mathbb{T}_\gamma(t-s)\left[\mathscr{L}\big(s,w(s)\big) - \mathscr{L}\big(s,v(s)\big)\right]\right\| ds \\
&\qquad + (t-t_k)^{1-\gamma}\int_0^t (t-s)^{\gamma-1}\left\|\mathbb{T}_\gamma(t-s)\int_0^s q(s-\tau)\right. \\
&\qquad (\times)\left.\left[\mathscr{P}\big(\tau,w(\tau)\big) - \mathscr{P}\big(\tau,v(\tau)\big)\right] d\tau\right\| ds \\
&\qquad + (t-t_k)^{1-\gamma}\left\|\sum_{0<t_k<t}\mathbb{T}_\gamma(t-t_k)(t-t_k)^{\gamma-1}\left[I_k\big(w(t_k^-)\big) - I_k\big(v(t_k^-)\big)\right]\right\| \\
&\quad\leq \left[\left(\left\|\mathscr{A}^{-\beta}\right\| + K(\gamma,\beta)\right)N_{\mathscr{D}} + \frac{\mathcal{M}}{\Gamma(\gamma+1)}\tau^{\gamma}\left[N_{\mathscr{L}} + q^{*}N_{\mathscr{P}}\right] + \frac{\mathcal{M}}{\Gamma(\gamma)}md_k\tau^{\gamma-1}\right] \\
&\qquad (\times)\|w-v\|_{C_{1-\gamma}} \\
&\quad\leq \epsilon.
\end{aligned}$$

Thus, we have that $\Upsilon : \Omega \to \Omega$ is a continuous operator. Next, we demonstrate that the operator $\Upsilon : \Omega \to \Omega$ is equicontinuous. For any $w \in \Omega$ and $0 \leq t_1 < t_2 \leq \tau_1$, we get that

$$\begin{aligned}
&\left\|(t_2-t_k)^{1-\gamma}(\Upsilon w)(t_2) - (t_1-t_k)^{1-\gamma}(\Upsilon w)(t_1)\right\| \\
&\quad\leq \left\|(t_2-t_k)^{1-\gamma}t_2^{\gamma-1}\mathbb{T}_\gamma(t_2)w_0 - (t_1-t_k)^{1-\gamma}t_1^{\gamma-1}\mathbb{T}_\gamma(t_1)w_0\right\| \\
&\qquad + \left\|(t_2-t_k)^{1-\gamma}\mathscr{D}\big(t_2,w(t_2)\big) - (t_1-t_k)^{1-\gamma}\mathscr{D}\big(t_1,w(t_1)\big)\right\| \\
&\qquad + \left\|(t_2-t_k)^{1-\gamma}\int_0^{t_2}(t_2-s)^{\gamma-1}\mathscr{A}\mathbb{T}_\gamma(t_2-s)\mathscr{D}\big(s,w(s)\big)\,ds\right. \\
&\qquad - \left.(t_1-t_k)^{1-\gamma}\int_0^{t_1}(t_1-s)^{\gamma-1}\mathscr{A}\mathbb{T}_\gamma(t_1-s)\mathscr{D}\big(s,w(s)\big)\,ds\right\| \\
&\qquad + \left\|(t_2-t_k)^{1-\gamma}\int_0^{t_2}(t_2-s)^{\gamma-1}\mathbb{T}_\gamma(t_2-s)\left[\mathscr{L}\big(s,w(s)\big)\right.\right. \\
&\qquad + \left.\int_0^s q(s-\tau)\mathscr{P}\big(\tau,w(\tau)\big)\,d\tau\right] ds
\end{aligned}$$

$$\begin{aligned}
&- (t_1 - t_k)^{1-\gamma} \int_0^{t_1} (t_1 - s)^{\gamma-1} \mathbb{T}_\gamma(t_1 - s) \biggl[\mathscr{L}\bigl(s, w(s)\bigr) \\
&+ \int_0^s q(s-\tau) \mathscr{P}\bigl(\tau, w(\tau)\bigr)\, d\tau \biggr] ds \biggr\| \\
&+ \Biggl\| (t_2 - t_k)^{1-\gamma} \sum_{i=1}^{k} \mathbb{T}_\gamma(t_2 - t_i)(t_2 - t_i)^{\gamma-1} I_i\bigl(w\bigl(t_i^-\bigr)\bigr) \\
&- (t_1 - t_k)^{1-\gamma} \sum_{i=1}^{k} \mathbb{T}_\gamma(t_1 - t_i)(t_1 - t_i)^{\gamma-1} I_i\bigl(w\bigl(t_i^-\bigr)\bigr) \Biggr\| \leq \sum_{j=5}^{9} J_j,
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
J_5 &= \bigl\| (t_2 - t_k)^{1-\gamma} t_2^{\gamma-1} \mathbb{T}_\gamma(t_2) w_0 - (t_1 - t_k)^{\gamma-1} t_1^{\gamma-1} \mathbb{T}_\gamma(t_1) w_0 \bigr\| \\
&\leq (t_2 - t_k)^{1-\gamma} \bigl[t_2^{\gamma-1} - t_1^{\gamma-1} \bigr] \bigl\| \mathbb{T}_\gamma(t_2) w_0 \bigr\| \\
&+ t_1^{\gamma-1} \bigl\| (t_2 - t_k)^{\gamma-1} \mathbb{T}_\gamma(t_2) - (t_1 - t_k)^{\gamma-1} \mathbb{T}_\gamma(t_1) \bigr\| \|w_0\| \\
&\leq \frac{\mathcal{M}}{\Gamma(\gamma)} (t_2 - t_k)^{1-\gamma} \bigl[t_2^{\gamma-1} - t_1^{\gamma-1} \bigr] \|w_0\| \\
&\quad + t_1^{\gamma-1} \bigl\| (t_2 - t_k)^{\gamma-1} \mathbb{T}_\gamma(t_2) - (t_1 - t_k)^{\gamma-1} \mathbb{T}_\gamma(t_1) \bigr\| \|w_0\|, \\
J_6 &= \bigl\| (t_2 - t_k)^{1-\gamma} \mathscr{D}\bigl(t_2, w(t_2)\bigr) - (t_1 - t_k)^{1-\gamma} \mathscr{D}\bigl(t_1, w(t_1)\bigr) \bigr\| \\
&\leq (t_1 - t_k)^{1-\gamma} \bigl\| \mathscr{A}^{-\beta} \bigr\| \bigl\| \mathscr{A}^{\beta} \mathscr{D}\bigl(t_2, w(t_2)\bigr) - \mathscr{A}^{\beta} \mathscr{D}\bigl(t_1, w(t_1)\bigr) \bigr\| \\
&\quad + \bigl[(t_2 - t_k)^{\gamma-1} - (t_1 - t_k)^{\gamma-1} \bigr] \bigl\| \mathscr{A}^{-\beta} \bigr\| \bigl\| \mathscr{A}^{\beta} \mathscr{D}\bigl(t_2, w(t_2)\bigr) \bigr\| \\
&\leq \mathcal{M}_0 (t_1 - t_k)^{1-\gamma} \bigl\| \mathscr{A}^{\beta} \mathscr{D}\bigl(t_2, w(t_2)\bigr) - \mathscr{A}^{\beta} \mathscr{D}\bigl(t_1, w(t_1)\bigr) \bigr\| \\
&\quad + \mathcal{M}_0 \bigl[(t_2 - t_k)^{\gamma-1} - (t_1 - t_k)^{\gamma-1} \bigr] \bigl\| \mathscr{A}^{\beta} \mathscr{D}\bigl(t_2, w(t_2)\bigr) \bigr\|, \\
J_7 &= \biggl\| (t_2 - t_k)^{1-\gamma} \int_0^{t_2} (t_2 - s)^{\gamma-1} \mathscr{A} \mathbb{T}_\gamma(t_2 - s) \mathscr{D}\bigl(s, w(s)\bigr)\, ds \\
&\quad - (t_1 - t_k)^{1-\gamma} \int_0^{t_1} (t_1 - s)^{\gamma-1} \mathscr{A} \mathbb{T}_\gamma(t_1 - s) \mathscr{D}\bigl(s, w(s)\bigr)\, ds \biggr\| \\
&\leq (t_2 - t_k)^{1-\gamma} \int_0^{t_1} \bigl[(t_2 - s)^{\gamma-1} - (t_1 - s)^{\gamma-1} \bigr] \\
&\quad (\times) \bigl\| \mathscr{A}^{1-\beta} \mathbb{T}_\gamma(t_2 - s) \bigr\| \bigl\| \mathscr{A}^{\beta} \mathscr{D}\bigl(s, w(s)\bigr) \bigr\|\, ds \\
&\quad + \int_0^{t_1} (t_1 - s)^{\gamma-1} \bigl\| (t_2 - s)^{1-\gamma} \mathscr{A}^{1-\beta} \mathbb{T}_\gamma(t_2 - s) \\
&\quad - (t_1 - s)^{1-\gamma} \mathscr{A}^{1-\beta} \mathbb{T}_\gamma(t_1 - s) \bigr\| \bigl\| \mathscr{A}^{\beta} \mathscr{D}\bigl(s, w(s)\bigr) \bigr\|\, ds \\
&\quad + (t_2 - t_k)^{1-\gamma} \int_{t_1}^{t_2} (t_2 - s)^{\gamma-1} \bigl\| \mathscr{A}^{1-\beta} \mathbb{T}_\gamma(t_2 - s) \bigr\| \bigl\| \mathscr{A}^{\beta} \mathscr{D}\bigl(s, w(s)\bigr) \bigr\|\, ds \\
&\leq \frac{\gamma \mathcal{M}_{1-\beta} \Gamma(1+\beta) N_{\mathscr{D}}}{\Gamma(1+\gamma\beta)} \int_0^{t_1} (t_2 - s)^{\gamma\beta-\gamma} \bigl[(t_2 - s)^{\gamma-1} - (t_1 - s)^{\gamma-1} \bigr]\, ds \\
&\quad + \int_0^{t_1} (t_1 - s)^{\gamma-1} \bigl\| (t_2 - s)^{1-\gamma} \mathscr{A}^{1-\beta} \mathbb{T}_\gamma(t_2 - s) \\
&\quad - (t_1 - s)^{1-\gamma} \mathscr{A}^{1-\beta} \mathbb{T}_\gamma(t_1 - s) \bigr\| \bigl\| \mathscr{A}^{\beta} \mathscr{D}\bigl(s, w(s)\bigr) \bigr\|\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\gamma\mathcal{M}_{1-\beta}\Gamma(1+\beta)N_{\mathscr{D}}}{\Gamma(1+\gamma\beta)}(t_2-t_k)^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\gamma\beta-1}\,ds,\\
J_8 ={}& \biggl\|(t_2-t_k)^{1-\gamma}\int_0^{t_2}(t_2-s)^{\gamma-1}\mathbb{T}_\gamma(t_2-s)\\
&(\times)\biggl[\mathscr{L}\bigl(s,w(s)\bigr)+\int_0^s q(s-\tau)\mathscr{P}\bigl(\tau,w(\tau)\bigr)\,d\tau\biggr]\,ds\\
&-(t_1-t_k)^{1-\gamma}\int_0^{t_1}(t_1-s)^{\gamma-1}\mathbb{T}_\gamma(t_1-s)\\
&(\times)\biggl[\mathscr{L}\bigl(s,w(s)\bigr)+\int_0^s q(s-\tau)\mathscr{P}\bigl(\tau,w(\tau)\bigr)\,d\tau\biggr]\,ds\biggr\|\\
\le{}& \frac{\mathcal{M}}{\Gamma(\gamma)}\bigl(N_{\mathscr{L}}+q^*N_{\mathscr{P}}\bigr)\int_0^{t_1}\bigl[(t_2-s)^{\gamma-1}-(t_1-s)^{\gamma-1}\bigr]\,ds\\
&+\int_0^{t_1}(t_1-s)^{\gamma-1}\bigl\|(t_2-s)^{1-\gamma}\mathbb{T}_\gamma(t_2-s)-(t_1-s)^{1-\gamma}\mathbb{T}_\gamma(t_1-s)\bigr\|\\
&(\times)\biggl\|\mathscr{L}\bigl(s,w(s)\bigr)+\int_0^s q(s-\tau)\mathscr{P}\bigl(\tau,w(\tau)\bigr)\,d\tau\biggr\|\,ds\\
&+\frac{\mathcal{M}}{\Gamma(\gamma)}\bigl(N_{\mathscr{L}}+q^*N_{\mathscr{P}}\bigr)\frac{(t_2-t_1)^\gamma}{\gamma},\\
J_9 ={}& \biggl\|(t_2-t_k)^{1-\gamma}\sum_{i=1}^{k}\mathbb{T}_\gamma(t_2-t_i)(t_2-t_i)^{\gamma-1}I_i\bigl(w\bigl(t_i^-\bigr)\bigr)\\
&-(t_1-t_k)^{1-\gamma}\sum_{i=1}^{k}\mathbb{T}_\gamma(t_1-t_i)(t_1-t_i)^{\gamma-1}I_i\bigl(w\bigl(t_i^-\bigr)\bigr)\biggr\|\\
\le{}& \frac{\mathcal{M}}{\Gamma(\gamma)}\sum_{i=1}^{k}N_I\bigl[(t_2-t_i)^{\gamma-1}-(t_1-t_i)^{\gamma-1}\bigr]+\sum_{i=1}^{k}(t_1-t_i)^{\gamma-1}\\
&(\times)\bigl[(t_2-t_i)^{1-\gamma}\mathbb{T}_\gamma(t_2-t_i)-(t_1-t_i)^{1-\gamma}\mathbb{T}_\gamma(t_1-t_i)\bigr]\bigl\|I_i\bigl(w\bigl(t_i^-\bigr)\bigr)\bigr\|.
\end{aligned}
$$

From (J_5)–(J_9) in equation (3.3), we have

$$
\begin{aligned}
&\bigl\|(t_2-t_k)^{1-\gamma}(\Upsilon w)(t_2)-(t_1-t_k)^{1-\gamma}(\Upsilon w)(t_1)\bigr\|\\
&\quad\le \frac{\mathcal{M}}{\Gamma(\gamma)}(t_2-t_k)^{1-\gamma}\bigl[t_2^{\gamma-1}-t_1^{\gamma-1}\bigr]\|w_0\|\\
&\qquad+t_1^{\gamma-1}\bigl\|(t_2-t_k)^{\gamma-1}\mathbb{T}_\gamma(t_2)-(t_1-t_k)^{\gamma-1}\mathbb{T}_\gamma(t_1)\bigr\|\|w_0\|\\
&\qquad+\mathcal{M}_0(t_1-t_k)^{1-\gamma}\bigl\|\mathscr{A}^\beta\mathscr{D}\bigl(t_2,w(t_2)\bigr)-\mathscr{A}^\beta\mathscr{D}\bigl(t_1,w(t_1)\bigr)\bigr\|\\
&\qquad+\mathcal{M}_0\bigl[(t_2-t_k)^{\gamma-1}-(t_1-t_k)^{\gamma-1}\bigr]\bigl\|\mathscr{A}^\beta\mathscr{D}\bigl(t_2,w(t_2)\bigr)\bigr\|\\
&\qquad+\frac{\gamma\mathcal{M}_{1-\beta}\Gamma(1+\beta)N_{\mathscr{D}}}{\Gamma(1+\gamma\beta)}\int_0^{t_1}(t_2-s)^{\gamma\beta-\gamma}\bigl[(t_2-s)^{\gamma-1}-(t_1-s)^{\gamma-1}\bigr]\,ds\\
&\qquad+\int_0^{t_1}(t_1-s)^{\gamma-1}\bigl\|(t_2-s)^{1-\gamma}\mathscr{A}^{1-\beta}\mathbb{T}_\gamma(t_2-s)\\
&\qquad-(t_1-s)^{1-\gamma}\mathscr{A}^{1-\beta}\mathbb{T}_\gamma(t_1-s)\bigr\|\bigl\|\mathscr{A}^\beta\mathscr{D}\bigl(s,w(s)\bigr)\bigr\|\,ds\\
&\qquad+\frac{\gamma\mathcal{M}_{1-\beta}\Gamma(1+\beta)N_{\mathscr{D}}}{\Gamma(1+\gamma\beta)}(t_2-t_k)^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\gamma\beta-1}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\mathcal{M}}{\Gamma(\gamma)} \big(N_{\mathscr{L}} + q^* N_{\mathscr{P}}\big) \int_0^{t_1} \big[(t_2 - s)^{\gamma-1} - (t_1 - s)^{\gamma-1}\big]\, ds \\
&+ \int_0^{t_1} (t_1 - s)^{\gamma-1} \big\| (t_2 - s)^{1-\gamma} \mathbb{T}_\gamma (t_2 - s) - (t_1 - s)^{1-\gamma} \mathbb{T}_\gamma (t_1 - s) \big\| \\
&(\times) \bigg\| \mathscr{L}\big(s, w(s)\big) + \int_0^s q(s-\tau) \mathscr{P}\big(\tau, w(\tau)\big)\, d\tau \bigg\|\, ds \\
&+ \frac{\mathcal{M}}{\Gamma(\gamma)} \Bigg[\big(N_{\mathscr{L}} + q^* N_{\mathscr{P}}\big) \frac{(t_2 - t_1)^\gamma}{\gamma} + \sum_{i=1}^{k} N_I \big[(t_2 - t_i)^{\gamma-1} - (t_1 - t_i)^{\gamma-1}\big] \Bigg] \\
&+ \sum_{i=1}^{k} (t_1 - t_i)^{\gamma-1} \big[(t_2 - t_i)^{1-\gamma} \mathbb{T}_\gamma (t_2 - t_i) - (t_1 - t_i)^{1-\gamma} \mathbb{T}_\gamma (t_1 - t_i)\big] \big\| I_i\big(w\big(t_i^-\big)\big) \big\| \\
&\to 0 \quad \text{as } t_2 - t_1 \to 0,
\end{aligned}
$$

which means that $\Upsilon : \Omega \to \Omega$ is equicontinuous.

Since $(t_2 - s)^{1-\gamma} \mathbb{T}_\gamma (t_2 - s) - (t_1 - s)^{1-\gamma} \mathbb{T}_\gamma (t_1 - s) \to 0$ as $t_2 \to t_1$ because $\mathbb{T}_\gamma(\cdot)$ is strongly continuous.

Let $B = \overline{\text{co}}\Upsilon(\omega)$. At that point it is anything but difficult to confirm that Υ maps B into itself and $B \subset PC_{1-\gamma}(\mathscr{I}, \mathbb{H})$ is equicontinuous. Now, we prove that $\Upsilon : B \to B$ is a condensing operator. For any $E \subset B$, by [8, Lemma 2.2], there exists a countable set $E_0 = \{w_n\} \subset E$ such that

$$
\omega\big(\Upsilon(E)\big) \le 2\omega\big(\Upsilon(E_0)\big). \tag{3.4}
$$

By the equicontinuity of B, we know that $E_0 \subset B$ is also equicontinuous. Therefore, by [8, Lemma 2.4], assumption (H6), we have

$$
\begin{aligned}
&\omega\big(\Upsilon(E_0)(t)\big) \\
&= \omega\Bigg\{ (t - t_k)^{1-\gamma} \bigg(t^{\gamma-1} \mathbb{T}_\gamma(t) w_0 + \mathscr{D}\big(t, w_n(t)\big) \\
&+ \int_0^t (t-s)^{\gamma-1} \mathscr{A} \mathbb{T}_\gamma(t-s) \mathscr{D}\big(s, w_n(s)\big)\, ds \\
&+ \int_0^t (t-s)^{\gamma-1} \mathbb{T}_\gamma(t-s) \bigg[\mathscr{L}\big(s, w_n(s)\big) + \int_0^s q(s-\tau) \mathscr{P}\big(\tau, w_n(\tau)\big)\, d\tau \bigg]\, ds \\
&+ \sum_{0<t_k<t} \mathbb{T}_\gamma(t - t_k)(t - t_k)^{\gamma-1} I_k\big(w\big(t_k^-\big)\big) \bigg) \Bigg\} \\
&\le 2 \big\| \mathscr{A}^{-\beta} \big\| \omega\big((t - t_k)^{1-\gamma} \mathscr{A}^\beta \mathscr{D}\big(t, w_n(t)\big)\big) \\
&+ 2\omega\bigg((t - t_k)^{1-\gamma} \int_0^t (t-s)^{\gamma-1} \mathscr{A} \mathbb{T}_\gamma(t-s) \mathscr{D}\big(s, w_n(s)\big)\, ds \bigg) \\
&+ \frac{2\mathcal{M}}{\Gamma(\gamma)} \omega\bigg((t - t_k)^{1-\gamma} \int_0^t (t-s)^{\gamma-1} \mathbb{T}_\gamma(t-s) \bigg[\mathscr{L}\big(s, w_n(s)\big) \\
&+ \int_0^s q(s-\tau) \mathscr{P}\big(\tau, w_n(\tau)\big)\, d\tau \bigg]\, ds \bigg) \\
&\le 2 \big\| \mathscr{A}^{-\beta} \big\| L_1 \omega\big((t - t_k)^{1-\gamma} w_n(t)\big) + \frac{\gamma \mathcal{M}_{1-\beta} \Gamma(1+\beta) t^{\gamma\beta}}{\gamma\beta \Gamma(1+\gamma\beta)} L_1 \omega\big((t - t_k)^{1-\gamma}
\end{aligned}
$$

$$(\times)w_n(t)\big) + \frac{2\mathcal{M}}{\Gamma(\gamma)}\frac{t^\gamma}{\gamma}\omega\big((t-t_k)^{1-\gamma}\big[\mathscr{L}\big(t,w_n(t)\big) + q^*\mathscr{P}\big(t,w_n(t)\big)\big]\big)$$
$$\le 2\left(\|\mathcal{M}_0L_1\| + K(\gamma,\beta)L_1 + \frac{\mathcal{M}}{\Gamma(\gamma+1)}\big[L_2 + q^*L_3\big]\tau^\gamma\right)\omega(E_0). \tag{3.5}$$

Since $\Upsilon(E_0) \subset B$ is bounded and equicontinuous, we know from [8, Lemma 2.3] that

$$\omega\big(\Upsilon(E_0)\big) = \max_{t\in\mathscr{I}}\omega\big(\Upsilon(E_0)(t)\big). \tag{3.6}$$

Therefore, from (3.1), (3.4)–(3.6), we know that

$$\omega\big(\Upsilon(E)\big) \le 4\left(\big(\mathcal{M}_0 + K(\gamma,\beta)\big)L_1 + \frac{\mathcal{M}}{\Gamma(\gamma+1)}\big[L_2 + q^*L_3\big]\tau^\gamma\right)\omega(E)$$
$$\le \omega(E). \tag{3.7}$$

Thus, $\Upsilon : B \to B$ is a condensing operator. It follows from [8, Lemma 2.5] that Υ has at least one fixed point $w(t_0) \in B$, which is the mild solution of model (1.1)–(1.3) on the interval $[0,\tau_1]$. □

4 Applications

Consider the subsequent initial-boundary value problem of impulsive fractional integro-differential model with RL fractional derivatives:

$${}^LD_{0+}^{\frac{3}{4}}\left[u(t,x) - \int_0^t e^{2s}\frac{\|u(s,x)\|}{1+\|u(s,x)\|}\,ds\right]$$
$$= \frac{\partial^2 u(t,x)}{\partial x^2} + \frac{2e^{-t}\|u(t,x)\|}{1+\|u(t,x)\|}$$
$$+ \int_0^t e^{t-s}\frac{e^s}{5+\|u(s,x)\|}\,ds, \quad t\in(0,2] \text{ and } t\ne 1, \tag{4.1}$$

$$\Delta I_{0+}^{\frac{1}{4}}u\big(1^-,x\big) = \sin\left(\frac{1}{7}\big\|u\big(1^-,x\big)\big\|\right), \tag{4.2}$$

$$u_0(x) = I_{0+}^{\frac{1}{4}}\left[u(t,x) - \int_0^t e^{2s}\frac{\|u(s,x)\|}{1+\|u(s,x)\|}\,ds\right]\Big|_{t=0}, \tag{4.3}$$

$$u(t,0) = u(t,\pi) = 0, \quad t\in(0,T], x\in[0,\pi], \tag{4.4}$$

where ${}^LD_{0^+}^{\frac{3}{4}}$ is the Riemann–Liouville fractional derivatives of order $\frac{3}{4}$, $0 < \frac{3}{4} \le 1$, $I_{0^+}^{\frac{1}{4}}$ is the RL integral of order $\frac{1}{4}$, $u_0(x) \in \mathbb{H}$. To study this problem, consider $\mathbb{H} = L^2([0,\pi],\mathbb{R})$. Let the operator $\mathscr{A}$ by $\mathscr{A}y = y''$, with the domain

$$D(\mathscr{A}) = \big\{y(\cdot) \in \mathbb{H} : y, y' \text{ are absolutely continuous}, y'' \in \mathbb{H}, y(t,0) = y(t,\pi) = 0\big\}.$$

Then $\mathscr{A}$ generates a c_0 semigroup $\{\mathbb{T}(t)\}_{t\ge 0}$ which is compact, analytic. Besides, $\mathscr{A}$ can be composed as

$$\mathscr{A}y = \sum_{n=1}^{\infty} n^2\langle y, e_n\rangle e_n, \quad y \in D(\mathscr{A}),$$

where $e_n(x) = \sqrt{\frac{2}{\pi}}\sin nx$, $0 \le x \le \pi$, $n = 1, 2, \ldots$, is an orthonormal basis of $\mathbb{H}$. We have

$$\mathbb{T}(t)y = \sum_{n=1}^{\infty} e^{-n^2 t}\langle y, e_n\rangle e_n, \quad y \in \mathbb{H}, \quad \text{and} \quad \|\mathbb{T}(t)\| \le e^{-t} \le 1 = \mathcal{M}, \quad t \ge 0.$$

For each $y \in \mathbb{H}$,

$$\mathscr{A}^{-\frac{1}{2}}y = \sum_{n=1}^{\infty} \frac{1}{n}\langle y, e_n\rangle e_n.$$

In specific, $\|\mathscr{A}^{-\frac{1}{2}}\| = 1$. The operator $\mathscr{A}^{\frac{1}{2}}$ is given by

$$\mathscr{A}^{\frac{1}{2}}y = \sum_{n=1}^{\infty} n\langle y, e_n\rangle e_n$$

on the space

$$D(\mathscr{A}^{\frac{1}{2}}) = \left\{ y(\cdot) \in \mathbb{H}, \sum_{n=1}^{\infty} n\langle y, e_n\rangle e_n \in \mathbb{H} \right\}.$$

From Theorem 2.1, the general solution of impulsive model (4.1)–(4.4) is obtained as follows:

$$u(t,x) = \begin{cases} \frac{u_0}{\Gamma(\frac{3}{4})}t^{\frac{-1}{4}} + \frac{1}{\Gamma(\frac{3}{4})}\int_0^t (t-s)^{\frac{-1}{4}}[\frac{\partial^2 u(s,x)}{\partial x^2} + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} \\ \quad + \int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau]\,ds + \int_0^t e^{2s}\frac{\|u(s,x)\|}{1+\|u(s,x)\|}\,ds, \\ \quad t \in (0,1], \\ \frac{u_0}{\Gamma(\frac{3}{4})}t^{\frac{-1}{4}} + \frac{1}{\Gamma(\frac{3}{4})}\int_0^t (t-s)^{\frac{-1}{4}}[\frac{\partial^2 u(s,x)}{\partial x^2} + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} \\ \quad + \int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau]\,ds + \int_0^t e^{2s}\frac{\|u(s,x)\|}{1+\|u(s,x)\|}\,ds \\ \quad + \frac{1}{\Gamma(\frac{3}{4})}\sin(\frac{1}{7}\|u(1^-,x)\|)(t-1)^{\frac{-1}{4}} \\ \quad - \frac{\xi}{\Gamma(\frac{3}{4})}\sin(\frac{1}{7}\|u(1^-,x)\|)u_0 t^{\frac{-1}{4}} \end{cases}$$

$$= \begin{cases} -\frac{\xi}{\Gamma(\frac{3}{4})}\sin(\frac{1}{7}\|u(1^-,x)\|)\{\frac{1}{\Gamma(\frac{3}{4})}\int_0^t (t-s)^{\frac{-1}{4}} \\ \quad (\times)[\frac{\partial^2 u(s,x)}{\partial x^2} + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} \\ \quad + \int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau]\,ds - (u_0 + \int_0^1 [\frac{\partial^2 u(s,x)}{\partial x^2} \\ \quad + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} + \int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau]\,ds)(t-1)^{\frac{-1}{4}} \\ \quad - \int_1^t (t-s)^{\frac{-1}{4}}[\frac{\partial^2 u(s,x)}{\partial x^2} + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} \\ \quad + \int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau]\,ds\}, \quad t \in (1,2]. \end{cases} \tag{4.5}$$

Next, it is verified that equation (4.5) satisfies the condition of system (4.1)–(4.4). Taking the Riemann–Liouville fractional derivative to both sides of equation (4.5), we have:

(i) for $t \in (0,1]$,

$$
\begin{aligned}
&D_{0^+}^{\frac{3}{4}}\left(u(t,x) - \int_0^t e^{2s}\frac{\|u(s,x)\|}{1+\|u(s,x)\|}\,ds\right)\\
&= D_{0^+}^{\frac{3}{4}}\left(\frac{u_0}{\Gamma(\frac{3}{4})}t^{\frac{-1}{4}} + \frac{1}{\Gamma(\frac{3}{4})}\int_0^t (t-s)^{\frac{-1}{4}}\left[\frac{\partial^2 u(s,x)}{\partial x^2} + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|}\right.\right.\\
&\quad \left.\left. + \int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau\right]ds\right)\\
&= \left\{\frac{1}{\Gamma(\frac{3}{4})\Gamma(1-\frac{3}{4})}\frac{d}{dt}\int_0^t (t-\eta)^{1-\frac{3}{4}-1}\left(u_0\eta^{\frac{-1}{4}} + \int_0^\eta (\eta-s)^{\frac{-1}{4}}\left[\frac{\partial^2 u(s,x)}{\partial x^2}\right.\right.\right.\\
&\quad \left.\left.\left. + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} + \int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau\right]ds\right)d\eta\right\}_{t\in(0,1]}\\
&= \left\{\int_0^t \left[\frac{\partial^2 u(s,x)}{\partial x^2} + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|}\right]ds\right\}_{t\in(0,1]}\\
&\quad + \left\{\int_0^t \left[\int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau\right]ds\right\}_{t\in(0,1]}\\
&= \frac{\partial^2 u(t,x)}{\partial x^2} + \frac{2e^{-t}\|u(t,x)\|}{1+\|u(t,x)\|} + \int_0^t e^{t-s}\frac{e^{s}}{5+\|u(s,x)\|}\,ds.
\end{aligned}
$$

(ii) for $t \in (1,2]$,

$$
\begin{aligned}
&D_{0^+}^{\frac{3}{4}}\left(u(t,x) - \int_0^t e^{2s}\frac{\|u(s,x)\|}{1+\|u(s,x)\|}\,ds\right)\\
&= D_{0^+}^{\frac{3}{4}}\left(\frac{u_0}{\Gamma(\frac{3}{4})}t^{\frac{-1}{4}} + \frac{1}{\Gamma(\frac{3}{4})}\int_0^t (t-s)^{\frac{-1}{4}}\left[\frac{\partial^2 u(s,x)}{\partial x^2} + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|}\right.\right.\\
&\quad \left.\left. + \int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau\right]ds\right) + D_{0^+}^{\frac{3}{4}}\left(\sin\left(\frac{1}{7}\left\|u(1^-,x)\right\|\right)\frac{(t-1)^{\frac{-1}{4}}}{\Gamma(\frac{3}{4})}\right)\\
&\quad - D_{0^+}^{\frac{3}{4}}\left(\frac{\xi\sin(\frac{1}{7}\|u(1^-,x)\|)}{\Gamma(\frac{3}{4})}\left\{u_0 t^{\frac{-1}{4}} + \frac{1}{\Gamma(\frac{3}{4})}\int_0^t (t-s)^{\frac{-1}{4}}\left[\frac{\partial^2 u(s,x)}{\partial x^2}\right.\right.\right.\\
&\quad \left. + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} + \int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau\right]ds\\
&\quad - \left(u_0 + \int_0^1\left[\frac{\partial^2 u(s,x)}{\partial x^2} + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} + \int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau\right]ds\right)(t-1)^{\frac{-1}{4}}\\
&\quad - \int_0^1 (t-s)^{\frac{-1}{4}}\left[\frac{\partial^2 u(s,x)}{\partial x^2} + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|}\right.\\
&\quad \left.\left.\left. + \int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau\right]ds\right\}\right)\\
&= \left\{\frac{1}{\Gamma(\frac{3}{4})\Gamma(1-\frac{3}{4})}\frac{d}{dt}\int_0^t (t-\eta)^{1-\frac{3}{4}-1}\left(u_0\eta^{\frac{-1}{4}} + \int_0^\eta (\eta-s)^{\frac{-1}{4}}\left[\frac{\partial^2 u(s,x)}{\partial x^2}\right.\right.\right.\\
&\quad \left.\left.\left. + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} + \int_0^s e^{s-\tau}\frac{e^{\tau}}{5+\|u(\tau,x)\|}\,d\tau\right]ds\right)d\eta\right\}_{t\in(1,2]}\\
&\quad + \left\{\frac{1}{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}\frac{d}{dt}\int_0^t (t-\eta)^{\frac{-3}{4}}(\eta-1)^{\frac{-1}{4}}\sin\left(\frac{1}{7}\left\|u(1^-,x)\right\|\right)d\eta\right\}_{t\in(1,2]}
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg\{\frac{1}{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}\frac{d}{dt}\int_0^t (t-\eta)^{\frac{-3}{4}}(\eta-1)^{\frac{-1}{4}}\frac{\xi\sin(\frac{1}{7}\|u(1^-,x)\|)}{\Gamma(\frac{3}{4})}\Bigg\{u_0\eta^{\frac{-1}{4}} \\
&+\int_0^\eta (\eta-s)^{\frac{-1}{4}}\bigg[\frac{\partial^2 u(s,x)}{\partial x^2}+\frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} \\
&+\int_0^s e^{s-\tau}\frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau\bigg]ds-\bigg(u_0+\int_0^1\bigg[\frac{\partial^2 u(s,x)}{\partial x^2}+\frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} \\
&+\int_0^s e^{s-\tau}\frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau\bigg]ds\bigg)(\eta-1)^{\frac{-1}{4}}-\int_0^1 (t-s)^{\frac{-1}{4}}\bigg[\frac{\partial^2 u(s,x)}{\partial x^2} \\
&+\frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|}+\int_0^s e^{s-\tau}\frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau\bigg]ds\Bigg\}d\eta\Bigg\}_{t\in(1,2]} \\
&=\bigg\{\int_0^t\bigg[\frac{\partial^2 u(s,x)}{\partial x^2}+\frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|}+\int_0^s e^{s-\tau}\frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau\bigg]ds\bigg\}_{t\in(1,2]} \\
&-\bigg\{\frac{\xi\sin(\frac{1}{7}\|u(1^-,x)\|)}{\Gamma(\frac{3}{4})}\int_0^t\bigg[\frac{\partial^2 u(s,x)}{\partial x^2}+\frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|}\bigg]ds\bigg\}_{t\in(1,2]} \\
&-\bigg\{\frac{\xi\sin(\frac{1}{7}\|u(1^-,x)\|)}{\Gamma(\frac{3}{4})}\int_0^t\int_0^s e^{s-\tau}\frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau\,ds\bigg\}_{t\in(1,2]} \\
&+\Bigg\{\frac{\xi\sin(\frac{1}{7}\|u(1^-,x)\|)}{\Gamma(\frac{3}{4})}\frac{1}{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}\frac{d}{dt}\int_1^t (t-\eta)^{\frac{-3}{4}}\Bigg\{\bigg(u_0+\int_0^1\bigg[\frac{\partial^2 u(s,x)}{\partial x^2} \\
&+\frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|}+\int_0^s e^{s-\tau}\frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau\bigg]ds\bigg)(\eta-1)^{\frac{-1}{4}} \\
&-\int_\eta^1 (t-s)^{\frac{-1}{4}}\bigg[\frac{\partial^2 u(s,x)}{\partial x^2}+\frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} \\
&+\int_0^s e^{s-\tau}\frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau\bigg]ds\Bigg\}d\eta\Bigg\}\}_{t\in(1,2]} \\
&=\bigg\{\bigg[\frac{\partial^2 u(t,x)}{\partial x^2}+\frac{2e^{-t}\|u(t,x)\|}{1+\|u(t,x)\|}+\int_0^t e^{t-s}\frac{e^s}{5+\|u(s,x)\|}\,ds\bigg]_{t\geq 0} \\
&-\frac{\xi\sin(\frac{1}{7}\|u(1^-,x)\|)}{\Gamma(\frac{3}{4})}\bigg\{\bigg[\frac{\partial^2 u(t,x)}{\partial x^2}+\frac{2e^{-t}\|u(t,x)\|}{1+\|u(t,x)\|} \\
&+\int_0^t e^{t-s}\frac{e^s}{5+\|u(s,x)\|}\,ds\bigg]_{t\geq 0}-\bigg[\frac{\partial^2 u(t,x)}{\partial x^2}+\frac{2e^{-t}\|u(t,x)\|}{1+\|u(t,x)\|} \\
&+\int_0^t e^{t-s}\frac{e^s}{5+\|u(s,x)\|}\,ds\bigg]_{t\geq 1}\bigg\}_{t\in(1,2]} \\
&=\bigg[\frac{\partial^2 u(t,x)}{\partial x^2}+\frac{2e^{-t}\|u(t,x)\|}{1+\|u(t,x)\|}+\int_0^t e^{t-s}\frac{e^s}{5+\|u(s,x)\|}\,ds\bigg]_{t\in(1,2]}.
\end{aligned}
$$

So, equation (4.5) satisfies the RL fractional derivative condition of system (4.1)–(4.4). By Definition 2.2, we obtain

$$
\begin{aligned}
&I_{0^+}^{1-\frac{3}{4}}u\big(1^+,x\big)-I_{0^+}^{1-\frac{3}{4}}u\big(1^-,x\big) \\
&=\bigg\{\frac{1}{\Gamma(\frac{1}{4})}\int_0^t (t-\eta)^{\frac{-3}{4}}u(\eta)\,d\eta\bigg\}_{t\to 1^+}-\bigg\{\frac{1}{\Gamma(\frac{1}{4})}\int_0^t (t-\eta)^{\frac{-3}{4}}u(\eta)\,d\eta\bigg\}_{t=1^+}
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{ \frac{\sin(\frac{1}{7}\|u(1^-,x)\|)}{\Gamma(\frac{3}{4})\Gamma(1-\frac{3}{4})} \int_0^t (t-\eta)^{\frac{-3}{4}}\,d\eta \right\}_{t\to 1^+} \\
&\quad - \xi \sin\biggl(\frac{1}{7}\bigl\|u\bigl(1^-,x\bigr)\bigr\|\biggr) \biggl\{ \frac{1}{\Gamma(\frac{3}{4})\Gamma(1-\frac{3}{4})} \int_0^t (t-\eta)^{\frac{-3}{4}} u_0 \eta^{\frac{-1}{4}} \\
&\quad + \int_1^\eta (\eta-s)^{\frac{-3}{4}} \biggl[\frac{\partial^2 u(s,x)}{\partial x^2} \\
&\quad + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} + \int_0^s e^{s-\tau} \frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau \biggr] ds - \biggl(u_0 + \int_0^1 \biggl[\frac{\partial^2 u(s,x)}{\partial x^2} \\
&\quad + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} + \int_0^s e^{s-\tau} \frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau \biggr] ds \biggr) (\eta-1)^{\frac{-1}{4}} \\
&\quad - \int_1^\eta (\eta-s)^{\frac{-1}{4}} \biggl[\frac{\partial^2 u(s,x)}{\partial x^2} + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} + \int_0^s e^{s-\tau} \frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau \biggr] ds \biggr\}\, d\eta \}_{t\to 1^+} \\
&= \sin\biggl(\frac{1}{7}\bigl\|u\bigl(1^-,x\bigr)\bigr\|\biggr) - \xi \sin\biggl(\frac{1}{7}\bigl\|u\bigl(1^-,x\bigr)\bigr\|\biggr) \biggl\{ u_0 + \int_0^t \biggl[\frac{\partial^2 u(s,x)}{\partial x^2} \\
&\quad + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} + \int_0^s e^{s-\tau} \frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau \biggr] ds - \biggl(u_0 + \int_0^1 \biggl[\frac{\partial^2 u(s,x)}{\partial x^2} \\
&\quad + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} + \int_0^s e^{s-\tau} \frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau \biggr] ds \biggr) \\
&\quad - \int_1^t \biggl[\frac{\partial^2 u(s,x)}{\partial x^2} + \frac{2e^{-s}\|u(s,x)\|}{1+\|u(s,x)\|} + \int_0^s e^{s-\tau} \frac{e^\tau}{5+\|u(\tau,x)\|}\,d\tau \biggr] ds \biggr\}_{t\to 1^+} \\
&= \sin\biggl(\frac{1}{7}\bigl\|u\bigl(1^-,x\bigr)\bigr\|\biggr).
\end{aligned}
$$

That is, equation (4.5) satisfies impulsive condition (4.2). Therefore, clearly equation (4.5) fulfills the following limit case:

$$
\lim_{\sin(\frac{1}{7}\|u(1^-,x)\|)\to 0}
\begin{cases}
D_{0^+}^{\frac{3}{4}}[u(t,x) - \int_0^t e^{2s} \frac{\|u(s,x)\|}{1+\|u(s,x)\|}\,ds] \\
\quad = \frac{\partial^2 u(t,x)}{\partial x^2} + \frac{2e^{-t}\|u(t,x)\|}{1+\|u(t,x)\|} + \int_0^t e^{t-s} \frac{e^s}{5+\|u(s,x)\|}\,ds, \\
\qquad t \in (0,2] \text{ and } t \neq 1,
\end{cases}
$$

$$
\lim_{\sin(\frac{1}{7}\|u(1^-,x)\|)\to 0}
\begin{cases}
\Delta(I_{0^+}^{\frac{1}{4}} u)|_{t=1} = I_{0^+}^{\frac{1}{4}} u(1^+,x) - I_{0^+}^{\frac{1}{4}} u(1^-,x) \\
\quad = \sin(\frac{1}{7}\|u(1^-,x)\|) \in \mathbb{H}, \\
I_{0+}^{\frac{1}{4}}[u(t,x) - \int_0^t e^{2s} \frac{\|u(s,x)\|}{1+\|u(s,x)\|}\,ds]|_{t=0} = u_0(x) \in \mathbb{H}, \\
D_{0^+}^{\frac{3}{4}}[u(t,x) - \int_0^t e^{2s} \frac{\|u(s,x)\|}{1+\|u(s,x)\|}\,ds] \\
\quad = \frac{\partial^2 u(t,x)}{\partial x^2} + \frac{2e^{-t}\|u(t,x)\|}{1+\|u(t,x)\|} \\
\qquad + \int_0^t e^{t-s} \frac{e^s}{5+\|u(s,x)\|}\,ds, \quad t \in (0,2], \\
I_{0+}^{\frac{1}{4}}[u(t,x) - \int_0^t e^{2s} \frac{\|u(s,x)\|}{1+\|u(s,x)\|}\,ds]|_{t=0} = u_0(x) \in \mathbb{H}.
\end{cases}
$$

So, equation (4.5) is the general solution of system (4.1)–(4.3). Characterize the administrators $\mathscr{L}, \mathscr{D}, \mathscr{P} : \mathscr{I} \times \mathbb{H} \to \mathbb{H}$ and $q : \mathscr{I} \to \mathbb{H}$ by

$$
\mathscr{L}(t,u) = \frac{2e^{-t}\|u(t,x)\|}{1+\|u(t,x)\|},
$$

$$\mathscr{P}(t,u) = \frac{e^t}{5 + \|u(s,x)\|},$$
$$\mathscr{D}(t,u) = \int_0^t e^{2s} \frac{\|u(s,x)\|}{1 + \|u(s,x)\|}\, ds,$$
$$q(t-s) = e^{t-s} \quad \text{and}$$
$$I_k\big(u\big(1^-,x\big)\big) = \sin\left(\frac{1}{7}\big\|u\big(1^-,x\big)\big\|\right).$$

Then the impulsive fractional differential system (4.1)–(4.4) can be converted into the abstract form problem (1.1)–(1.3). Next, we shall show that hypotheses (H2)–(H5) are satisfied. For this, $u, v \in PC_{1-\gamma}((0,2],\mathbb{H})$.

(i)

$$\begin{aligned}\big\|\mathscr{L}(t,u) - \mathscr{L}(t,v)\big\| &= \left\| \frac{2e^{-t}u}{1+u} - \frac{2e^{-t}v}{1+v} \right\| \\ &= 2e^{-t}\left\| \frac{u}{1+u} - \frac{v}{1+v} \right\| \\ &= 2e^{-t}\frac{\|u-v\|}{(1+u)(1+v)} \\ &\le 2\|u-v\|.\end{aligned}$$

Hypothesis (H2) holds if $N_{\mathscr{L}} = 2$.

(ii)

$$\begin{aligned}\big\|\mathscr{P}(t,u) - \mathscr{P}(t,v)\big\| &= \left\| \frac{e^t}{5+u} - \frac{e^t}{5+v} \right\| \\ &= e^t\frac{\|u-v\|}{(5+u)(5+v)} \\ &\le \frac{e}{25}\|u-v\|.\end{aligned}$$

If $N_{\mathscr{L}} = \frac{e}{25}$, condition (H3) is satisfied.

(iii) Choose $\beta = \frac{1}{2}$, we have

$$\begin{aligned}\big\|(\mathscr{A})^{\frac{1}{2}}\mathscr{D}(t,u) - (\mathscr{A})^{\frac{1}{2}}\mathscr{D}(t,v)\big\| &= \int_0^t e^{2s}\left\| \frac{u}{1+u} - \frac{v}{1+v} \right\| ds \\ &= \frac{e^{2t}-1}{2}\frac{\|u-v\|}{(1+u)(1+v)} \\ &\le \frac{e-1}{2}\|u-v\|.\end{aligned}$$

Here, condition (H4) holds with $N_{\mathscr{P}} = \frac{e-1}{2}$.

(iv) Finally,

$$\begin{aligned}\|I_k(u(1^-)) - I_k(u(1^-))\| &= \left\|\sin\left(\frac{1}{7}\|u(1^-,x)\|\right) - \sin\left(\frac{1}{7}\|v(1^-,x)\|\right)\right\| \\ &= \left\|\sin\left(\frac{1}{7}u\right) - \sin\left(\frac{1}{7}u\right)\right\| \\ &\leq \frac{1}{7}\|u-v\|.\end{aligned}$$

And

$$\begin{aligned}\|I_k(u)\| &= \left\|\sin\left(\frac{1}{7}\|u(1^-,x)\|\right)\right\| \\ &\leq \frac{1}{7}\|u\|.\end{aligned}$$

From (iv), we notice that hypothesis (H5) holds with $d_i = N_I = \frac{1}{7}$.

Let $t \in \mathscr{I}$, then we have

$$q^* = \sup_{t\in\mathscr{I}} \int_0^t |q(t-s)|\,ds = \int_0^t e^{t-s}\,ds = e^t - 1 \leq e - 1.$$

Therefore, all the conditions of Theorem 3.1 are verified. Hence, problem (4.1)–(4.4) has a solution in $(0,2]$.

5 Conclusion

In this manuscript, we have studied the local existence for an impulsive fractional neutral integro-differential system with Riemann–Liouville fractional derivatives in a Banach space. More precisely, by utilizing the semigroup theory, fractional powers of operators, and condensing fixed point theorem, we investigate the impulsive fractional neutral integro-differential system in a Banach space. To validate the obtained theoretical results, an example is analyzed. The fractional differential equations are very efficient to describe real life phenomena; thus, it is essential to extend the present study to establish the other qualitative and quantitative properties such as stability and approximate controllability.

There are two direct issues which require further study. First, we will investigate the global existence of a mild solution to impulsive fractional semilinear integro-differential equations with noncompact semigroup. Secondly, we will be devoted to studying the approximate controllability of impulsive fractional neutral integro-differential systems with Riemann–Liouville fractional derivatives in a Banach space both in the case of a noncompact operator and a normal topological space.

Acknowledgements
The authors are grateful to the anonymous referees for their constructive comments and helpful suggestions.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equal contributions. All authors read and approved the final manuscript.

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Publisher's Note

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Received: 2 January 2018 Accepted: 30 October 2018 Published online: 12 November 2018

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