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Solution of fractional differential equations via $\alpha - \psi$-Geraghty type mappings

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Abstract

Using fixed point results of $\alpha - \psi$-Geraghty contractive type mappings, we examine the existence of solutions for some fractional differential equations in b-metric spaces. By some concrete examples we illustrate the obtained results.

Keywords: Fractional differential equation; Normal cone; $\alpha - \psi$-Geraghty contractive type mapping

1 Introduction

In 2012, Samet et al. [11] presented the concepet of α-admissible mappings, which was expanded by several authors (see [5, 6, 9]). Baleanu, Rezapour, and Mohammadi [3] studied the existence of a solution for problem $D^{\nu}w(\xi) = h(\xi, w(\xi))$ ($\xi \in [0,1], 1 < \nu \leq 2$). Afshari, Aydi, and Karapinar [1, 2] considered generalized $\alpha - \psi$-Geraghty contractive mappings in b-metric spaces.

We investigate the existence of solutions for some fractional differential equations in b-metric spaces. We denote $I = [0,1]$.

Definition 1.1 ([7, 10]) The Caputo derivative of order ν of a continuous function $h : [0,\infty) \to \mathbb{R}$ is defined by

$$ {}^{c}D^{\nu}h(\xi) = \frac{1}{\Gamma(n-\nu)} \int_0^{\xi} (\xi - \zeta)^{n-\nu-1} h^{(n)}(\zeta)\, d\zeta, $$

where $n - 1 < \nu < n$, $n = [\nu] + 1$, $[\nu]$ is the integer part of ν, and

$$ \Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x}\, dx. \tag{1} $$

Definition 1.2 ([7, 10]) The Riemann–Liouville derivative of a continuous function h is defined by

$$ D^{\nu}h(\xi) = \frac{1}{\Gamma(n-\nu)} \left(\frac{d}{d\xi}\right)^{n} \int_0^{\xi} \frac{h(\zeta)}{(\xi - \zeta)^{\nu-n-1}}\, d\zeta \quad \bigl(n = [\nu] + 1\bigr), $$

where the right-hand side is defined on $(0,\infty)$.

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Let Ψ be the set of all increasing continuous functions $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(\lambda x) \leq \lambda\psi(x) \leq \lambda x$ for $\lambda > 1$, and let $\mathcal{B}$ be the family of nondecreasing functions $\gamma : [0,\infty) \to [0, \frac{1}{s^2})$ for some $s \geq 1$.

Definition 1.3 ([1]) Let (X, d) be a b-metric space (with constant s). A function $g : X \to X$ is a generalized $\alpha - \psi$-Geraghty contraction if there exists $\alpha : X \times X \to [0,\infty)$ such that

$$\alpha(z,t)\psi\bigl(s^3 d(gz,gt)\bigr) \leq \gamma\bigl(\psi\bigl(d(z,t)\bigr)\bigr)\psi\bigl(d(z,t)\bigr) \tag{2}$$

for all $z, t \in X$, where $\gamma \in \mathcal{B}$ and $\psi \in \Psi$.

Definition 1.4 ([11]) Let $g : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be given. Then g is called α-admissible if for $z, t \in X$,

$$\alpha(z,t) \geq 1 \quad \Longrightarrow \quad \alpha(gz,gt) \geq 1. \tag{3}$$

Theorem 1.5 ([1]) *Let (X, d) be a complete b-metric space, and let $f : X \to X$ be a generalized $\alpha - \psi$-Geraghty contraction such that*

(i) *f is α-admissible;*
(ii) *there exists $u_0 \in X$ such that $\alpha(u_0, fu_0) \geq 1$;*
(iii) *if $\{u_n\} \subseteq X$, $u_n \to u$ in X, and $\alpha(u_n, u_{n+1}) \geq 1$, then $\alpha(u_n, u) \geq 1$.*

Then f has a fixed point.

2 Main result

By $X = C(I)$ we denote the set of continuous functions. Let $d : X \times X \to [0,\infty)$ be given by

$$d(y,z) = \bigl\|(y-z)^2\bigr\|_\infty = \sup_{\xi\in I}\bigl(y(\xi) - z(\xi)\bigr)^2. \tag{4}$$

Evidently, (X, d) is a complete b-metric space with $s = 2$ but is not a metric space.

Now we study the problem

$$\frac{D^\nu}{D\xi} w(\xi) = h\bigl(\xi, w(\xi)\bigr), \quad \xi \in I, 3 < \nu \leq 4, \tag{5}$$

under the conditions

$$w(0) = w'(0) = w(1) = w'(1) = 0, \tag{6}$$

where D^ν is the Riemann–Liouville derivative, and $h : I \times X \to \mathbb{R}$ is continuous.

Lemma 2.1 ([13]) *Given $h \in C(I \times X, \mathbb{R})$ and $3 < \nu \leq 4$, the unique solution of*

$$\frac{D^\nu}{D\xi} w(\xi) = h\bigl(\xi, w(\xi)\bigr), \quad \xi \in I, 3 < \nu \leq 4, \tag{7}$$

where

$$w(0) = w'(0) = w(1) = w'(1) = 0, \tag{8}$$

is given by $w(\xi) = \int_0^1 G(\xi,\zeta)h(s,w(s))\,ds$, *where*

$$G(\xi,\zeta) = \begin{cases} \frac{(\xi-1)^{\nu-1}+(1-\zeta)^{\nu-2}\xi^{\nu-2}[(\zeta-\xi)+(\nu-2)(1-\xi)\zeta]}{\Gamma(\nu)}, & 0 \le \zeta \le \xi \le 1, \\ \frac{(1-\zeta)^{\nu-2}\xi^{\nu-2}[(\zeta-\xi)+(\nu-2)(1-\xi)\zeta]}{\Gamma(\nu)}, & 0 \le \xi \le \zeta \le 1. \end{cases} \tag{9}$$

If $h(\xi,w(\xi)) = 1$, *then the unique solution of* (7)–(8) *is given by*

$$f(\xi) = \int_0^1 G(\xi,\zeta)\,ds = \frac{1}{\Gamma(\nu+1)}\xi^{\nu-2}(1-\xi)^2.$$

Lemma 2.2 ([13]) *In Lemma* 2.1, $G(\xi,\zeta)$ *given in* (9) *satisfies the following conditions*:

(1) $G(\xi,\zeta) > 0$, *and* $G(\xi,\zeta)$ *is continuous for* $\xi,\zeta \in I$;

(2) $\frac{(\nu-2)\sigma(\xi)\rho(\zeta)}{\Gamma(\nu)} \le G(\xi,\zeta) \le \frac{r_0\rho(\zeta)}{\Gamma(\nu)}$,

where

$$r_0 = \max\{\nu-1,(\nu-2)^2\}, \qquad \sigma(\xi) = \xi^{\nu-2}(1-\xi)^2, \quad \text{and} \quad \rho(\zeta) = \zeta^2(1-\zeta)^{\nu-2}.$$

Theorem 2.3 *Suppose*

(i) *there exist* $\theta : \mathbb{R}^2 \to \mathbb{R}$ *and* $\psi \in \Psi$ *such that*

$$\left|h(\xi,c) - h(\xi,d)\right| \le \frac{1}{2\sqrt{2}} \frac{\Gamma(\nu+1)}{4\nu} \frac{\psi(|c-d|^2)}{\sqrt{4\|(c-d)^2\|_\infty + 1}}$$

for $\xi \in I$ *and* $c,d \in \mathbb{R}$ *with* $\theta(c,d) \ge 0$;

(ii) *there exists* $y_0 \in C(I)$ *such that* $\theta(y_0(\xi), \int_0^1 G(\xi,\zeta)h(\zeta,y_0(\xi))\,d\zeta) \ge 0$, $\xi \in I$;

(iii) *for* $\xi \in I$ *and* $y,z \in C(I)$, $\theta(y(\xi),z(\xi)) \ge 0$ *implies*

$$\theta\left(\int_0^1 G(\xi,\zeta)h\big(\zeta,y(\zeta)\big)d\zeta, \int_0^1 G(\xi,\zeta)h\big(\zeta,z(\xi)\big)d\zeta\right) \ge 0;$$

(iv) *if* $\{y_n\} \subseteq C(I)$, $y_n \to y$ *in* $C(I)$, *and* $\theta(y_n,y_{n+1}) \ge 0$, *then* $\theta(y_n,y) \ge 0$.

Then problem (7) *has at least one solution.*

Proof By Lemma 2.1 $y \in C(I)$ is a solution of (7) if and only if it is a solution of $y(\xi) = \int_0^1 G(\xi,\zeta)h(\zeta,y(\zeta))\,d\zeta$, and we define $A : C(I) \to C(I)$ by $Ay(\xi) = \int_0^1 G(\xi,\zeta)h(\zeta,y(\zeta))\,d\zeta$ for $\xi \in I$. For this purpose, we find a fixed point of A. Let $y,z \in C(I)$ be such that $\theta(y(\xi),z(\xi)) \ge 0$ for $\xi \in I$. Using (i), we get

$$\begin{aligned} \left|Ay(\xi) - Az(\xi)\right|^2 &= \left|\int_0^1 G(\xi,\zeta)\big(h\big(\zeta,y(\zeta)\big) - h\big(\zeta,z(\zeta)\big)\big)\,d\zeta\right|^2 \\ &\le \left[\int_0^1 G(\xi,\zeta)\left|h\big(\zeta,y(\zeta)\big) - h\big(\zeta,z(\zeta)\big)\right|d\zeta\right]^2 \\ &\le \left[\int_0^1 G(\xi,\zeta)\frac{1}{2\sqrt{2}}\frac{\Gamma(\nu+1)}{4\nu}\frac{\psi(|y(\zeta)-z(\zeta)|^2)}{\sqrt{4\|(y-z)^2\|_\infty+1}}\,d\zeta\right]^2 \\ &\le \frac{1}{8}\frac{(\psi(\|(y-z)^2\|_\infty))^2}{4\|(y-z)^2\|_\infty+1}. \end{aligned}$$

Hence, for $y, z \in C(I)$ and $\xi \in I$ with $\theta(y(\xi), z(\xi)) \geq 0$, we have

$$\|(Ay - Az)^2\|_\infty \leq \frac{1}{8} \frac{(\psi(\|(y-z)^2\|_\infty))^2}{4\|(y-z)^2\|_\infty + 1}.$$

Let $\alpha : C(I) \times C(I) \to [0, \infty)$ be defined by

$$\alpha(y,z) = \begin{cases} 1, & \theta(y(\xi), z(\xi)) \geq 0, \xi \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\gamma : [0, \infty) \to [0, \frac{1}{4})$ by $\gamma(q) = \frac{q}{4q+1}$ and $s = 2$.

So

$$\begin{aligned} \alpha(y,z)\psi\bigl(8d(Ay, Az)\bigr) \leq 8\alpha(y,z)\psi\bigl(d(Ay, Az)\bigr) &\leq \frac{(\psi(d(y,z)))^2}{4d(y,z) + 1} \\ &\leq \frac{(\psi(d(y,z)))^2}{4\psi(d(y,z)) + 1} \\ &= \frac{1}{\gamma(\psi(d(y,z)))}\gamma\bigl(\psi\bigl(d(y,z)\bigr)\bigr)\frac{(\psi(d(y,z)))^2}{4\psi(d(y,z)) + 1} \\ &\leq \gamma\bigl(\psi\bigl(d(y,z)\bigr)\bigr)\psi\bigl(d(y,z)\bigr), \quad \gamma \in \mathcal{B}. \end{aligned}$$

Then A is an $\alpha - \psi$–contractive mapping. From (iii) and the definition of α we have

$$\begin{aligned} \alpha(y,z) \geq 1 \quad &\Rightarrow \quad \theta\bigl(y(\xi), z(\xi)\bigr) \geq 0 \\ &\Rightarrow \quad \theta\bigl(A(y), A(z)\bigr) \geq 0 \\ &\Rightarrow \quad \alpha\bigl(A(y), A(z)\bigr) \geq 1, \end{aligned}$$

for $y, z \in C(I)$. Thus, A is α-admissible. By (ii) there exists $y_0 \in C(I)$ such that $\alpha(y_0, Ay_0) \geq 1$. By (iv) and Theorem 1.5 there is $y^* \in C(I)$ such that $y^* = Ay^*$. Hence y^* is a solution of the problem. □

Corollary 2.4 *Suppose that there exist $\theta : \mathbb{R}^2 \to \mathbb{R}$ and $\psi \in \Psi$ such that*

$$\bigl|h(\xi, c) - h(\xi, d)\bigr| \leq \frac{10^3}{4\sqrt{8}} \frac{\psi(|c-d|^2)}{\sqrt{4\|(c-d)^2\|_\infty + 1}} \tag{10}$$

for $\xi \in I$ and $c, d \in \mathbb{R}$ with $\theta(c,d) \geq 0$. Also, suppose that conditions (ii)–(iv) from Theorem 2.3 hold for h, where $G(\xi, \zeta)$ is given in (9). Then the problem

$$\frac{D^{\frac{7}{2}}}{D\xi} w(\xi) = h\bigl(\xi, w(\xi)\bigr), \quad \xi \in I, \tag{11}$$

where

$$w(0) = w'(0) = w(1) = w'(1) = 0,$$

has at least one solution.

Proof By Lemma 2.2

$$\min \int_0^1 G(\xi,\zeta)\,d\zeta = 10^{-5} \quad \text{and} \quad \max \int_0^1 G(\xi,\zeta)\,d\zeta = 4\times 10^{-3}. \tag{12}$$

Using (10) and (12), by Theorem 2.3 we obtain

$$\left|Ay(\xi)-Az(\xi)\right|^2 \le \frac{1}{8}\frac{(\psi(|y-z|^2))^2}{4\|(y-z)^2\|_\infty + 1}.$$

The rest of the proof is according to Theorem 2.3. □

Lemma 2.5 ([8]) *If* $h \in C(I\times X,\mathbb{R})$ *and* $h(\xi,w(\xi)) \le 0$, *then the problem*

$$\begin{aligned} &-D^{\nu}_{0+}w(\xi) = h\bigl(\xi,w(\xi)\bigr), \quad (0<\xi<1, 3<\nu\le 4),\\ &w(0)=w'(0)=w''(0)=w''(1)=0 \end{aligned} \tag{13}$$

has a unique positive solution

$$w(\xi) = \int_0^1 G(\xi,\zeta)h\bigl(\zeta,w(\zeta)\bigr)\,d\zeta,$$

where $G(\xi,\zeta)$ *is given by*

$$G(\xi,\zeta) = \frac{1}{\Gamma(\nu)}\begin{cases} \xi^{\nu-1}(1-\zeta)^{\nu-3}-(\xi-\zeta)^{\nu-1}, & 0\le\zeta\le\xi\le 1,\\ \xi^{\nu-1}(1-\zeta)^{\nu-3}, & 0\le\xi\le\zeta\le 1. \end{cases} \tag{14}$$

Lemma 2.6 ([12]) *The function* $G(\xi,\zeta)$ *in Lemma* 2.5 *has the following property*:

$$\frac{1}{\Gamma(\nu)}\zeta(2-\zeta)(1-\zeta)^{\nu-3}\xi^{\nu-1} \le G(\xi,\zeta) \le \frac{1}{\Gamma(\nu)}(1-\zeta)^{\nu-3}\xi^{\nu-1},$$

where $\xi,\zeta\in I$ *and* $3<\nu\le 4$.

Based on Theorem 2.3, we get the following result.

Corollary 2.7 *Assume that there exist* $\theta:\mathbb{R}^2\to\mathbb{R}$ *and* $\psi\in\Psi$ *such that*

$$\left|h(\xi,c)-h(\xi,d)\right| \le \frac{1}{2\sqrt{2}M}\frac{\psi(|c-d|^2)}{\sqrt{4\|(c-d)^2\|_\infty+1}},$$

where $M=\sup_{\xi\in I}\int_0^1 G(\xi,\zeta)\,d\zeta$. *Also, suppose that conditions* (ii)–(iv) *from Theorem* 2.3 *are satisfied, where* $G(\xi,\zeta)$ *is given in* (14). *Then problem* (13) *has at least one solution.*

Proof By Lemma 2.5 $y\in C(I)$ is a solution of (13) if and only if a solution of $y(\xi) = \int_0^1 G(\xi,\zeta)h(\zeta,y(\zeta))\,d\zeta$. Define $A: C(I)\to C(I)$ by $Ay(\xi)=\int_0^1 G(\xi,\zeta)h(\zeta,y(\zeta))\,d\zeta$ for $\xi\in I$. We find a fixed point of A. Let $y,z\in C(I)$ be such that $\theta(y(\xi),z(\xi))\ge 0$ for $\xi\in I$. By (i) and

Lemma 2.6 we get

$$\begin{aligned}
&\left|Ay(\xi) - Az(\xi)\right|^2 \\
&= \left|\int_0^1 G(\xi,\zeta)\big(h(\zeta,y(\zeta)) - h(\zeta,z(\zeta))\big)\,d\zeta\right|^2 \\
&\leq \left[\int_0^1 G(\xi,\zeta)\big|h(\zeta,y(\zeta)) - h(\zeta,z(\zeta))\big|\,d\zeta\right]^2 \\
&\leq \left[\int_0^1 G(\xi,\zeta)\frac{1}{2\sqrt{2}M}\frac{\psi(|y(\zeta)-z(\zeta)|^2)}{\sqrt{4\|(y-z)^2\|_\infty + 1}}\,d\zeta\right]^2 \\
&\leq \left[\int_0^1 G(\xi,\zeta)\frac{1}{2\sqrt{2}(\sup_{\xi\in I}\int_0^1 G(\xi,\zeta)\,d\zeta)}\frac{\psi(|y(\zeta)-z(\zeta)|^2)}{\sqrt{4\|(y-z)^2\|_\infty + 1}}\,d\zeta\right]^2 \\
&\leq \left[\int_0^1 G(\xi,\zeta)\frac{1}{2\sqrt{2}(\int_0^1 G(\xi,\zeta)\,d\zeta)}\frac{\psi(|y(\zeta)-z(\zeta)|^2)}{\sqrt{4\|(y-z)^2\|_\infty + 1}}\,d\zeta\right]^2 \\
&\leq \left[\int_0^1 \frac{1}{\Gamma(\nu)}(1-\zeta)^{\nu-3}\xi^{\nu-1}\frac{\Gamma(\nu)}{2\sqrt{2}(\int_0^1 \zeta(2-\zeta)(1-\zeta)^{\nu-3}\xi^{\nu-1}\,d\zeta)}\right. \\
&\quad\left.\times\frac{\psi(|y(\zeta)-z(\zeta)|^2)}{\sqrt{4\|(y-z)^2\|_\infty + 1}}\,d\zeta\right]^2 \\
&\leq \frac{1}{8}\frac{(\psi(\|(y-z)^2\|_\infty))^2}{4\|(y-z)^2\|_\infty + 1}.
\end{aligned}$$

Suppose that conditions (ii)–(iv) from Theorem 2.3 are satisfied, where $G(\xi,\zeta)$ is given in (14). By Theorem 2.3 problem (13) has at least one solution.

Let (X,d) be given in (4). For the equation

$$ {}^cD^\nu y(\xi) = h(\xi,y(\xi)), \quad (\xi\in I, 1<\nu\leq 2), \tag{15}$$

via

$$y(0) = 0, \qquad y(1) = \int_0^\eta y(\zeta)\,d\zeta \quad (0<\eta<1),$$

where $h : I\times X \to \mathbb{R}$ is continuous, we have the following result. □

Theorem 2.8 *Assume that there exist $\theta : \mathbb{R}^2 \to \mathbb{R}$, $\gamma \in \mathcal{B}$, and $\psi \in \Psi$ such that*

$$\left|h(\xi,c) - h(\xi,d)\right| \leq \frac{\Gamma(\nu+1)}{5}\sqrt{\frac{1}{8}\gamma\big(\psi(|c-d|^2)\big)\psi\big(|c-d|^2\big)}.$$

Suppose conditions (ii)–(iv) *from Theorem* 2.3 *hold, where $A : C(I)\to C(I)$ is defined by*

$$\begin{aligned}
Ay(\xi) :={}& \frac{1}{\Gamma(\nu)}\int_0^1 (\xi-\zeta)^{\nu-1}h(\zeta,y(\zeta))\,d\zeta - \frac{2\xi}{(2-\eta^2)\Gamma(\nu)}\int_0^1 (1-\zeta)^{\nu-1}h(\zeta,y(\zeta))\,d\zeta \\
&+ \frac{2\xi}{(2-\eta^2)\Gamma(\nu)}\int_0^\eta\left(\int_0^\zeta (\zeta-n)^{\nu-1}h(n,y(n))dn\right)d\zeta \quad (\xi\in I);
\end{aligned}$$

Then (15) *has at least one solution.*

Proof A function $y \in C(I)$ is a solution of (15) if and only if it is a solution of

$$y(\xi) = \frac{1}{\Gamma(\nu)} \int_0^1 (\xi - \zeta)^{\nu-1} h\bigl(\zeta, y(\zeta)\bigr)\, d\zeta - \frac{2\xi}{(2-\eta^2)\Gamma(\nu)} \int_0^1 (1-\zeta)^{\nu-1} h\bigl(\zeta, y(\zeta)\bigr)\, d\zeta + \frac{2\xi}{(2-\eta^2)\Gamma(\nu)} \int_0^\eta \biggl(\int_0^\zeta (\zeta - n)^{\nu-1} h\bigl(n, y(n)\bigr) dn \biggr) d\zeta \quad (\xi \in I).$$

Then (15) is equivalent to finding $y^* \in C(I)$ that is a fixed point of A. Let $y, z \in C(I)$ with $\theta(y(\xi), z(\xi)) \geq 0$, $\xi \in I$. By (i) we have

$$\begin{aligned}
&\bigl|Ay(\xi) - Az(\xi)\bigr|^2 \\
&= \Biggl| \frac{1}{\Gamma(\nu)} \int_0^1 (\xi - \zeta)^{\nu-1} h\bigl(\zeta, y(\zeta)\bigr)\, d\zeta \\
&\quad - \frac{2\xi}{(2-\eta^2)\Gamma(\nu)} \int_0^1 (1-\zeta)^{\nu-1} h\bigl(\zeta, y(\zeta)\bigr)\, d\zeta \\
&\quad + \frac{2\xi}{(2-\eta^2)\Gamma(\nu)} \int_0^\eta \biggl(\int_0^\zeta (\zeta - n)^{\nu-1} h\bigl(n, y(n)\bigr) dn \biggr) d\zeta \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^1 (\xi - \zeta)^{\nu-1} h\bigl(\zeta, z(\zeta)\bigr)\, d\zeta \\
&\quad + \frac{2\xi}{(2-\eta^2)\Gamma(\nu)} \int_0^1 (1-\zeta)^{\nu-1} h\bigl(\zeta, z(\zeta)\bigr)\, d\zeta \\
&\quad - \frac{2\xi}{(2-\eta^2)\Gamma(\nu)} \int_0^\eta \biggl(\int_0^\zeta (\zeta - n)^{\nu-1} h\bigl(n, z(n)\bigr) dn \biggr) d\zeta \Biggr|^2 \\
&\leq \Biggl| \frac{1}{\Gamma(\nu)} \int_0^1 |\xi - \zeta\Bigr|^{\nu-1} \bigl| h\bigl(\zeta, y(\zeta)\bigr) - h\bigl(\zeta, z(\zeta)\bigr) \bigr|\, d\zeta \\
&\quad + \frac{2\xi}{(2-\eta^2)\Gamma(\nu)} \int_0^1 |1-\zeta|^{\nu-1} \bigl| h\bigl(\zeta, y(\zeta)\bigr) - h\bigl(\zeta, z(\zeta)\bigr) \bigr|\, d\zeta \\
&\quad + \frac{2\xi}{(2-\eta^2)\Gamma(\nu)} \int_0^\eta \biggl| \int_0^\zeta |\zeta - n|^{\nu-1} \bigl| h\bigl(n, y(n)\bigr) - h\bigl(n, z(n)\bigr) \bigr| |dn|\, d\zeta \Biggr|^2 \\
&\leq \Biggl| \frac{1}{\Gamma(\nu)} \int_0^1 |\xi - \zeta|^{\nu-1} \frac{\Gamma(\nu+1)}{5} \sqrt{\frac{1}{8} \gamma \bigl(\psi \bigl(|y(\zeta) - z(\zeta)|^2\bigr)\bigr) \psi \bigl(|y(\zeta) - z(\zeta)|^2\bigr)}\, d\zeta \\
&\quad + \frac{2\xi}{(2-\eta^2)\Gamma(\nu)} \int_0^1 |1-\zeta|^{\nu-1} \frac{\Gamma(\nu+1)}{5} \\
&\quad \times \sqrt{\frac{1}{8} \gamma \bigl(\psi \bigl(|y(\zeta) - z(\zeta)|^2\bigr)\bigr) \psi \bigl(|y(\zeta) - z(\zeta)|^2\bigr)}\, d\zeta \\
&\quad + \frac{2\xi}{(2-\eta^2)\Gamma(\nu)} \int_0^\eta \int_0^\zeta |\zeta - n|^{\nu-1} \\
&\quad \times \sqrt{\frac{1}{8} \gamma \bigl(\psi \bigl(|y(n) - z(n)|^2\bigr)\bigr) \psi \bigl(|y(n) - z(n)|^2\bigr)}\, d\zeta \Biggr|^2 \\
&\leq \biggl(\frac{\Gamma(\nu+1)}{5} \biggr)^2 \frac{1}{8} \gamma \bigl(\psi \bigl(\|y - z\|_\infty^2\bigr)\bigr) \psi \bigl(\|y - z\|_\infty^2\bigr) \biggl[\sup \biggl(\int_0^1 |\xi - \zeta|^{\nu-1}\, d\zeta \\
&\quad + \frac{2\xi}{(2-\eta^2)\Gamma(\nu)} \int_0^1 |1-\zeta|^{\nu-1}\, d\zeta
\end{aligned}$$

$$+\frac{2\xi}{(2-\eta^2)\Gamma(\nu)}\int_0^{\eta}\left(\int_0^{\zeta}|\zeta-n|^{\nu-1}dn\right)d\zeta\right)\Bigg]^2$$
$$\leq\frac{1}{8}\gamma\left(\psi\left(\|y-z\|_\infty^2\right)\right)\psi\left(\|y-z\|_\infty^2\right)$$

for all $y,z\in C(I)$ with $\theta(y(\xi),z(\xi))\geq 0$, $\xi\in I$, so that

$$\left\|(Ay-Az)^2\right\|_\infty\leq\frac{1}{8}\gamma\left(\psi\left(\|y-z\|_\infty^2\right)\right)\psi\left(\|y-z\|_\infty^2\right).$$

Let $\alpha: C(I)\times C(I)\to[0,\infty)$ be defined by

$$\alpha(y,z)=\begin{cases}1 & \theta(y(\xi),z(\xi))\geq 0,\xi\in I,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\alpha(y,z)\psi\big(8d(Ay,Az)\big)&\leq 8\alpha(y,z)\psi\big(d(Ay,Az)\big)\\&\leq\alpha(y,z)\psi\Big(\gamma\Big(\psi\big(d(y,z)\big)\Big)\psi\big(d(y,z)\big)\Big)\\&\leq\gamma\Big(\psi\big(d(y,z)\big)\Big)\psi\big(d(y,z)\big)\end{aligned}$$

for all $y,z\in C(I)$, and thus A is an $\alpha-\psi-$contractive mapping. From Theorem 1.5, based on the proof of Theorem 2.3, we can deduce the proof of Theorem 2.8. □

Here we find a positive solution for

$$\frac{{}^cD^\nu}{D\xi}w(\xi)=h\big(\xi,w(\xi)\big),\quad 0<\nu\leq 1,\xi\in I,\tag{16}$$

where

$$w(0)+\int_0^1 w(\zeta)\,d\zeta=w(1).$$

Note that ${}^cD^\nu$ is the Caputo derivative of order ν. We consider the Banach space of continuous functions on I endowed with the sup norm. We have the following lemma.

Lemma 2.9 ([4]) *Let* $0<\nu\leq 1$ *and* $h\in C([0,T]\times X,\mathbb{R})$ *be given. Then the equation*

$${}^cD^\nu w(\xi)=h\big(\xi,w(\xi)\big)\quad\big(\xi\in[0,T],T\geq 1\big)$$

with

$$w(0)+\int_0^T w(\zeta)\,d\zeta=w(T)$$

has a unique solution given by

$$w(\xi)=\int_0^T G(\xi,\zeta)h\big(\zeta,w(\zeta)\big)\,d\zeta,$$

where $G(\xi,\zeta)$ is defined by

$$G(\xi,\zeta)=\begin{cases}\frac{-(T-\zeta)^{\nu}+\nu T(\xi-\zeta)^{\nu-1}}{T\Gamma(\nu+1)}+\frac{(T-\zeta)^{\nu-1}}{T\Gamma(\nu)}, & 0\le\zeta<\xi,\\ \frac{-(T-\zeta)^{\nu}}{T\Gamma(\nu+1)}+\frac{(T-\zeta)^{\nu-1}}{T\Gamma(\nu)}, & \xi\le\zeta<T.\end{cases} \tag{17}$$

By Lemma 2.9 and Theorem 2.4 we get the following conclusion.

Corollary 2.10 *Assume that there exist $\theta:\mathbb{R}^2\to\mathbb{R}$ and $\psi\in\Psi$ such that*

$$\bigl|h(\xi,c)-h(\xi,d)\bigr|\le\frac{51}{80\sqrt{8}}\frac{\psi(|c-d|^2)}{\sqrt{4\|(c-d)^2\|_\infty+1}}$$

for $\xi\in I$ and $c,d\in\mathbb{R}$ with $\theta(c,d)\ge0$. Suppose conditions (ii)–(iv) *from Theorem 2.3 are satisfied, where $G(\xi,\zeta)$ is given in (17). Then the following problem has at least one solution:*

$${}^{c}D^{\frac{1}{2}}w(\xi)=h\bigl(\xi,w(\xi)\bigr),\quad \bigl(\xi\in[0,1]\bigr),\qquad w(0)+\int_0^1 w(\zeta)\,d\zeta=w(1).$$

Proof It is easily that $\min_{t\in[0,1]}\int_0^1 G(t,s)\,ds=\frac{1}{3}$ and $\max_{t\in[0,1]}\int_0^1 G(t,s)\,ds=\frac{80}{51}$. By Theorem 2.3 we conclude the desired result. □

Example 2.11 Let $\psi(r)=r$, $\theta(x,z)=xz$, and $y_n(\xi)=\frac{\xi}{n^2+1}$. We consider $h:I\times[-2,2]\to[-2,2]$ and the periodic boundary value problem

$$\frac{D^{\frac{7}{2}}}{D\xi}w(\xi)=h\bigl(\xi,w(\xi)\bigr)=w(\xi),\quad \xi\in I, \tag{18}$$

with

$$w(0)=w'(0)=w(1)=w'(1)=0.$$

Then

$$\bigl|h(\xi,c)-h(\xi,d)\bigr|=|c-d|\le\frac{10^3}{4\sqrt{8}}\frac{\psi(|c-d|^2)}{\sqrt{4\|(c-d)^2\|_\infty+1}}$$

for $\xi\in I$ and $c,d\in[-2,2]$ with $\theta(c,d)\ge0$. Because $y_0(\xi)=\xi$, thus

$$\theta\left(y_0(\xi),\int_0^1 G(\xi,\zeta)h\bigl(\zeta,y_0(\zeta)\bigr)\,d\zeta\right)\ge0$$

for all $\xi\in I$. Also, $\theta(y(\xi),z(\xi))=y(\xi)z(\xi)\ge0$ implies that

$$\theta\left(\int_0^1 G(\xi,\zeta)h\bigl(\zeta,y_(\zeta)\bigr)\,d\zeta,\int_0^1 G(\xi,\zeta)h\bigl(\zeta,z(\zeta)\bigr)\,d\zeta\right)\ge0.$$

It is obvious that condition (iv) in Corollary (2.4) holds. Hence by Corollary 2.4 problem (18) has at least one solution.

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