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The extended fractional Caputo–Fabrizio derivative of order $0 \leq \sigma < 1$ on $C_{\mathbb{R}}[0,1]$ and the existence of solutions for two higher-order series-type differential equations

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Abstract

We extend the fractional Caputo–Fabrizio derivative of order $0 \leq \sigma < 1$ on $C_{\mathbb{R}}[0,1]$ and investigate two higher-order series-type fractional differential equations involving the extended derivation. Also, we provide an example to illustrate one of the main results.

MSC: 34A08; 34A99

Keywords: The extended Caputo–Fabrizio derivative of order $0 \leq \sigma < 1$; Higher-order fractional differential equation; Series-type equation

1 Introduction

Recently, Caputo and Fabrizio suggested a new fractional derivative [15, 16]), and Losada and Nieto [21] investigated some of its properties. Later, some authors tried to utilized it for solving various equations (see [2–14, 17], and [26]), whereas some researchers studied some singular fractional integro-differential equations [22–25]. As you know, the fractional Caputo–Fabrizio derivative is defined on the space H^1 (which is not necessarily a Banach space), and because of this reason, a researcher has to investigate approximate solutions for some problems [11, 13]. It seems that Caputo and Fabrizio tried to give a formula for an extension of their definition (see formula (3) in [15]), but they did not use it in their investigation. In 2016, Alqahtani tried to extend the Caputo–Fabrizio derivative by using formula (2.2) in [5]. Again, he did not use it for investigating the problems reported in [5]. In this manuscript, we extend the fractional Caputo–Fabrizio derivative on $C_{\mathbb{R}}[0,1]$. Using it, we discuss some higher-order series-type fractional integro-differential equations.

The properties of the fractional Caputo–Fabrizio derivative were investigated very recently in [7]. Specifically, the Caputo–Fabrizio fractional derivative is discussed in he distributional setting [7]. For more detail about physical interpretation of the Caputo–Fabrizio derivative, the reader can see the new results presented recently in [19]. Specifically, the physical origin of Caputo–Fabrizio derivative is demonstrated in [18]. Besides, very recently the determination of the fractional order (relation to physical characteristics of the process) was investigated in [20].

Having all the mentioned things in mind, in this paper, we extend the fractional Caputo–Fabrizio derivative on $C_{\mathbb{R}}[0,1]$. Using it, we investigate some higher-order series-type fractional integro-differential equations.

Let $b>0$, $\kappa \in H^1(0,b)$, and $\sigma \in [0,1]$. Thus, for the function κ, its Caputo–Fabrizio fractional derivative is written as ${}^{CF}D^{\sigma}\kappa(t) = \frac{B(\sigma)}{1-\sigma}\int_0^t \exp(\frac{-\sigma}{1-\sigma}(t-s))\kappa'(p)\,dp$, where $t \geq 0$, and $B(\sigma)$ denotes a normalization constant obeying $B(0)=B(1)=1$ [1, 15]. The associated fractional integral of order σ for the function κ is defined by ${}^{CF}I^{\sigma}\kappa(t) = \frac{1-\sigma}{B(\sigma)}\kappa(t) + \frac{\sigma}{B(\sigma)}\int_0^t \kappa(s)\,ds$ for $0<\sigma<1$ [1, 21].

If $n \geq 1$ and $\sigma \in [0,1]$, then the fractional derivative ${}^{CF}D^{\sigma+n}$ of order $n+\sigma$ is defined by ${}^{CF}D^{\sigma+n}\kappa := {}^{CF}D^{\sigma}(D^n\kappa(t))$ [21]. Also, we have $\lim_{\sigma\to 0}{}^{CF}D^{\sigma}\kappa(t) = \kappa(t)-\kappa(0)$, $\lim_{\sigma\to 1}{}^{CF}D^{\sigma}\kappa(t) = \kappa'(t)$, and ${}^{CF}D^{\sigma}(\lambda\kappa(t)+\gamma\upsilon(t)) = \lambda{}^{CF}D^{\sigma}\kappa(t)+\gamma{}^{CF}D^{\sigma}\upsilon(t)$ for all $\kappa,\upsilon \in H^1$ and $\lambda,\gamma \in \mathbb{R}$ [15]. We now present the following important results.

Lemma 1.1 ([21]) *Let $0<\sigma<1$. Then the unique solution of ${}^{CF}D^{\sigma}\kappa(p) = \upsilon(p)$ such that $\kappa(0)=c$ is written as $\kappa(p) = c + a_\sigma(\upsilon(p)-\upsilon(0)) + b_\sigma\int_0^p \upsilon(s)\,ds$, where $a_\sigma = \frac{1-\sigma}{B(\sigma)}$ and $b_\sigma = \frac{\sigma}{B(\sigma)}$. Note that $\upsilon(0)=0$.*

Lemma 1.2 ([27]) *Let $t \in \mathbb{R}$ and $0 \leq |t| < \infty$. Then $t\prod_{i=1}^{\infty}(1-\frac{t^2}{i^2\pi^2}) = \sin t$, $\prod_{i=1}^{\infty}(1-\frac{4t^2}{(2i-1)^2\pi^2}) = \cos t$ and $e^t = \sum_{i=0}^{\infty}\frac{t^i}{i!}$ for $0<|t|<\infty$.*

2 Results and discussion

We further show our main results. Let $\kappa \in C_{\mathbb{R}}[0,b]$, $b>0$, and $\sigma \in (0,1)$. We define the expended fractional Caputo–Fabrizio derivative of order σ by

$$\begin{aligned}{}^{CF}_{N}D^{\sigma}\kappa(p) &= \frac{B(\sigma)}{1-\sigma}\big(\kappa(p)-\kappa(0)\big)\exp\left(\frac{-\sigma}{1-\sigma}p\right)\\ &\quad + \frac{\sigma B(\sigma)}{(1-\sigma)^2}\int_0^p \big(\kappa(p)-\kappa(s)\big)\exp\left(\frac{-\sigma}{1-\sigma}(p-s)\right)ds.\end{aligned}$$

If $\kappa(0)=0$, then we have ${}^{CF}_{N}D^{\sigma}\kappa(p) = \frac{B(\sigma)}{1-\sigma}\kappa(p) - \frac{\sigma B(\sigma)}{(1-\sigma)^2}\int_0^p \exp(-\frac{\sigma}{1-\sigma}(p-s))\kappa(s)\,ds$. We recall that

$$\begin{aligned}J^n\kappa(p) &= \underbrace{\int_0^{p=p_n}\int_0^{s=p_{n-1}}\int_0^{p_{n-2}}\cdots\int_0^{p_1}}_{n\text{ times}}\kappa(p_0)\,dp_0\,dp_1\cdots d(p_{n-2})\,ds\\ &= \frac{1}{(n-1)!}\int_0^p \kappa(s)(p-s)^{n-1}\,ds.\end{aligned}$$

Now, let us po define ${}^{CF}_{N}D^{\sigma^{[n]}}\kappa(p) := \underbrace{{}^{CF}_{N}D^{\sigma}({}^{CF}_{N}D^{\sigma}({}^{CF}_{N}D^{\sigma}\cdots({}^{CF}_{N}D^{\sigma}}_{n\text{ times}}\kappa(p))\cdots))$ for $n \geq 1$, $a,p \in \mathbb{R}$, and $p>0$. Also, we define $J^0\kappa(p) = \int_0^{p^{[0]}}\kappa(s)\,ds = \kappa(p)$,

$$\int_0^{p^{[n]}}\kappa(s)\,ds = \underbrace{\int_0^{p=p_n}\int_0^{s=p_{n-1}}\int_0^{p_{n-2}}\cdots\int_0^{p_1}}_{n\text{ times}}\kappa(p_0)\,dp_0\,dp_1\cdots d(p_{n-2})\,ds = J^n\kappa(p),$$

and

$$\begin{aligned}
\left(a_\sigma + b_\sigma J\kappa(p)\right)^{[n]} &= \left(a_\sigma + b_\sigma \int_0^p \kappa(s)\,ds\right)^{[n]} \\
&= \binom{n}{0} a_\sigma^n b_\sigma^0 \int_0^{p^{[0]}} \kappa(s)\,ds + \binom{n}{1} a_\sigma^{n-1} b_\sigma^1 \int_0^{p^{[1]}} \kappa(s)\,ds \\
&\quad + \cdots + \binom{n}{n-1} a_\sigma^1 b_\sigma^{n-1} \int_0^{p^{[n-1]}} \kappa(s)\,ds + \binom{n}{n} a_\sigma^0 b_\sigma^n \int_0^{p^{[n]}} \kappa(s)\,ds \\
&= \sum_{i=0}^{n} \binom{n}{i} a_\sigma^{n-i} b_\sigma^i \int_0^{p^{[i]}} \kappa(s)\,ds \\
&= \sum_{i=0}^{n} \binom{n}{i} a_\sigma^{n-i} b_\sigma^i J^i \kappa(p).
\end{aligned}$$

The following result shows that our definition is a generalization of the Caputo–Fabrizio derivative.

Lemma 2.1 *Let $\kappa \in H^1(0,b)$, $b > 0$, and $\sigma \in (0,1)$. Then ${}_N^{CF}D^\sigma \kappa(t) = {}^{CF}D^\sigma \kappa(t)$. If $\kappa \in C_{\mathbb{R}}[0,b]$, then there exists a sequence $(\kappa_n)_{n=1}^\infty$ of $H^1(0,b)$ such that ${}_N^{CF}D^\sigma \kappa(t) = \lim_{n\to\infty} {}_N^{CF}D^\sigma \kappa_n(t)$ and $\lim_{\sigma\to 0} {}_N^{CF}D^\sigma \kappa(t) = \kappa(t) - \kappa(0)$.*

Proof Let $\kappa \in H^1(0,b)$. Note that

$$\begin{aligned}
&{}^{CF}D^\sigma \kappa(t) \\
&\quad = \frac{B(\sigma)}{1-\sigma} \int_0^t \exp\left(-\frac{\sigma}{1-\sigma}(t-p)\right) \kappa'(p)\,dp \\
&\quad = \frac{B(\sigma)}{1-\sigma} \exp\left(-\frac{\sigma}{1-\sigma}(t-p)\right) \kappa(p)\Big|_0^t - \frac{B(\sigma)}{1-\sigma} \int_0^t \frac{\sigma}{1-\sigma} \exp\left(-\frac{\sigma}{1-\sigma}(t-p)\right) \kappa(p)\,dp \\
&\quad = \frac{B(\sigma)}{1-\sigma}\kappa(t) - \frac{B(\sigma)}{1-\sigma} \exp\left(-\frac{\sigma}{1-\sigma}t\right)\kappa(0) - \frac{\sigma B(\sigma)}{(1-\sigma)^2} \int_0^t \exp\left(-\frac{\sigma}{1-\sigma}(t-p)\right)\kappa(p)\,dp \\
&\quad = \frac{B(\sigma)}{1-\sigma}\kappa(t) - \frac{B(\sigma)}{1-\sigma} \exp\left(-\frac{\sigma}{1-\sigma}t\right)\kappa(0) - \frac{\sigma B(\sigma)}{(1-\sigma)^2} \int_0^t \exp\left(-\frac{\sigma}{1-\sigma}(t-p)\right)\kappa(p)\,dp \\
&\qquad + \frac{\sigma B(\sigma)}{(1-\sigma)^2} \int_0^t \exp\left(-\frac{\sigma}{1-\sigma}(t-p)\right)\kappa(t)\,dp \\
&\qquad - \frac{\sigma B(\sigma)}{(1-\sigma)^2} \int_0^t \exp\left(-\frac{\sigma}{1-\sigma}(t-p)\right)\kappa(t)\,dp \\
&\quad = \frac{B(\sigma)}{1-\sigma}\left(\kappa(t) - \kappa(0)\right) \exp\left(\frac{-\sigma}{1-\sigma}t\right) \\
&\qquad + \frac{\sigma B(\sigma)}{(1-\sigma)^2} \int_0^t \left(\kappa(t) - \kappa(p)\right) \exp\left(\frac{-\sigma}{1-\sigma}(t-p)\right) dp.
\end{aligned}$$

Now, let $\kappa \in C_{\mathbb{R}}[0,b]$. Choose a sequence of polynomials $\{\kappa_n = P_n\}_{n=1}^\infty$ that converges uniformly to κ. Hence ${}_N^{CF}D^\sigma \kappa(t) = \lim_{n\to\infty} {}_N^{CF}D^\sigma \kappa_n(t)$. Since $P_n \in H^1$, we conclude that $\lim_{\sigma\to 0} {}_N^{CF}D^\sigma \kappa(t) = \lim_{\sigma\to 0}\lim_{n\to\infty} {}^{CF}D^\sigma P_n(t) = \lim_{n\to\infty}\lim_{\sigma\to 0} {}^{CF}D^\sigma P_n(t) = \lim_{n\to\infty}[P_n(t) - P_n(0)] = \kappa(t) - \kappa(0)$. □

Note that if $\kappa \in H^1(0,b)$, then $\lim_{\sigma\to 1} {}^{CF}_{N}D^{\sigma}\kappa(t) = \kappa'(t)$. But this may be not true for $\kappa \in C_{\mathbb{R}}[0,b]$.

Lemma 2.2 *A solution of the problem* ${}^{CF}_{N}D^{\sigma}\kappa(t) = \upsilon(t)$ *such that* $\kappa(0) = 0$ *is of the form* $\kappa(t) = a_\sigma \upsilon(t) + b_\sigma \int_0^t \upsilon(s)\,ds$ *for* $0 < \sigma < 1$.

Proof Note that ${}^{CF}_{N}D^{\sigma}\kappa(t) = \frac{B(\sigma)}{1-\sigma}\kappa(t) - \frac{\sigma B(\sigma)}{(1-\sigma)^2}\int_0^t \exp(-\frac{\sigma}{1-\sigma}(t-s))\kappa(s)\,ds = \upsilon(t)$. Hence $\frac{\sigma B(\sigma)}{(1-\sigma)^2}\int_0^t \exp(\frac{\sigma}{1-\sigma}s)\kappa(s)\,ds = \exp(\frac{\sigma}{1-\sigma}t)[\frac{B(\sigma)}{1-\sigma}\kappa(t) - \upsilon(t)]$. By differentiating both sides we get $\upsilon(t) = \frac{1-\sigma}{\sigma}[\frac{B(\sigma)}{1-\sigma}\kappa(t) - \upsilon(t)]'$. Now by integrating we obtain $\kappa(t) = a_\sigma \upsilon(t) + b_\sigma \int_0^t \upsilon(s)\,ds$. □

It is crucial to check that in the last result the equation ${}^{CF}_{N}D^{\sigma}\kappa(t) = 0$ with $\kappa(0) = 0$ possesses a unique solution. Using this note and Lemma 2.2, we deduce the next result.

Lemma 2.3 *Let* $0 < \sigma < 1$. *Then the unique solution of* ${}^{CF}_{N}D^{\sigma}\kappa(t) = \upsilon(t)$ *with* $\kappa(0) = 0$ *is written as* $\kappa(t) = a_\sigma \upsilon(t) + b_\sigma \int_0^t \upsilon(p)\,dp$.

Lemma 2.4 *Let* $0 < \sigma < 1$ *and* $\kappa(0) = 0$. *Then the unique solution for the problem* ${}^{CF}_{N}D^{\sigma^{[n]}}\kappa(t) = \upsilon(t)$ *is given by* $\kappa(t) = (a_\sigma + b_\sigma J\upsilon(t))^{[n]}$.

Proof Applying Lemma 2.3 to ${}^{CF}_{N}D^{\sigma}\kappa(t) = \upsilon(t)$, we conclude that $\kappa(t) = a_\sigma \upsilon(t) + b_\sigma \int_0^t \upsilon(p)\,dp$. Using Lemma 2.3 for equation ${}^{CF}_{N}D^{\sigma^{[2]}}\kappa(t) = \upsilon(t)$, we get ${}^{CF}_{N}D^{\sigma}\kappa(t) = a_\sigma \upsilon(t) + b_\sigma \int_0^t \upsilon(s)\,ds$, and so

$$\begin{aligned}
\kappa(t) &= a_\sigma\left(a_\sigma \upsilon(t) + b_\sigma \int_0^t \upsilon(s)\,ds\right) + b_\sigma \int_0^t \left(a_\sigma \upsilon(t) + b_\sigma \int_0^s \upsilon(r)\,dr\right) ds \\
&= a_\sigma^2 \upsilon(t) + 2a_\sigma b_\sigma \int_0^t \upsilon(s)\,ds + b_\sigma^2 \int_0^t \int_0^s \upsilon(r)\,dr\,ds = \left(a_\sigma + b_\sigma \int_0^t \upsilon(s)\,ds\right)^{[2]}.
\end{aligned}$$

Now, suppose that $\kappa(t) = (a_\sigma + b_\sigma J\upsilon(t))^{[n]}$ is the solution of ${}^{CF}_{N}D^{\sigma^{[n]}}\kappa(t) = \upsilon(t)$. We prove that $\kappa(t) = (a_\sigma + b_\sigma J\upsilon(t))^{[n+1]}$ is the solution of ${}^{CF}_{N}D^{\sigma^{[n+1]}}\kappa(t) = \upsilon(t)$. If ${}^{CF}_{N}D^{\sigma^{[n]}}({}^{CF}_{N}D^{\sigma}\kappa(t)) = \upsilon(t)$, then ${}^{CF}_{N}D^{\sigma}\kappa(t)) = (a_\sigma + b_\sigma J\upsilon(t))^{[n]}$, and so

$$\begin{aligned}
\kappa(t) &= a_\sigma\big(a_\sigma + b_\sigma J\upsilon(t)\big)^{[n]} + b_\sigma \int_0^t \big(a_\sigma + b_\sigma J\upsilon(s)\big)^{[n]}\,ds \\
&= a_\sigma\left[\binom{n}{0} a_\sigma^n b_\sigma^0 \int_0^{t^{[0]}} \upsilon(s)\,ds + \binom{n}{1} a_\sigma^{n-1} b_\sigma^1 \int_0^{t^{[1]}} \upsilon(s)\,ds \right. \\
&\quad \left. + \cdots + \binom{n}{n-1} a_\sigma^1 b_\sigma^{n-1} \int_0^{t^{[n-1]}} \upsilon(s)\,ds + \binom{n}{n} a_\sigma^0 b_\sigma^n \int_0^{t^{[n]}} \upsilon(s)\,ds\right] \\
&\quad + b_\sigma\left[\binom{n}{0} a_\sigma^n b_\sigma^0 \int_0^{t^{[1]}} \upsilon(s)\,ds + \binom{n}{1} a_\sigma^{n-1} b_\sigma^1 \int_0^{t^{[2]}} \upsilon(s)\,ds \right. \\
&\quad \left. + \cdots + \binom{n}{n-1} a_\sigma^1 b_\sigma^{n-1} \int_0^{t^{[n]}} \upsilon(s)\,ds + \binom{n}{n} a_\sigma^0 b_\sigma^n \int_0^{t^{[n+1]}} \upsilon(s)\,ds\right] \\
&= \binom{n}{0} a_\sigma^{n+1} b_\sigma^0 \int_0^{t^{[0]}} \upsilon(s)\,ds + \left[\binom{n}{1} + \binom{n}{0}\right] a_\sigma^n b_\sigma^1 \int_0^{t^{[1]}} \upsilon(s)\,ds
\end{aligned}$$

$$+\cdots+\left[\binom{n}{n}+\binom{n}{n-1}\right]a_\sigma^1 b_\sigma^n\int_0^{t^{[n]}}\upsilon(s)\,ds+\binom{n}{n}a_\sigma^0 b_\sigma^{n+1}\int_0^{t^{[n+1]}}\upsilon(s)\,ds$$
$$=\binom{n+1}{0}a_\sigma^{n+1}b_\sigma^0\int_0^{t^{[0]}}\upsilon(s)\,ds+\binom{n+1}{1}a_\sigma^n b_\sigma^1\int_0^{t^{[1]}}\upsilon(s)\,ds$$
$$+\cdots+\binom{n+1}{n}+a_\sigma^1 b_\sigma^n\int_0^{t^{[n]}}\upsilon(s)\,ds+\binom{n+1}{n+1}a_\sigma^0 b_\sigma^{n+1}\int_0^{t^{[n+1]}}\upsilon(s)\,ds$$
$$=\big(a_\sigma+b_\sigma J\upsilon(t)\big)^{[n+1]}.$$
□

By using the details of the proof of the last result, we can easily deduce the following results (see [13]).

Lemma 2.5 *Let $\kappa,\upsilon\in C_{\mathbb{R}}[0,1]$. If there exists a real number K such that $|\kappa(t)-\upsilon(t)|\le K$ for all $t\in[0,1]$, then $|{}^{CF}_{N}D^\sigma\kappa(t)-{}^{CF}_{N}D^\sigma\upsilon(t)|\le\frac{(2-\sigma)B(\sigma)}{(1-\sigma)^2}K$ for all $t\in[0,1]$. If $\kappa(0)=\upsilon(0)$, then $|{}^{CF}_{N}D^\sigma\kappa(t)-{}^{CF}_{N}D^\sigma\upsilon(t)|\le\frac{B(\sigma)}{(1-\sigma)^2}K$.*

This result implies that $|{}^{CF}_{N}D^\sigma\kappa(t)|\le\frac{(2-\sigma)B(\sigma)}{(1-\sigma)^2}K$ for all $t\in[0,1]$ whenever $\kappa\in C_{\mathbb{R}}[0,1]$ with $|\kappa(t)|\le K$ for some $K\ge 0$ and all $t\in[0,1]$.

Lemma 2.6 *Suppose that $\kappa,\upsilon\in C_{\mathbb{R}}[0,1]$ and there exists a real number K such that $|\kappa(t)-\upsilon(t)|\le K$ for all $t\in[0,1]$. Then $|{}^{CF}_{N}D^{\sigma^{[n]}}\kappa(t)-{}^{CF}_{N}D^{\sigma^{[n]}}\upsilon(t)|\le(\frac{(2-\sigma)B(\sigma)}{(1-\sigma)^2})^nK$ for all $t\in[0,1]$. If $\kappa(0)=\upsilon(0)$, then $|{}^{CF}_{N}D^{\sigma^{[n]}}\kappa(t)-{}^{CF}_{N}D^{\sigma^{[n]}}\upsilon(t)|\le(\frac{B(\sigma)}{(1-\sigma)^2})^nK$ for all $t\in[0,1]$.*

In [21], the nonlinear fractional differential problem ${}^{CF}D^\sigma\kappa(t)=f(t,\kappa(t))$ with $0\le\sigma<1$ was studied. In the proof of the related result (Theorem 1), the self-map $F:\in C_{\mathbb{R}}[0,1]\to C_{\mathbb{R}}[0,1]$ defined by $(F\kappa)(t)=a_\sigma f(t,\kappa(t))+b_\sigma\int_0^t f(s,\kappa(s))\,ds$ is well defined, and there is no problem. Note that this method of proofs cannot be useful for the problem

$${}^{CF}D^\sigma\kappa(t)=f\big(t,\kappa(t),{}^{CF}D^\sigma\kappa(t)\big)$$

because the map F cannot be defined on the space H^1. Here $\sigma,\beta\in(0,1)$. For finding a new method for solving such problems, we defined a new notion by replacing ${}^{CF}D^\sigma\kappa(t)$ with ${}^{CF}_{N}D^\sigma\kappa(t)$. Note that our extended derivative can only be used for order $\sigma\in(0,1)$. First, we investigate the fractional differential problem

$${}^{CF}_{N}D^\sigma\kappa(t)=f\big(t,\kappa(t),g(t)\,{}^{CF}_{N}D^\beta\kappa(t)\big)\tag{1}$$

with $\kappa(0)=0$, where $\sigma,\beta\in(0,1)$.

Theorem 2.7 *Let $\sigma,\beta\in(0,1)$, and let $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ be a continuous function such that $|f(t,x,y)-f(t,x',y')|\le\eta(t)(|x-x'|+|y-y'|)$ for all $t\in[0,1]$ and $x,y,x',y'\in\mathbb{R}$. Then problem (1) has a unique solution in $H^1(0,1)$ whenever $\Delta=\frac{1}{B(\sigma)}[\eta^*[1+\frac{MB(\beta)}{(1-\beta)^2}]<1$.*

Proof Consider the map $F:\in C_{\mathbb{R}}[0,1]\to C_{\mathbb{R}}[0,1]$ defined by

$$(F\kappa)(t)=a_\sigma f\big(t,\kappa(t),g(t)\,{}^{CF}_{N}D^\beta\kappa(t)\big)+b_\sigma\int_0^t f\big(s,\kappa(s),g(s)\,{}^{CF}_{N}D^\beta\kappa(s)\big)\,ds.$$

By Lemma 2.5 we get

$$\begin{aligned}&\left|f\left(t,\kappa(t),g(t)\,{}^{CF}_{N}D^{\beta}\kappa(t)\right)-f\left(t,\upsilon(t),g(t)\,{}^{CF}_{N}D^{\beta}\upsilon(t)\right)\right|\\&\quad\le\eta^{*}\left(\|\kappa-\upsilon\|+\frac{MB(\beta)}{(1-\beta)^{2}}\|\kappa-\upsilon\|\right)=\eta^{*}\left[1+\frac{MB(\beta)}{(1-\beta)^{2}}\right]\|\kappa-\upsilon\|\end{aligned}$$

for all $\kappa,\upsilon\in C_{\mathbb{R}}[0,1]$ and $t\in[0,1]$. Hence

$$\begin{aligned}\left|(F\kappa)(t)-(F\upsilon)(t)\right|&\le a_{\sigma}\eta^{*}\left[1+\frac{MB(\beta)}{(1-\beta)^{2}}\right]\|\kappa-\upsilon\|+b_{\sigma}\int_{0}^{t}\eta^{*}\left[1+\frac{MB(\beta)}{(1-\beta)^{2}}\right]\|\kappa-\upsilon\|\,ds\\&\le[a_{\sigma}+b_{\sigma}][\eta^{*}\left[1+\frac{MB(\beta)}{(1-\beta)^{2}}\right]\|\kappa-\upsilon\|\\&=\frac{1}{B(\sigma)}[\eta^{*}\left[1+\frac{MB(\beta)}{(1-\beta)^{2}}\right]\|\kappa-\upsilon\|\end{aligned}$$

for all $\kappa,\upsilon\in C_{\mathbb{R}}[0,1]$. Since $\Delta<1$, F is a contraction. By the Banach contraction principle F has a unique fixed point, which is the unique solution for problem (1). □

Let h and g be bounded functions on $[0,1]$ with $M_1=\sup_{t\in I}|h(t)|<\infty$ and $M_2=\sup_{t\in I}|g(t)|<\infty$. Now, we investigate the fractional higher-order series-type differential problem

$$ {}^{CF}_{N}D^{\sigma}\kappa(t)=\sum_{j=0}^{\infty}\frac{{}^{CF}_{N}D^{\varrho^{[j]}}f(t,\kappa(t),(\phi\kappa)(t),h(t)\,{}^{CF}_{N}D^{\nu}\kappa(t),g(t)\,{}^{CF}_{N}D^{\delta}\kappa(t))}{2^{j}} \tag{2}$$

with boundary condition $\kappa(0)=0$, where $\sigma,\nu,\varrho,\delta\in(0,1)$. Note that the functions h and g may be discontinuous. Since the left side of equation (2) is continuous, so is the right side as problem (2) should be a well defined equation (check our example). For this reason, we add the continuity of the function f to the assumptions of the next two results in order equations (2) and (3) to be well defined. Consider the Banach space $C_{\mathbb{R}}[0,1]$ endowed with the norm $\|\kappa\|=\sup_{t\in I}|\kappa(t)|$.

Theorem 2.8 *Let $f:[0,1]\times\mathbb{R}^{4}\to\mathbb{R}$ be a continuous function such that*

$$\left|f(t,x,y,w,v)-f_{1}\left(t,x',y',w',v'\right)\right|\le\xi_{1}\left|x-x'\right|+\xi_{2}\left|y-y'\right|+\xi_{3}\left|w-w'\right|+\xi_{4}\left|v-v'\right|$$

for some nonnegative real numbers ξ_1, ξ_2, ξ_3, ξ_4 and all $x,y,w,v,x',y',w',v'\in\mathbb{R}$ and $t\in[0,1]$. If $\Delta=\frac{1}{B(\sigma)}\sum_{j=0}^{\infty}\frac{1}{2^{j}}(\frac{B(\varrho)}{(1-\varrho)^{2}})^{j}(\xi_{1}+\xi_{2}\gamma_{0}+\xi_{3}\frac{M_{1}B(\nu)}{(1-\nu)^{2}}+\xi_{4}\frac{M_{2}B(\delta)}{(1-\delta)^{2}})<1$, then problem (2) has a unique solution.

Proof Define the map $F:C_{\mathbb{R}}[0,1]\to C_{\mathbb{R}}[0,1]$ by

$$\begin{aligned}(F\kappa)(t)&=a_{\sigma}\sum_{j=0}^{\infty}\frac{{}^{CF}_{N}D^{\varrho^{[j]}}f(t,\kappa(t),(\phi\kappa)(t),h(t)\,{}^{CF}_{N}D^{\nu}\kappa(t),g(t)\,{}^{CF}_{N}D^{\delta}\kappa(t))}{2^{j}}\\&\quad+b_{\sigma}\int_{0}^{t}\sum_{j=0}^{\infty}\frac{{}^{CF}_{N}D^{\varrho^{[j]}}f(s,\kappa(s),(\phi\kappa)(s),h(s)\,{}^{CF}_{N}D^{\nu}\kappa(s),g(s)\,{}^{CF}_{N}D^{\delta}\kappa(s))}{2^{j}}\,ds,\end{aligned}$$

where a_σ and b_σ are introduced in Lemma 1.1. By Lemmas 2.5 and 2.6 we get

$$\begin{aligned}
&\left|\left[\sum_{j=0}^{\infty}\frac{1}{2^j}\,{}^{CF}_{N}D^{\varrho^{[j]}}f\big(t,\kappa(t),(\phi\kappa)(t),h(t)\,{}^{CF}_{N}D^{\nu}\kappa(t),g(t)\,{}^{CF}_{N}D^{\delta}\kappa(t)\big)\right.\right.\\
&\qquad\left.-\sum_{j=0}^{\infty}\frac{1}{2^j}\,{}^{CF}_{N}D^{\varrho^{[j]}}f\big(t,\upsilon(t),(\phi\upsilon)(t),h(t)\,{}^{CF}_{N}D^{\nu}\upsilon(t),g(t)\,{}^{CF}_{N}D^{\delta}\upsilon(t)\big)\right|\\
&\quad\le\sum_{j=0}^{\infty}\frac{1}{2^j}\left(\frac{B(\varrho)}{(1-\varrho)^2}\right)^j\bigg(\xi_1\|\kappa-\upsilon\|+\xi_2\gamma_0\|\kappa-\upsilon\|\\
&\qquad+\xi_3\frac{M_1B(\nu)}{(1-\nu)^2}\|\kappa-\upsilon\|+\xi_4\frac{M_2B(\delta)}{(1-\delta)^2}\|\kappa-\upsilon\|\bigg)\\
&\qquad\le\sum_{j=0}^{\infty}\frac{1}{2^j}\left(\frac{B(\varrho)}{(1-\varrho)^2}\right)^j\left(\xi_1+\xi_2\gamma_0+\xi_3\frac{M_1B(\nu)}{(1-\nu)^2}+\xi_4\frac{M_2B(\delta)}{(1-\delta)^2}\right)\|\kappa-\upsilon\|
\end{aligned}$$

for all $\kappa,\upsilon\in C_{\mathbb{R}}[0,1]$ and $t\in[0,1]$. Hence

$$\begin{aligned}
&\big|(F\kappa)(t)-(F\upsilon)(t)\big|\\
&\quad\le a_\sigma\left|\sum_{j=0}^{\infty}\frac{1}{2^j}\,{}^{CF}_{N}D^{\varrho^{[j]}}f\big(t,\kappa(t),(\phi\kappa)(t),h(t)\,{}^{CF}_{N}D^{\nu}\kappa(t),g(t)\,{}^{CF}_{N}D^{\delta}\kappa(t)\big)\right.\\
&\qquad\left.-\sum_{j=0}^{\infty}\frac{1}{2^j}\,{}^{CF}_{N}D^{\varrho^{[j]}}f\big(t,\upsilon(t),(\phi\upsilon)(t),h(t)\,{}^{CF}_{N}D^{\nu}\upsilon(t),g(t)\,{}^{CF}_{N}D^{\delta}\upsilon(t)\big)\right|\\
&\qquad+b_\sigma\left[\int_0^t\left|\sum_{j=0}^{\infty}\frac{1}{2^j}\,{}^{CF}_{N}D^{\varrho^{[j]}}f\big(s,\kappa(s),(\phi\kappa)(s),h(s)\,{}^{CF}_{N}D^{\nu}\kappa(s),g(s)\,{}^{CF}_{N}D^{\delta}\kappa(s)\big)\right.\right.\\
&\qquad\left.\left.-\sum_{j=0}^{\infty}\frac{1}{2^j}\,{}^{CF}_{N}D^{\varrho^{[j]}}f\big(s,\upsilon(s),(\phi\upsilon)(s),h(s)\,{}^{CF}_{N}D^{\nu}\upsilon(s),g(s)\,{}^{CF}_{N}D^{\delta}\upsilon(s)\big)\right|ds\right]\\
&\quad\le\left(a_\sigma\sum_{j=0}^{\infty}\frac{1}{2^j}\left(\frac{B(\varrho)}{(1-\varrho)^2}\right)^j\left(\xi_1+\xi_2\gamma_0+\xi_3\frac{M_1B(\nu)}{(1-\nu)^2}+\xi_4\frac{M_2B(\delta)}{(1-\delta)^2}\right)]\right.\\
&\qquad\left.+b_\sigma\int_0^t\sum_{j=0}^{\infty}\frac{1}{2^j}\left(\frac{B(\varrho)}{(1-\varrho)^2}\right)^j\left(\xi_1+\xi_2\gamma_0+\xi_3\frac{M_1B(\nu)}{(1-\nu)^2}+\xi_4\frac{M_2B(\delta)}{(1-\delta)^2}\right)ds\right)\\
&\quad\le[a_\sigma+b_\sigma]\left[\sum_{j=0}^{\infty}\frac{1}{2^j}\left(\frac{B(\varrho)}{(1-\varrho)}\right)^j\left(\xi_1+\xi_2\gamma_0+\xi_3\frac{M_1B(\nu)}{(1-\nu)^2}+\xi_4\frac{M_2B(\delta)}{(1-\delta)^2}\right)\right]\|\kappa-\upsilon\|
\end{aligned}$$

for all $\kappa,\upsilon\in C_{\mathbb{R}}[0,1]$ and $t\in[0,1]$. This implies that

$$\|F\kappa-F\upsilon\|\le\frac{1}{B(\sigma)}\sum_{j=0}^{\infty}\frac{1}{2^j}\left(\frac{B(\varrho)}{(1-\varrho)^2}\right)^j\left(\xi_1+\xi_2\gamma_0+\xi_3\frac{M_1B(\nu)}{(1-\nu)^2}+\xi_4\frac{M_2B(\delta)}{(1-\delta)^2}\right)\|\kappa-\upsilon\|$$

for all $\kappa,\upsilon\in C_{\mathbb{R}}[0,1]$. Note that $\Delta<1$, so that F is a contraction. The Banach contraction principle implies that F has a unique fixed point, which is the unique solution for (2). $\square$

Note that we used the boundary conditions $\kappa(0) = 0$ for obtaining the key inequality

$$\begin{aligned}&\left|(F\kappa)(t) - (F\upsilon)(t)\right| \\ &\quad \leq \frac{1}{B(\sigma)} \sum_{j=0}^{\infty} \frac{1}{2^j} \left(\frac{B(\varrho)}{(1-\varrho)^2} \right)^j \left(\xi_1 + \xi_2 \gamma_0 + \xi_3 \frac{M_1 B(\nu)}{(1-\nu)^2} + \xi_4 \frac{M_2 B(\delta)}{(1-\delta)^2} \right) \|\kappa - \upsilon\|.\end{aligned}$$

Let k, s, h, and g be bounded functions on $[0,1]$ with $M_1 = \sup_{t\in I} |k(t)| < \infty$, $M_2 = \sup_{t\in I} |s(t)| < \infty$, $M_3 = \sup_{t\in I} |h(t)| < \infty$, and $M_4 = \sup_{t\in I} |g(t)| < \infty$. Now, we investigate the fractional higher-order series-type differential problem

$$\begin{aligned}&\frac{{}^{CF}_{N}D^{\sigma^{[n]}}\kappa(t) - [\lambda k(t)\,{}^{CF}_{N}D^{\beta^{[m]}}\kappa(t) + \mu s(t)\,{}^{CF}_{N}D^{\rho^{[p]}}\kappa(t)]}{\sum_{j=0}^{\infty}\frac{1}{j!}\,{}^{CF}_{N}D^{\theta^{[j]}} f(t,\kappa(t),(\phi\kappa)(t),h(t)\,{}^{CF}_{N}D^{\nu^{[q]}}\kappa(t),g(t)\,{}^{CF}_{N}D^{\delta^{[r]}}\kappa(t))} \\ &\quad = \prod_{i=1}^{\infty}\left(1 - \frac{[\sum_{j=0}^{\infty}\frac{1}{j!}\,{}^{CF}_{N}D^{\theta^{[j]}} f(t,\kappa(t),(\phi\kappa)(t),h(t)\,{}^{CF}_{N}D^{\nu^{[q]}}\kappa(t),g(t)\,{}^{CF}_{N}D^{\delta^{[r]}}\kappa(t))]^2}{i^2\pi^2}\right) \\ &\qquad + \prod_{i=1}^{\infty}\left(1 - \frac{4[\sum_{j=0}^{\infty}\frac{1}{j!}\,{}^{CF}_{N}D^{\theta^{[j]}} f(t,\kappa(t),(\phi\kappa)(t),h(t)\,{}^{CF}_{N}D^{\nu^{[q]}}\kappa(t),g(t)\,{}^{CF}_{N}D^{\delta^{[r]}}\kappa(t))]^2}{(2i-1)^2\pi^2}\right)\end{aligned} \tag{3}$$

such that $\kappa(0) = 0$, where $\lambda, \mu \geq 0$, $\sigma, \beta, \rho, \theta, \nu, \delta, \in (0,1)$, and n, m, p, q, and r are natural numbers.

Theorem 2.9 *Assume that $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function such that*

$$\begin{aligned}&\left|f(t,x,y,w,z) - f\left(t,x',y',w',z'\right)\right| \\ &\quad \leq \xi_1\left(\left|x - x'\right| + \xi_2\left|y - y'\right| + \xi_3\left|w - w'\right| + \xi_4\left|z - z'\right|\right)\end{aligned}$$

for some nonnegative real numbers ξ_1, ξ_2, ξ_3, ξ_4 and all $x, y, w, z, x', y', w', z' \in \mathbb{R}$ and $t \in [0,1]$. If $\Delta = (\frac{1}{B(\sigma)})^n([\lambda \frac{M_1 B(\beta)}{(1-\beta)^{2m}} + \mu \frac{M_2 B(\rho)}{(1-\rho)^{2p}}] + [e^{\frac{B(\theta)}{(1-\theta)^2}}(\xi_1 + \xi_2\gamma_0 + \xi_3 \frac{M_3 B(\nu)}{(1-\nu)^{2q}} + \xi_4 \frac{M_4 B(\delta)}{(1-\delta)^{2r}})]) < 1$, then problem (3) has a unique solution.

Proof Define $(G\kappa)(s) = \sum_{j=0}^{\infty}\frac{1}{j!}\,{}^{CF}_{N}D^{\theta^{[j]}} f(s,\kappa(s),(\phi\kappa)(s),h(s)\,{}^{CF}_{N}D^{\nu^{[q]}}\kappa(s),g(s)\,{}^{CF}_{N}D^{\delta^{[r]}}\kappa(s))$ for all $\kappa \in C_{\mathbb{R}}[0,1]$ and $s \in [0,1]$. Consider the map $F : C_{\mathbb{R}}[0,1] \to C_{\mathbb{R}}[0,1]$ defined by

$$\begin{aligned}F(\kappa) = \Biggl(a_\sigma + b_\sigma \int_0^t \Biggl[&(G\kappa)(s)\prod_{i=1}^{\infty}\left(1 - \frac{[(G\kappa)(s)]^2}{i^2\pi^2}\right) + (G\kappa)(t)\prod_{i=1}^{\infty}\left(1 - \frac{4[(G\kappa)(t)]^2}{(2i-1)^2\pi^2}\right) \\ &+ \left(\lambda k(s)\,{}^{CF}_{N}D^{\beta^{[m]}}\kappa(s) + \mu s(s)\,{}^{CF}_{N}D^{\rho^{[p]}}\kappa(s)\right)\Biggr]\,ds\Biggr)^{[n]}.\end{aligned}$$

By Lemma 1.2 we have

$$\begin{aligned}{}^{CF}_{N}D^{\sigma^{[n]}}\kappa(t) = {}&(G\kappa)(t)\prod_{i=1}^{\infty}\left(1 - \frac{[(G\kappa)(t)]^2}{i^2\pi^2}\right) + (G\kappa)(t)\prod_{i=1}^{\infty}\left(1 - \frac{4[(G\kappa)(t)]^2}{(2i-1)^2\pi^2}\right) \\ &+ \left(\lambda k(t)\,{}^{CF}_{N}D^{\beta^{[m]}}\kappa(t) + \mu s(t)\,{}^{CF}_{N}D^{\rho^{[p]}}\kappa(t)\right)\end{aligned}$$

$$= \sin(G\kappa)(t) + \cos(G\kappa)(t) + \left(\lambda k(t){}_N^{CF}D^{\beta^{[m]}}\kappa(t) + \mu s(t){}_N^{CF}D^{\rho^{[p]}}\kappa(t)\right)$$
$$= 2^{\frac{1}{2}}\sin\left((G\kappa)(t) + \frac{\pi}{4}\right) + \left(\lambda k(t){}_N^{CF}D^{\beta^{[m]}}\kappa(t) + \mu s(t){}_N^{CF}D^{\rho^{[p]}}\kappa(t)\right)$$

for all $\kappa \in C_{\mathbb{R}}[0,1]$ and $s \in [0,1]$. Hence

$$\begin{aligned}
&\left|(F\kappa)(t) - (F\upsilon)(t)\right| \\
&\leq \left(a_\sigma + b_\sigma \int_0^t \left[2^{\frac{1}{2}}\left|\sin\left((G\kappa)(s) + \frac{\pi}{4}\right) - \sin\left((G\upsilon)(s) + \frac{\pi}{4}\right)\right|\right.\right. \\
&\quad + \left|\left(\lambda k(s){}_N^{CF}D^{\beta^{[m]}}\kappa(s) + \mu s(s){}_N^{CF}D^{\rho^{[p]}}\kappa(s)\right) - \left(\lambda k(s){}_N^{CF}D^{\beta^{[m]}}\upsilon(s)\right.\right. \\
&\quad \left.\left.\left.\left. + \mu s(t){}_N^{CF}D^{\rho^{[p]}}\upsilon(s)\right)\right|\right] ds\right)^{[n]} \\
&\leq \left(a_\sigma + \int_0^t 2^{\frac{1}{2}}\left|(G\kappa)(s) - (G\upsilon)(s)\right|\right) \\
&\quad + \left|\left(\lambda k(s){}_N^{CF}D^{\beta^{[m]}}\kappa(s) + \mu s(s){}_N^{CF}D^{\rho^{[p]}}\kappa(s)\right) - \left(\lambda k(s){}_N^{CF}D^{\beta^{[m]}}\upsilon(s)\right.\right. \\
&\quad \left.\left. + \mu s(t){}_N^{CF}D^{\rho^{[p]}}\upsilon(s)\right)\right| ds)^{[n]} \\
&\leq \left(a_\sigma + \int_0^t 2^{\frac{1}{2}}\left|(G\kappa)(s) - (G\upsilon)(s)\right|\right) \\
&\quad + \lambda\left|k(s)\right|\left|{}_N^{CF}D^{\beta^{[m]}}\left(\kappa(s) - \upsilon(s)\right)\right| + \mu\left|s(s)\right|\left|{}_N^{CF}D^{\rho^{[p]}}\left(\kappa(s) - \upsilon(s)\right)\right| ds)^{[n]}
\end{aligned}$$

for all $\kappa, \upsilon \in C_{\mathbb{R}}[0,1]$ and $t \in [0,1]$. Also, by Lemma 2.6 we get

$$\left|(G\kappa)(s) - (G\upsilon)(s)\right| \leq \sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{B(\theta)}{(1-\theta)^2}\right)^i\left[\xi_1 + \xi_2\gamma_0 + \xi_3\frac{M_3B(\nu)}{(1-\nu)^{2q}} + \xi_4\frac{M_4B(\delta)}{(1-\delta)^{2r}}\right]\|\kappa - \upsilon\|$$

for all $\kappa \in C_{\mathbb{R}}[0,1]$ and $s \in [0,1]$. Thus we get

$$\begin{aligned}
&\|F\kappa - F\upsilon\| \\
&\leq (a_\sigma + b_\sigma)^n\left[\lambda\frac{M_1B(\beta)}{(1-\beta)^{2m}} + \mu\frac{M_2B(\rho)}{(1-\rho)^{2p}}\right] \\
&\quad + (a_\sigma + b_\sigma)^n\left[e^{\frac{B(\theta)}{(1-\theta)^2}}\left(\xi_1 + \xi_2\gamma_0 + \xi_3\frac{M_3B(\nu)}{(1-\nu)^{2q}} + \xi_4\frac{M_4B(\delta)}{(1-\delta)^{2r}}\right)\right]\|\kappa - \upsilon\| \\
&= \left(\frac{1}{B(\sigma)}\right)^n\left(\left[\lambda\frac{M_1B(\beta)}{(1-\beta)^{2m}} + \mu\frac{M_2B(\rho)}{(1-\rho)^{2p}}\right]\right. \\
&\quad \left. + \left[e^{\frac{B(\theta)}{(1-\theta)^2}}\left(\xi_1 + \xi_2\gamma_0 + \xi_3\frac{M_3B(\nu)}{(1-\nu)^{2q}} + \xi_4\frac{M_4B(\delta)}{(1-\delta)^{2r}}\right)\right]\right)\|\kappa - \upsilon\|
\end{aligned}$$

for all $\kappa, \upsilon \in C_{\mathbb{R}}[0,1]$.

Since $\Delta < 1$, F is a contraction. By the Banach contraction principle, F possesses a unique fixed point, which is the unique solution for (3). □

Now, we explicitly show an example to illustrate our aim.

Example 2.1 Consider $\gamma : [0,1] \times [0,1] \to [0,\infty)$ defined by $\gamma(t,s) = \frac{e^{2t-s}}{e}$. Note that $\gamma_0 \leq e$. Put $\sigma = \frac{1}{4}$, $\nu = \frac{1}{4}$, $\delta = \frac{1}{4}$, $\varrho = \frac{1}{8}$, $\xi_1 = \frac{2}{41}$, $\xi_2 = \frac{1}{48}$, $\xi_3 = \frac{1}{e^2}$, and $\xi_4 = \frac{1}{2e^2}$. Let $B(\sigma) = 1$ for $\sigma \in (0,1)$, $h(t) = 1$ for $x \in Q \cap [0,1]$ and $h(t) = 0$ for $x \in Q^c \cap [0,1]$, $g(t) = 0$ for $x \in Q \cap [0,1]$ and $g(t) = 2$ for $x \in Q^c \cap [0,1]$. Then, $M_1 = \sup_{t\in[0,1]} |h(t)| = 1$ and $M_2 = \sup_{t\in[0,1]} |g(t)| = 2$. We further discuss the fractional problem

$$\begin{aligned} {}^{CF}D^{\frac{1}{4}}\kappa(t) = \sum_{j=0}^{\infty} \frac{1}{2^j} {}^{CF}_{N}D^{\varrho^{[j]}} \biggl(t + \frac{2}{41}\kappa(t) + \frac{1}{48}\int_0^t \frac{e^{2t-s}}{e}\kappa(s)\,ds \\ + \frac{1}{e^2}h(t)^{CF}D^{\frac{1}{4}}\kappa(t) + \frac{1}{2e^2}g(t)^{CF}D^{\frac{1}{4}}\kappa(t)\biggr))) \end{aligned} \tag{4}$$

such that $\kappa(0) = 0$. Consider $f(t,x,y,w,v) = t + \frac{2}{41}x + \frac{1}{48}y + \frac{1}{e^2}w + \frac{1}{2e^2}v$ for all $t \in I$ and $x,y,w,v \in \mathbb{R}$. Note that $\Delta = \frac{1}{B(\alpha)}\sum_{j=0}^{\infty} \frac{1}{2^j}(\frac{B(\varrho)}{(1-\varrho)^2})^j(\xi_1 + \xi_2\gamma_0 + \xi_3\frac{M_1B(\nu)}{(1-\nu)^2} + \xi_4\frac{M_2B(\delta)}{(1-\delta)^2}) < 0.3 < 1$. By Theorem 2.8 we conclude that problem (4) has a unique solution.

3 Conclusion

The nonlinear fractional differential problem ${}^{CF}D^{\sigma}\kappa(t) = f(t,\kappa(t))$ with $0 \leq \sigma < 1$ has been studied in some works. The method used in the proofs cannot be used for the problem ${}^{CF}D^{\sigma}\kappa(t) = f(t,\kappa(t), {}^{CF}D^{\sigma}\kappa(t))$ for technical reasons. For finding a new method for solving such problems, we define a new extended fractional derivative ${}^{CF}_{N}D^{\sigma}\kappa(t)$ replacing ${}^{CF}D^{\sigma}\kappa(t)$, and we study two higher-order series-type fractional differential equations involving the extended derivative. We emphasize that our extended derivative can be used only for order $\sigma \in (0,1)$.

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