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Sturm-Liouville difference equations having Bessel and hydrogen atom potential type

Abstract: In this work, we bring a different approach for Sturm-Liouville problems having Bessel and hydrogen atom type and we provide a basis for direct and inverse problems. From this point of view, we find representations of solutions and asymptotic expansions for eigenfunctions. Furthermore, some numerical estimations are given to illustrate the necessity of the Sturm-Liouville difference equations with the potential function for the convenience to the spectral theory. The behavior of eigenfunctions for the Sturm-Liouville problem having Bessel and hydrogen atom potential type is analyzed and compared to each other. And then, comparisons are showed by tables and figures.

Keywords: Sturm-Liouville; Difference equation; Bessel potential; Hydrogen atom potential; Schrödinger equation

PACS: 02.30.Tb; 02.90.+p; 02.30.Zz

1 Introduction

The Sturm-Liouville problem is a one-particle Schrödinger equation, and it has an important place in mathematical physics. The potential function $q(t)$ determines the type of equation, like Bessel and hydrogen atom equations. In this study, we consider Sturm-Liouville problems with two different potentials and investigate the behaviors of its eigenfunctions. The Liouville normal form of Bessel equations, also called Sturm-Liouville equation having Bessel poten-

tial type, is studied in [1]–[9] , and it is defined as follows,

$$y'' + \left(\lambda - \frac{p^2 - \frac{1}{4}}{x^2} - q(x) \right) y = 0.$$

Bessel functions are obtained by series solutions while $q(x) = 0$ in the equation above, accordingly, the equation above is more general than classical Bessel equation. As $q(x)$ changes, the structure of the equation will change and hence, a different investigation will require. Lately, the Bessel difference equation has been studied by [10], and obtained discrete Bessel functions, which are the discrete analogue of Bessel functions. The radial Schrödinger equation, also called hydrogen atom equation, is studied by [3, 7, 8, 11, 12], and it is defined as follows,

$$y'' + \left(\lambda - \frac{\ell(\ell+1)}{x^2} + \frac{2}{x} - q(x) \right) y = 0.$$

Physical interpretation of hydrogen atom and Bessel equations:

Bessel equations seem to have a lot of application areas in physics and mathematics. For example, the solution of two-dimensional wave equation, the heat equation and the Dirichlet problem in a cylinder are obtained with the help of Bessel functions [4, 13, 14]. Also, vibrational, gravitational and electromagnetic potential problems with cylindrical symmetry, diffraction problems (astronomy) resolving power of optical instruments, heavy chain, certain subjects in chemistry and biochemistry are expressed by the means of Bessel equations [5].

The radial Schrödinger equation enables to calculate the development of quantum systems with time, also it can give analytical solutions for the non-relativistic hydrogen atom.

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\ell(\ell+1)}{r^2}R + \left(E + \frac{2}{r} \right) R = 0 \qquad (0 < r < \infty),$$

where R is the distance of the mass center to the origin, ℓ is the orbital quantum number and a positive integer, E is energy constant and r is the distance between the nucleus with the electron.

The hydrogen atom is a system consists of a two-particle system, and it forms of one electron and one proton. Internal motion of two particles around the center of

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mass corresponds to the movement of a single particle by a reduced mass. If $R = \frac{y}{x}$ and $E = \lambda$ is taken in the equation above, it forms of

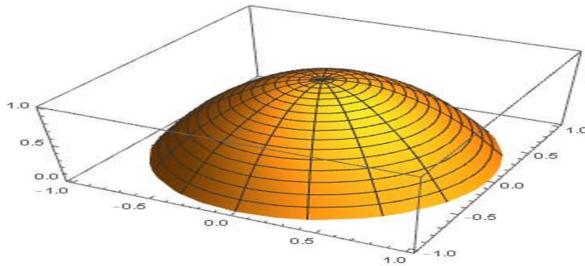$$y'' + \left(\lambda - \frac{\ell(\ell+1)}{x^2} + \frac{2}{x} - q(x) \right) y = 0.$$

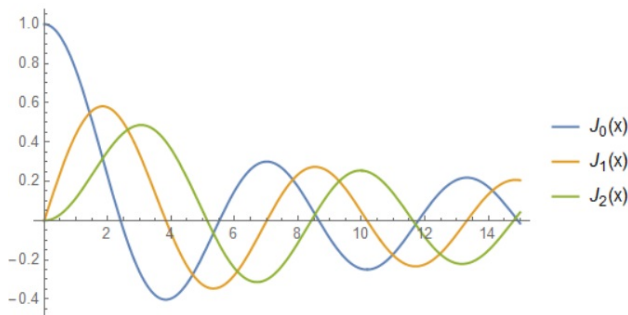**Figure 1:** Bessel functions are the radial part of modes of vibrations of a circular drum[9]

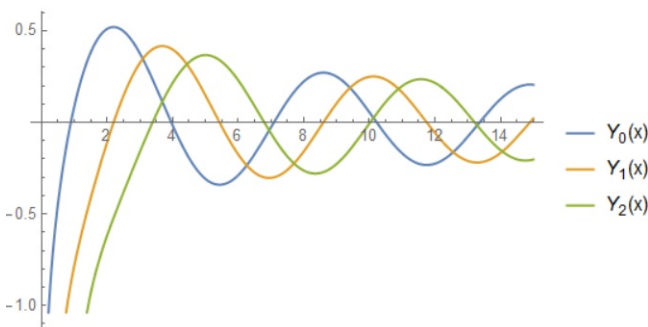**Figure 2:** The Bessel functions of the first kind

Figure 3: The Bessel functions of the second kind

Difference equations have always been an interesting subject due to the fact that the discrete analogue of differential equations. The theory of linear ordinary difference equations was improved by [15]–[19]. Recently, spectral analysis of difference equations has attracted great attention. Especially, many scientists study on Sturm-Liouville difference equations, see [16]–[18],[20]–[25].

The zeros of Bessel and hydrogen atom problems cannot be calculated directly because of having closed form solutions, and this type of solution is called the representation of solution. Accordingly, solution function $y(x)$, also called eigenfunction, can be found only by asymptotic estimations. Based on this, the number of eigenvalues λ can be found only by asymptotic estimations. Thereafter, the other spectral data, like norming constants, normalized eigenfunctions and spectral function, are found by asymptotic estimations. If we close attention, we have no knowledge about potential function $q(x)$ except for it is a continuous function. This type of problem is named "direct problem". Viceversa, while there is knowledge about spectral data, one is made estimations about potential function $q(x)$ and this type of problem is named "inverse problem". Additionally, Bessel functions are obtained by series solutions while $q(x) = 0$ in the equation above, accordingly the equation above is more general than classical Bessel equation. As $q(x)$ changes, structure of the equation will change and hence, a different investigation will require. Our main aim is to apply the spectral theory of these type of differential equations to the discrete case.

In this paper, we are concerned with the discrete analogue of Sturm-Liouville equations having hydrogen atom and Bessel potentials, and we provide a basis for direct and inverse problems. From this point of view, we obtain representation of solutions, asymptotic estimations of eigenfunctions and some numerical estimations about behaviors of eigenfunctions and eigenvalues. The numerical results for the eigenfunctions corresponding to the certain significant eigenvalues for Sturm-Liouville problem having Bessel and hydrogen atom potential type are shown and compared to the each other. The integral representation and asymptotic formulae for eigenfunctions of Sturm-Liouville differential problem are found in [3]. Similarly, the sum representation and asymptotic formulae for eigenfunctions of Sturm-Liouville difference problem are acquired in [26]–[28]. Also, numerical computations of Sturm-Liouville problem are considered in [21], [29].

Now, let's introduce Liouville normal form of Bessel's difference equations

$$\Delta^2 u(n-1) + \left(\lambda + q(n) - \frac{p^2 - \frac{1}{4}}{n^2}\right) u(n) = 0, \quad n = 1, \dots, b, \tag{1}$$

where $p \in \mathbb{R}$, $u(n)$, $q(n) \in l^2[0, b]$, b is a finite integer, Δ is the forward difference operator, $\Delta x(n) = x(n+1) - x(n)$, assume that λ is the positive spectral parameter, $q(n) - \frac{p^2 - \frac{1}{4}}{n^2}$ are called potential function, n is a finite integer. Then, let's introduce hydrogen atom difference equation

$$\Delta^2 v(n-1) + \left(\lambda - q(n) + \frac{2}{n} - \frac{\ell(\ell+1)}{n^2}\right) v(n) = 0,$$
$$n = 1, \dots, b, \tag{2}$$

where ℓ is the orbital quantum number and a positive integer, $v(n) \in \ell^2[0, b]$, $q(n)$, b, λ and n is as defined above, $-q(n) + \frac{2}{n} - \frac{\ell(\ell+1)}{n^2}$ are called potential function.

Our object is to give discrete analogue results to the studies mentioned before [2], [3], [26]–[28], [30].

2 Preliminaries

Definition 1. *[25] The matrix of Casoratian is defined by*

$$w(n) =$$
$$\begin{pmatrix} u_1(n) & u_2(n) \dots & u_r(n) \\ u_1(n+1) & u_2(n+1) \dots & u_r(n+1) \\ \vdots & \vdots & \vdots \\ u_1(n+r-1) & u_2(n+r-1) \dots & u_r(n+r-1) \end{pmatrix}$$

where $u_1(n), u_2(n), \dots, u_r(n)$ are known functions. The determinant

$$W(n) = \det w(n)$$

is called Casoratian.

Theorem 1. *[16] (Wronskian-Type Identity) Assume y and y are linearly independent solutions of Sturm-Liouville difference equation, mentioned in [26]. Then, for $a \le n \le b$*

$$W[y, z](n) = [y(n)\Delta z(n-1) - y(n)\Delta z(n-1)] \tag{3}$$
$$= -[y(n)z(n-1) - y(n-1)z(n)]$$

is a constant (Particularly equal to $W[r, u](a)$).

Definition 2. *Let's define Sturm-Liouville problem having Bessel potential type (1) as following, $n \in [1, b]$*

$$L_1 u(n) = \Delta^2 u(n-1) + \left(q(n) - \frac{p^2 - \frac{1}{4}}{n^2}\right) u(n) = -\lambda u(n), \tag{4}$$

with initial conditions

$$u(1) = -h, \quad u(2) = 1, \tag{5}$$

where L_1 is a self-adjoint difference operator, $h = \cot \alpha$, $\alpha \in \mathbb{R}$.

Definition 3. *Let's define Sturm-Liouville problem having hydrogen atom potential (2) as following, $n \in [1, b]$,*

$$L_2 v(n) = \Delta^2 v(n-1) + \left(-q(n) + \frac{2}{n} - \frac{\ell(\ell+1)}{n^2}\right) v(n)$$
$$= -\lambda v(n), \tag{6}$$

with initial conditions

$$v(1) = -h, \quad v(2) = 1, \tag{7}$$

where L_2 is a self-adjoint difference operator, $h = \cot \alpha$, $\alpha \in \mathbb{R}$.

Theorem 2. *[25] (**Summation by parts**) Assume $m < n$, then*

$$\sum_{k=m}^{n-1} u(k)\Delta r(k) = [u(k)r(k)]_m^n - \sum_{k=m}^{n-1} \Delta u(k) r(k+1). \tag{8}$$

Theorem 3. *[25] Assume z_n is an indefinite sum of u_n, then*

$$\sum_{k=m}^{n-1} z(k) = u(n) - u(m). \tag{9}$$

3 Main results

In this paper, we are concerned with the discrete analogue of Sturm-Liouville problem with Bessel and hydrogen atom potentials $(6)-(7)$, $(4)-(5)$, and we provide a basis for direct and inverse problems. From this point of view, we obtain the representation of solutions, asymptotic estimations of eigenfunction and some numerical estimations about behaviors of eigenfunctions and eigenvalues. The numerical results for eigenfunctions corresponding to the certain significant eigenvalues for Sturm-Liouville problem having Bessel and hydrogen atom potential type are shown and compared to the each other. Our main aim is to apply the spectral theory of these type of differential equations to the discrete case.

In $\ell^2(1, b)$, the Hilbert space of sequences of complex numbers $u(1), \dots, u(b)$ with the inner product,

$$< u(n), r(n) > = \sum_{n=1}^{b} u(n) r(n),$$

for every $u \in D_L$, let's define as follows

$$D_L = \left\{ u(n), \, v(n) \in \ell^2(1, b) : L_1 u(n), \right.$$
$$L_2 v(n) \in \ell^2(1, b), \, u(1) = v(1) = -h,$$
$$\left. u(2) = v(2) = 1. \right\}$$

In this section, we show the sum representations of solutions of Sturm-Liouville problem having Bessel and hydrogen atom potential by the variation of parameters method and prove that the solutions obtained hold for the problems.

Theorem 4. *Sturm-Liouville problem having Bessel potential type is defined as follows*

$$\Delta^2 \varphi(n-1) + \left(q(n) - \frac{p^2 - \frac{1}{4}}{n^2} \right) \varphi(n) = -\lambda \varphi(n), \quad (10)$$

$$\varphi(1) = -h, \quad \varphi(2) = 1, \quad (11)$$

then the problem $(10) - (11)$ *has a unique solution for* $\varphi(n)$ *as follows,*

$$\varphi(n, \lambda) = \left(\frac{-h \sin 2\theta - \sin \theta \left[1 + h \left(q(1) + p^2 - \frac{1}{4} \right) \right]}{\sin \theta} \right) \cos n\theta$$

$$+ \left(\frac{\cos \theta \left[1 + h \left(q(1) + p^2 - \frac{1}{4} \right) \right] + h \cos 2\theta}{\sin \theta} \right) \sin n\theta$$

$$+ \frac{1}{\sin \theta} \sum_{i=1}^{n-1} \left(-q(i) + \frac{p^2 - \frac{1}{4}}{i^2} \right) \varphi(i) \sin(n-i)\theta,$$

where, $\sum_{i=a}^{b} . = 0$ *if* $a > b$.

Proof. We restrict the eigenvalues with $0 < \lambda < 4$ to be able to obtain more advantageous results for the spectral theory. Let $u_1(n)$ and $u_2(n)$ be linearly idependent solutions for homogeneous part of (10), so by characteristic equation, we can find characteristic roots, and since $|\lambda - 2| < 2$, we can take

$$\lambda = 2 - 2\cos\theta,$$

from here, the solution of the homogeneous part of (10) is

$$\varphi_h(n) = c_1 \cos n\theta + c_2 \sin n\theta.$$

By means of variation of parameters method for difference equations [2, 17], we get

$$\varphi_p(n) = c_1(n) \cos n\theta + c_2(n) \sin n\theta, \quad (12)$$

if we insert the equation (12) in the equation (10) and make necessary operations, then we can find the parameters as,

$$c_1(n) = \sum_{i=1}^{n} \frac{\left[-q(i) + \frac{p^2 - \frac{1}{4}}{i^2} \right] \varphi(i) \varphi_2(i)}{W},$$

$$c_2(n) = -\sum_{i=1}^{n} \frac{\left[-q(i) + \frac{p^2 - \frac{1}{4}}{i^2} \right] \varphi(i) \varphi_1(i)}{W},$$

where $W = \sin\theta$ by Theorem 1.

Eventually, we get the general solution

$$\varphi(n, \lambda) = c_1 \cos n\theta + c_2 \sin n\theta$$

$$+ \frac{1}{\sin \theta} \sum_{i=1}^{n} \left[-q(i) + \frac{p^2 - \frac{1}{4}}{i^2} \right] \varphi(i) \sin(n-i)\theta,$$

by the initial conditions, we get the sum representation of solution, $c_1 \neq 0, \, c_2 \neq 0$,

$$\varphi(n, \lambda) = \left(\frac{-h \sin 2\theta - \sin \theta \left[1 + h \left(q(1) + p^2 - \frac{1}{4} \right) \right]}{\sin \theta} \right) \cos n\theta$$

$$+ \left(\frac{\cos \theta \left[1 + h \left(q(1) + p^2 - \frac{1}{4} \right) \right] + h \cos 2\theta}{\sin \theta} \right) \sin n\theta +$$

$$+ \frac{1}{\sin \theta} \sum_{i=1}^{n-1} \left(-q(i) + \frac{p^2 - \frac{1}{4}}{i^2} \right) \varphi(i) \sin(n-i)\theta.$$

We can show that the solution holds for the problem $(10) - (11)$ by inserting in (10) by virtue of Theorem 2 and Theorem 3.

The proof completes. $\qquad\qquad \square$

Theorem 5. *Sturm-Liouville problem having hydrogen atom potential type is defined as follows*

$$\Delta^2 \psi(n-1) + \left(-q(n) + \frac{2}{n} - \frac{\ell(\ell+1)}{n^2} \right) \psi(n) = -\lambda \psi(n), \quad (14)$$

$$\psi(1) = -h, \quad \psi(2) = 1, \quad (15)$$

then the problem $(14) - (15)$ *has a unique solution for* $\psi(n)$ *as follows,*

$$\psi(n, \lambda) =$$
$$\left(\frac{-h \sin 2\theta - \sin \theta \left[1 + h \left(q(1) + 2 + \ell(\ell+1) \right) \right]}{\sin \theta} \right) \cos n\theta$$

$$+ \left(\frac{\cos \theta \left[1 + h \left(q(1) + 2 + \ell(\ell+1) \right) \right] + h \cos 2\theta}{\sin \theta} \right) \sin n\theta$$

$$+ \frac{1}{\sin \theta} \sum_{i=1}^{n-1} \left(q(i) + \frac{2}{i} + \frac{\ell(\ell+1)}{i^2} \right) \psi(i) \sin(n-i)\theta.$$

where, $\sum_{i=a}^{b} . = 0$ *if* $a > b$.

Proof. Similar arguments used in the proof of Theorem 5 are applied to easily obtain the conclusion. $\qquad \square$

4 Asymptotic formulas for Bessel and hydrogen atom difference equations

In this section, we give the asymptotic formulas for the eigenfunctions of Bessel and hydrogen atom potential, respectively.

Theorem 6. *Sturm-Liouville problem having Bessel potential* (10) − (11) *has the estimate*

$$\varphi(n) = O\left(e^{|\tau|n}\right),$$

and more precisely

$$\varphi(n) = \left(\frac{-h\sin 2\theta - \sin\theta\left[1 + h\left(q(1) + p^2 - \frac{1}{4}\right)\right]}{\sin\theta}\right)\cos n\theta$$
$$+ O\left(\frac{e^{|\tau|n}}{\sin\frac{\theta}{2}\sin\theta}\right),$$

where $n \in \mathbb{Z}^+$, $\{\varphi(n)\}$ *is a complex sequence,* $\sum_{i=0}^{\infty} i\,|q(i)| < \infty$

Proof. Let's insert $\varphi(n) = f(n)\,e^{|\tau|n}$ in (10). Hence, we get

$$f(n) = \left(\frac{-h\sin 2\theta - \sin\theta\left[1 + h\left(q(1) + p^2 - \frac{1}{4}\right)\right]}{\sin\theta}\right)e^{-|\tau|n}$$
$$+ \left(\frac{\cos\theta\left[1 + h\left(q(1) + p^2 - \frac{1}{4}\right)\right] + h\cos 2\theta}{\sin\theta}\right)e^{-|\tau|n}$$
$$+ \frac{1}{\sin\theta}\sum_{i=1}^{n-1}\left(-q(i) + \frac{p^2 - \frac{1}{4}}{i^2}\right)f(i)\,e^{|\tau|(n-i)}\sin(n-i)\theta.$$

Let $\xi = \max_{1\le n\le\infty}|f(n)|$, then if we insert the last equation in that

$$\xi \le 2\left(|h| + 1 + |h|\left(|q(1)| + p^2 + \frac{1}{4}\right)\right)$$
$$+ \frac{\xi}{|\sin\theta|}\sum_{i=1}^{\infty}\left(|q(i)| + \frac{p^2 + \frac{1}{4}}{i^2}\right),$$

and thus,

$$\xi \le \frac{2\left(|h| + 1 + |h|\left(|q(1)| + p^2 + \frac{1}{4}\right)\right)}{1 - \frac{1}{|\sin\theta|}\sum_{i=1}^{\infty}\left|\,|q(i)| + \frac{p^2 + \frac{1}{4}}{i^2}\right|},$$

the proof completes under the assumption that nominator of the last inequality is large enough and

$$|\sin\theta| > \sum_{i=1}^{\infty}\left|\,|q(i)| + \frac{p^2 + \frac{1}{4}}{i^2}\right|.$$

The proof is complete under the assumption.   □

Theorem 7. *Sturm-Liouville problem having hydrogen atom potential* (14) − (15) *has the estimate*

$$\psi(n) = O\left(e^{|\tau|n}\right),$$

and more precisely

$$\psi(n) = \left(\frac{-h\sin 2\theta - \sin\theta\left[1 + h\left(q(1) + 2 + \ell(\ell+1)\right)\right]}{\sin\theta}\right)\cos n\theta$$
$$+ O\left(\frac{e^{|\tau|n}}{\sin\frac{\theta}{2}\sin\theta}\right),$$

where $n \in \mathbb{Z}^+$, $\{\psi(n)\}$ *is a complex sequence,* $\sum_{i=0}^{\infty} i\,|q(i)| < \infty$.

*Proof.* Similar arguments used in the proof of Theorem 6 are applied to easily obtain the conclusion.   □

5 Applications

Assuming that $h = p = 1$, $l = 2$, $b = 24$ in all of the applications.

Application 1. Taking $q_1(n) = \dfrac{1}{\sqrt{n}}$ *in the problems* (10) − (11) *and* (14) − (15), *then we have the eigenfunctions as in Figure 4, Figure 5, Figure 6, Table 1, Table 2, Table 3*

Application 2. Taking $h = p = l = 1$, $b = 24$ *and* $q_2(n) = 1$ *in the problem* (10) − (11) *and* (14) − (15), *we have the eigenfunctions as in Table4, Table5.*

Application 3. Taking $h = p = 1$, $l = 2$, $b = 24$ *and* $q_3(n) = \dfrac{1}{n}$ *in the problem* (10) − (11) *and* (14) − (15), *we have the eigenfunctions as in Fig8−Fig17.*
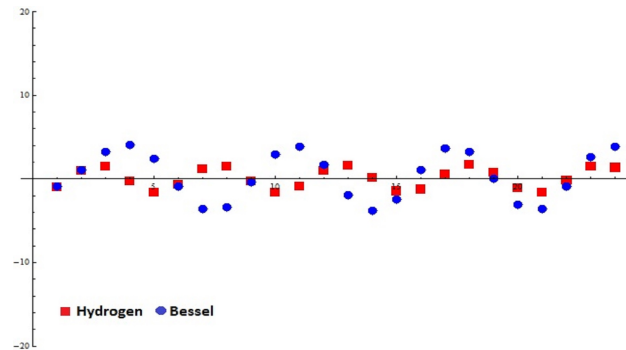
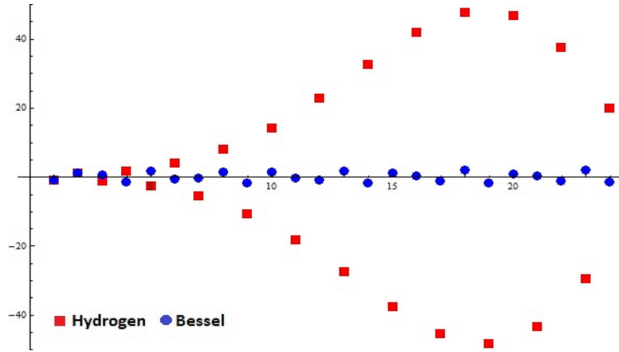**Figure 4:** Comparison of datas in Table 1
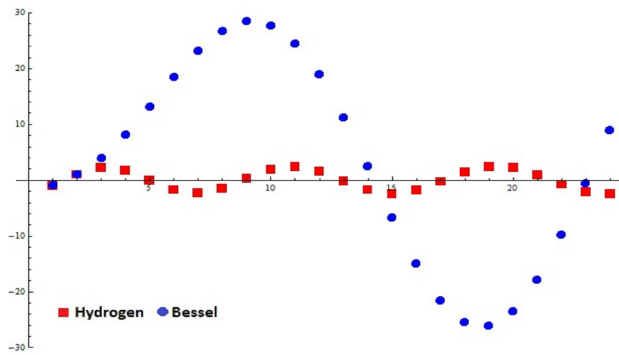
Figure 5: Comparison of datas in Table 3

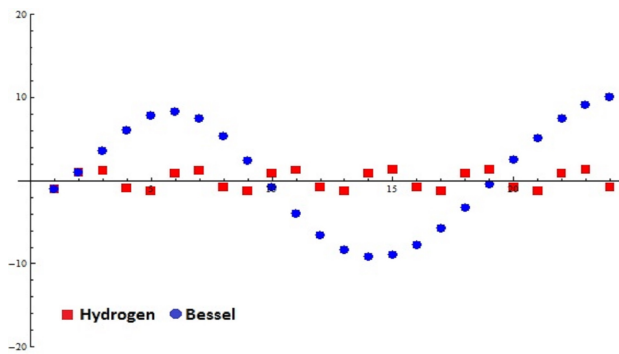**Figure 6:** Comparison of datas in Table 2

Figure 7: Comparison of datas in Table 4

Table 1

Eigenfunctions correspond to $\lambda_1 = 1$		
n	Bessel: $\varphi(n)$	Hydrogen: $\psi(n)$
1	−1	−1
2	1	1
3	1.48	3.207
4	−0.25	4.058
5	−1.617	2.373
6	−0.691	−1.003
13	1.59	−1.960
14	0.161	−3.86
15	−1.471	−2.50
16	−1.257	0.981
21	−1.586	−3.62
22	−0.162	−0.937
23	1.458	2.556
24	1.318	3.834

Table 2

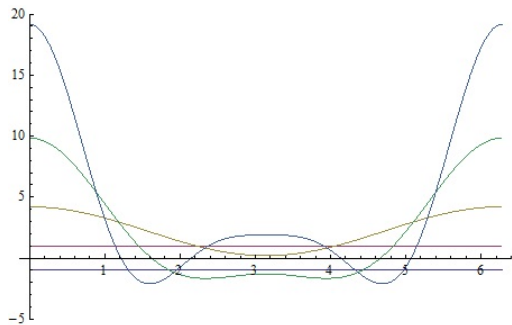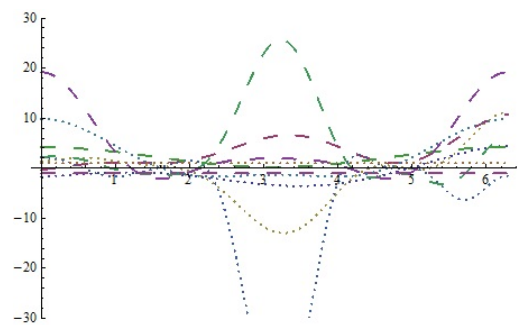| Eigenfunctions correspond to $\lambda_2 = 2 - \sqrt{3}$		
n	Bessel: $\varphi(n)$	Hydrogen: $\psi(n)$
1	−1	−1
2	1	1
3	2.21244	3.93916
4	1.73908	8.09709
5	0.0117111	13.1218
6	−1.72368	18.3994
13	−0.145179	11.2468
14	−1.75401	2.4093
15	−2.43079	−6.70029
16	−1.83671	−15.0299
21	0.930821	−17.9065
22	−0.785314	−9.8949
23	−2.12481	−0.564677
24	−2.45493	8.84181

Figure 8: $\psi(n)$, $n = 1, 2, \ldots, 5$

Figure 9: Comparison of $\varphi(n)$, $n = 1, 2, \ldots, 5$

Table 3

Eigenfunctions correspond to $\lambda_3 = 2 + \sqrt{3}$		
n	Bessel: $\varphi(n)$	Hydrogen: $\psi(n)$
1	−1	−1
2	1	1
3	−1.25166	0.475056
4	1.78627	−1.54855
5	−2.65167	1.6264
6	3.91286	−0.801338
13	−27.6413	1.69851
14	32.6601	−1.76153
15	−37.5314	1.07949
16	41.9117	0.055379
21	−43.325	0.271728
22	37.5582	−1.32168
23	−29.6769	1.83948
24	19.9896	−1.61992

Table 5

Eigenfunctions correspond to $\lambda_5 = 2 - \sqrt{3}$		
n	Bessel: $\varphi(n)$	Hydrogen: $\psi(n)$
1	−1	−1
2	1	1
3	2.41955	3.73205
4	2.5859	6.70812
5	1.53409	8.72527
6	−0.189564	8.7535
13	2.73757	−4.85591
14	2.81137	−0.212425
15	1.94182	4.49665
16	0.428975	7.82098
21	−1.20795	−6.02999
22	0.42675	−8.51653
23	1.92836	−8.43953
24	2.83217	−5.82995

Table 4

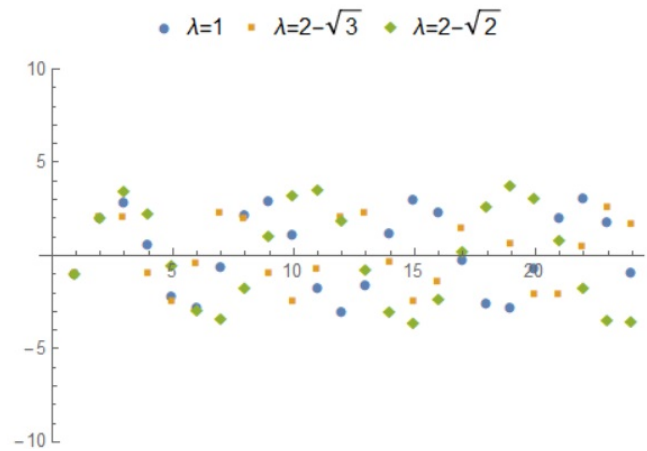| Eigenfunctions correspond to $\lambda_1 = 1$		
n	Bessel: $\varphi(n)$	Hydrogen: $\psi(n)$
1	−1	−1
2	1	1
3	1.1875	3.5
4	−0.901042	6
5	−1.22974	7.75
6	0.86415	8.26
13	−1.26819	−8.39808
14	0.819945	−9.19431
15	1.27133	−8.95851
16	−0.815707	−7.76715
21	−1.27712	5.13985
22	0.807579	7.3874
23	1.27837	9.05495
24	−0.805767	10.0378

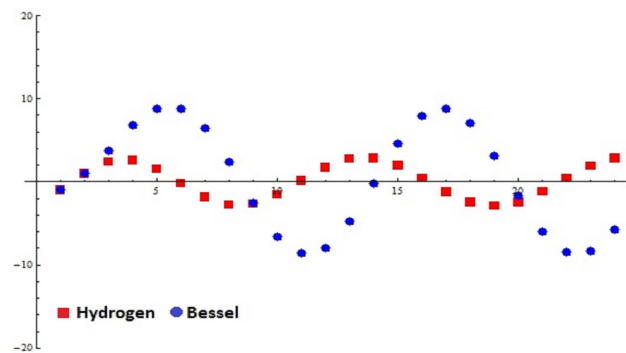**Figure 10:** Comparison of $\varphi(n)$ and $\psi(n)$,n=1,2,...,5

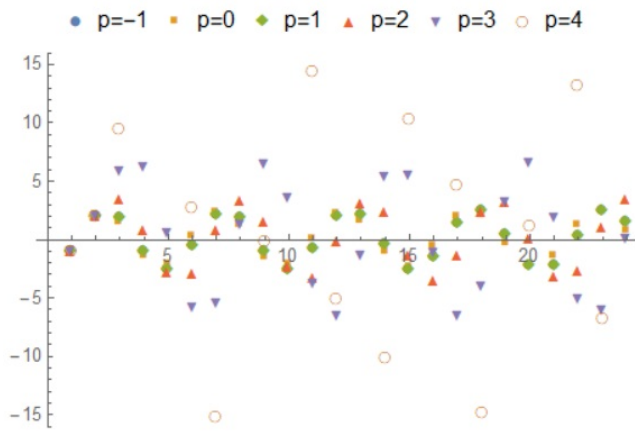**Figure 11:** Comparison of datas in Table 5

Figure 12: Comparison of $varphi(n)$ for different p values, $\lambda = 1$, $q = \frac{1}{\sqrt{n}}$
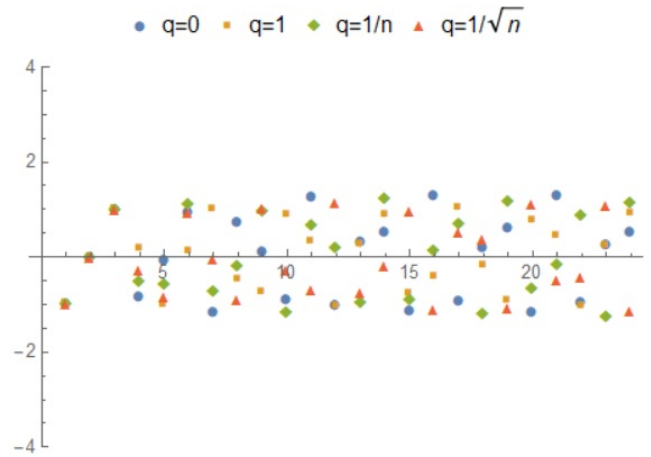

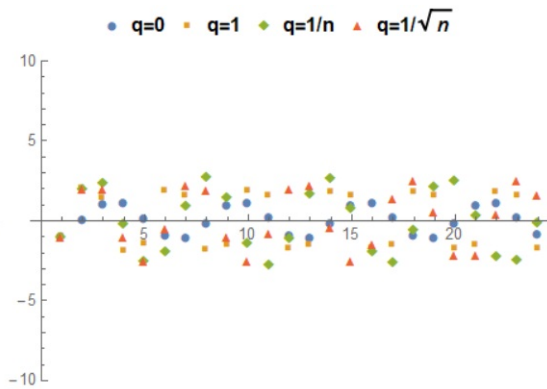**Figure 15:** Comparison of $\psi(n)$ for different potentials $\lambda = 2 + \sqrt{3}$, $l = 1$

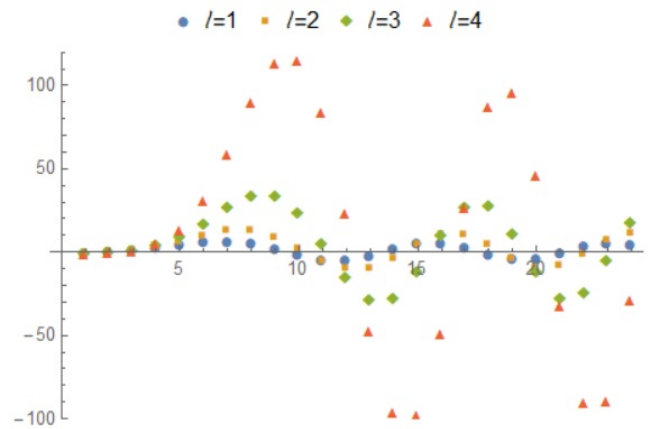**Figure 13:** Comparison of $\varphi(n)$ for different potentials, $\lambda = 1$

Figure 16: Comparison of $\psi(n)$ for different l orbital quantum numbers, $\lambda = 2 + \sqrt{3}$, $q = \frac{1}{\sqrt{n}}$
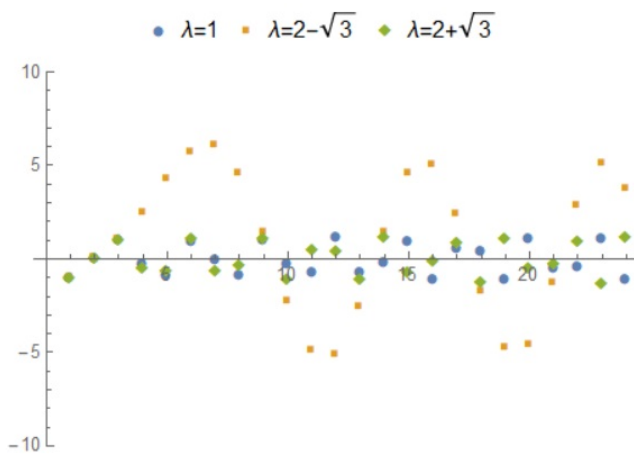

Figure 14: Comparison of $\psi(n)$ for different eigenvalues, $q = \frac{1}{\sqrt{n}}$
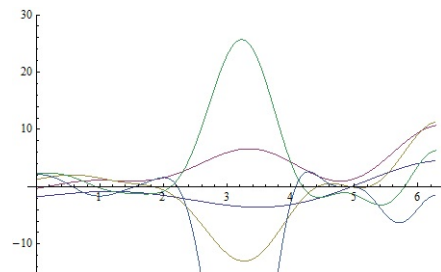

Figure 17: $\varphi(n)$, $n = 1, 2, \ldots, 5$

6 Conclusion

In this study, we are concerned with discrete analogue of Sturm-Liouville problem with Bessel and hydrogen atom potentials $(6) - (7)$, $(4) - (5)$, and we provide a basis for direct and inverse problems. From this point of view, we obtain representation of solutions, asymptotic estimations of eigenfunction and some numerical estimations about behaviors of eigenfunctions and eigenvalues. The numerical results for eigenfunctions corresponding to the certain significant eigenvalues for Sturm-Liouville problem having Bessel and hydrogen atom potential type are shown and compared to the each other. Our main aim is to apply the spectral theory of these type of differential equations to the discrete case.

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