A MODIFICATION FRACTIONAL VARIATIONAL ITERATION METHOD FOR SOLVING NON-LINEAR GAS DYNAMIC AND COUPLED KdV EQUATIONS INVOLVING LOCAL FRACTIONAL OPERATORS

by

Dumitru BALEANU[a,b,c]***, Hassan Kamil JASSIM***[d]***, and Hasib KHAN***[a,e*]

[a] College of Engineering Mechanics and Soft Materials, Hohai University, Jiangning, Nanjing, China
[b] Department of Mathematics, Faculty of Arts and Sciences, Cankaya University, Ankara, Turkey
[c] Institute of Space Sciences, Magurele-Bucharest, Romania
[d] Department of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar, Nasiriyah, Iraq
[e] Department of Mathematics, Shaheed BB University, Sheringal, Dir Upper, Khyber Pakhtunkhwa, Pakistan

Original scientific paper
https://doi.org/10.2298/TSCI170804283B

In this paper, we apply a new technique, namely local fractional variational iteration transform method on homogeneous/non-homogeneous non-linear gas dynamic and coupled KdV equations to obtain the analytical approximate solutions. The iteration procedure is based on local fractional derivative and integral operators. This method is the combination of the local fractional Laplace transform and variational iteration method. The method in general is easy to implement and yields good results. Illustrative examples are included to demonstrate the validity and applicability of the new technique.

Key words: *coupled KdV equation, non-linear gas dynamic equation, local fractional variational iteration method, local fractional Laplace transform, local fractional operator*

Introduction

The variational iteration method was first proposed by He [1, 2] and was applied to deal with Helmholtz equations in [3], Burger's and coupled Burger's equations in [4], Klein-Gordon equations in [5], in KdV in [6], the oscillation equations in [7], Schrodinger equation in [8], diffusion equation in [9], Bernoulli equation in [10], and others. The extended variational iteration method, called the fractional variational iteration method, was developed and applied to handle some fractional differential equations within the modified Riemann-Liouville derivative in [11-15]. More recently, the local fractional variational iteration method, initiated in [16], was used to find the non-differentiable solutions for the heat conduction equation, Poisson equation in [17], coupled KdV equation in [18], damped and dissipative wave equation in [19], Fokker-Planck equation in [20], and non-linear PDE in [21] with local fractional derivative operators.

* Corresponding author, e-mail: hasibkhan13@yahoo.com

In recent years, a many of approximate and analytical methods have been utilized to solve the PDE with local fractional derivative operators such as the local fractional Adomian decomposition method in [22, 23], local fractional differential transform method in [24-26], local fractional series expansion method in [27], local fractional Sumudu transform method in [28], local fractional Laplace transform method in [29], local fractional reduced differential transform method in [30, 32], local fractional Laplace variational iteration method in [31].

The standard form of non-linear gas dynamic equation involving local fractional derivative operators can be written:

$$\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}+\frac{1}{2}\frac{\partial^\alpha u^2(x,y)}{\partial y^\alpha}-u(x,y)\left[1-u(x,y)\right]=g(x,y),\quad 0<\alpha\le 1 \tag{1}$$

subject to the initial conditions

$$u(0,y)=\phi(y) \tag{2}$$

The KdV equation describes the theory of water waves in shallow channels. It is a non-linear equation which exhibits special solutions, known as Solutions, which are stable and do not disperse with time. The coupled KdV equations involving local fractional derivative operators can be written:

$$\begin{aligned}&\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}+\frac{\partial^{3\alpha} u(x,y)}{\partial y^{3\alpha}}+2u(x,y)\frac{\partial^\alpha u(x,y)}{\partial y^\alpha}+2v(x,y)\frac{\partial^\alpha u(x,y)}{\partial y^\alpha}=0\\&\frac{\partial^\alpha v(x,y)}{\partial x^\alpha}+\frac{\partial^{3\alpha} v(x,y)}{\partial y^{3\alpha}}+2v(x,y)\frac{\partial^\alpha v(x,y)}{\partial y^\alpha}+2u(x,y)\frac{\partial^\alpha v(x,y)}{\partial y^\alpha}=0\end{aligned} \tag{3}$$

subject to the initial conditions

$$\begin{aligned}u(0,y)&=\kappa_1(y)\\v(0,y)&=\kappa_2(y)\end{aligned} \tag{4}$$

In this paper, our aims are to present the coupling method of local fractional Laplace transform and variational iteration method, which is called as the local fractional variational iteration transform method, and to use it to solve the non-linear gas dynamic and coupled KdV equations with local fractional derivative.

Analysis of the method

We consider a general non-linear local PDE:

$$L_\alpha u(x,y)+R_\alpha u(x,y)+N_\alpha u(x,y)=g(x,y),\quad 0<\alpha\le 1 \tag{5}$$

where $L_a=\partial^{n\alpha}/\partial x^{n\alpha}$, $n\in N$ is the linear LFDO, R_α denotes a lower order LFDO, N_α represented the general non-linear LFDO, and $g(x,y)$ is the non-differentiable source term.

Applying the Yang-Laplace transform (denoted by Ł$_\alpha$) on both sides of eq. (5), we get:

$$Ł_\alpha\left\{L_\alpha u(x,y)\right\}+Ł_\alpha\left\{R_\alpha u(x,y)\right\}+Ł_\alpha\left\{N_\alpha u(x,y)\right\}=Ł_\alpha\left\{g(x,y)\right\} \tag{6}$$

Using the property of the Yang-Laplace transform, we have:

$$s^{n\alpha} Ł_\alpha \{u(x,y)\} - s^{(n-1)\alpha} u(0,y) - s^{(n-2)} u_x^{(\alpha)}(0,y) - \ldots - u_x^{[(n-1)\alpha]}(0,y) = \\ = Ł_\alpha \{g(x,y) - R_\alpha u(x,y) - N_\alpha u(x,y)\} \tag{7}$$

or

$$Ł_\alpha \{u(x,y)\} = \frac{1}{s^\alpha} u(0,y) + \frac{1}{s^{2\alpha}} u_x^{(\alpha)}(0,y) + \ldots + \frac{1}{s^{n\alpha}} u_x^{[(n-1)\alpha]}(0,y) + \\ + \frac{1}{s^{n\alpha}} Ł_\alpha \{g(x,y) - R_\alpha u(x,y) - N_\alpha u(x,y)\} \tag{8}$$

Operating with the Yang-Laplace inverse on both sides of eq. (8) gives:

$$u(x,y) = u(0,y) + \frac{x^\alpha}{\Gamma(1+\alpha)} u_x^{(\alpha)}(0,y) + \ldots + \frac{x^{(n-1)\alpha}}{\Gamma[1+(n-1)\alpha]} u_x^{[(n-1)\alpha]}(0,y) + \\ + Ł_\alpha^{-1} \left\{ \frac{1}{s^{n\alpha}} Ł_\alpha [g(x,y) - R_\alpha u(x,y) - N_\alpha u(x,y)] \right\} \tag{9}$$

Deriving both side eq. (9) with respect to x, we have:

$$\frac{\partial^\alpha u(x,y)}{\partial x^\alpha} - \frac{\partial^\alpha}{\partial x^\alpha} Ł_\alpha^{-1} \left\{ \frac{1}{s^{n\alpha}} Ł_\alpha [g(x,y) - R_\alpha u(x,y) - N_\alpha u(x,y)] \right\} - \\ - u_x^{(\alpha)}(0,y) - \ldots - \frac{x^{(n-2)\alpha}}{\Gamma[1+(n-2)\alpha]} u_x^{[(n-1)\alpha]}(0,y) = 0 \tag{10}$$

We now structure the correctional local fractional function in the form:

$$u_{m+1}(x,y) = u_m(x,y) + \frac{1}{\Gamma(1+\alpha)} \cdot \\ \cdot \int_0^x \frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} \left(\begin{array}{l} \frac{\partial^\alpha u_m(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha} Ł_\alpha^{-1} \left\{ \frac{1}{s^{k\alpha}} Ł_\alpha [g(\xi,y) - R_\alpha u_m(\xi,y) - N_\alpha u_m(\xi,y)] \right\} - \\ - (u_m)_\xi^{(\alpha)}(0,y) - \ldots - \frac{\xi^{(n-2)\alpha}}{\Gamma[1+(n-2)\alpha]} (u_m)_\xi^{[(n-1)\alpha]}(0,y) \end{array} \right) (\mathrm{d}\xi)^\alpha \tag{11}$$

Making the local fractional variation, we get:

$$\delta^\alpha u_{m+1}(x,y) = \delta^\alpha u_m(x,y) + \delta^\alpha \frac{1}{\Gamma(1+\alpha)} \cdot \\ \cdot \int_0^x \frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} \left(\begin{array}{l} \frac{\partial^\alpha u_m(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha} Ł_\alpha^{-1} \left\{ \frac{1}{s^{k\alpha}} Ł_\alpha [g(\xi,y) - R_\alpha u_m(\xi,y) - N_\alpha u_m(\xi,y)] \right\} - \\ - (u_m)_\xi^{(\alpha)}(0,y) - \ldots \frac{\xi^{(n-2)\alpha}}{\Gamma[1+(n-2)\alpha]} (u_m)_\xi^{[(n-1)\alpha]}(0,y) \end{array} \right) (\mathrm{d}\xi)^\alpha \tag{12}$$

The extremum condition of $u_{n+1}(x,y)$ is given:

$$\delta^{\alpha}u_{m+1}\left(x,y\right)=0 \tag{13}$$

In view of eq. (13), we have the following stationary conditions:

$$1+\left.\frac{\lambda\left(\xi\right)^{\alpha}}{\Gamma\left(1+\alpha\right)}\right|_{\xi=x}=0,\quad \left.\left[\frac{\lambda\left(\xi\right)^{\alpha}}{\Gamma\left(1+\alpha\right)}\right]^{(\alpha)}\right|_{\xi=x}=0 \tag{14}$$

This is turn gives:

$$\frac{\lambda\left(\xi\right)^{\alpha}}{\Gamma\left(1+\alpha\right)}=-1 \tag{15}$$

Substituting eq. (15) into eq. (11), we obtained:

$$u_{m+1}\left(x,y\right)=u_{m}\left(x,y\right)-\frac{1}{\Gamma\left(1+\alpha\right)}\cdot$$

$$\cdot\int_{0}^{x}\left(\begin{array}{c}\dfrac{\partial^{\alpha}u_{m}\left(\xi,y\right)}{\partial\xi^{\alpha}}-\dfrac{\partial^{\alpha}}{\partial\xi^{\alpha}}Ł_{\alpha}^{-1}\left\{\dfrac{1}{s^{k\alpha}}Ł_{\alpha}\left[g\left(\xi,y\right)-R_{\alpha}u_{m}\left(\xi,y\right)-N_{\alpha}u_{m}\left(\xi,y\right)\right]\right\}-\\ -\left(u_{m}\right)_{\xi}^{(\alpha)}\left(0,y\right)-\cdots-\dfrac{\xi^{(n-2)\alpha}}{\Gamma\left[1+\left(n-2\right)\alpha\right]}\left(u_{m}\right)_{\xi}^{[(n-1)\alpha]}\left(0,y\right)\end{array}\right)\left(\mathrm{d}\xi\right)^{\alpha} \tag{16}$$

Finally, the solution $u(x,y)$ is given:

$$u\left(x,y\right)=\lim_{m\to\infty}u_{m}\left(x,y\right) \tag{17}$$

Applications

Example 1. Consider the following homogeneous non-linear gas dynamic equation involving local fractional derivative operator:

$$\frac{\partial^{\alpha}u\left(x,y\right)}{\partial x^{\alpha}}+\frac{1}{2}\frac{\partial^{\alpha}u^{2}\left(x,y\right)}{\partial y^{\alpha}}-u\left(x,y\right)\left[1-u\left(x,y\right)\right]=0 \tag{18}$$

and the initial condition

$$u\left(0,y\right)=E_{\alpha}\left(-y^{\alpha}\right) \tag{19}$$

In view of eqs. (16) and (18) the local fractional iteration algorithm can be written:

$$u_{n+1}\left(x,y\right)=u_{n}\left(x,y\right)-\frac{1}{\Gamma\left(1+\alpha\right)}\cdot$$

$$\cdot\int_{0}^{x}\left(\frac{\partial^{\alpha}u_{n}\left(\tau,y\right)}{\partial\tau^{\alpha}}+\frac{\partial^{\alpha}}{\partial\tau^{\alpha}}\left[Ł_{\alpha}^{-1}\left(\frac{1}{s^{\alpha}}Ł_{\alpha}\left\{\frac{1}{2}\frac{\partial^{\alpha}u^{2}{}_{n}\left(\tau,y\right)}{\partial y^{\alpha}}-u_{n}\left(\tau,y\right)\left[1-u_{n}\left(\tau,y\right)\right]\right\}\right)\right]\right)\left(\mathrm{d}\tau\right)^{\alpha} \tag{20}$$

We can use the initial condition to select $u_0(x,y)=u(0,y)=E_{\alpha}(-y^{\alpha})$. Using this selection into the correction functional (20) gives the following successive approximations:

$$u_0(x,y) = E_\alpha\left(-y^\alpha\right) \tag{21}$$

$$u_1(x,y) = u_0(x,y) - \frac{1}{\Gamma(1+\alpha)} \cdot$$

$$\cdot\int_0^x \left(\frac{\partial^\alpha u_0(\tau,y)}{\partial\tau^\alpha} + \frac{\partial^\alpha}{\partial\tau^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^\alpha} Ł_\alpha \left\{ \frac{1}{2}\frac{\partial^\alpha u^2{}_0(\tau,y)}{\partial y^\alpha} - u_0(\tau,y)\left[1-u_0(\tau,y)\right] \right\} \right) \right] \right) (\mathrm{d}\tau)^\alpha =$$

$$= E_\alpha\left(-y^\alpha\right)\left[1 + \frac{x^\alpha}{\Gamma(1+\alpha)}\right] \tag{22}$$

$$u_2(x,y) = u_1(x,y) - \frac{1}{\Gamma(1+\alpha)} \cdot$$

$$\cdot\int_0^x \left(\frac{\partial^\alpha u_1(\tau,y)}{\partial\tau^\alpha} + \frac{\partial^\alpha}{\partial\tau^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^\alpha} Ł_\alpha \left\{ \frac{1}{2}\frac{\partial^\alpha u^2{}_1(\tau,y)}{\partial y^\alpha} - u_1(\tau,y)\left[1-u_1(\tau,y)\right] \right\} \right) \right] \right) (\mathrm{d}\tau)^\alpha =$$

$$= E_\alpha\left(-y^\alpha\right)\left[1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right] \tag{23}$$

$$u_n(x,y) = E_\alpha\left(-y^\alpha\right)\sum_{k=0}^{n} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)} \tag{24}$$

Finally, the solution $u(x,y)$ is given:

$$u(x,y) = \lim_{n\to\infty} u_n(x,y) = E_\alpha\left(-y^\alpha\right)\sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)} \tag{25}$$

and in closed form:

$$u(x,y) = E_\alpha\left(-y^\alpha\right)E_\alpha\left(x^\alpha\right) = E_\alpha\left(x^\alpha - y^\alpha\right) \tag{26}$$

Example 2. Consider the following non-homogeneous non-linear gas dynamic equation involving local fractional derivative operator:

$$\frac{\partial^\alpha u(x,y)}{\partial x^\alpha} + \frac{1}{2}\frac{\partial^\alpha u^2(x,y)}{\partial y^\alpha} - u(x,y)\left[1-u(x,y)\right] = -E_\alpha\left(x^\alpha - y^\alpha\right) \tag{27}$$

with the initial condition:

$$u(0,y) = 1 - E_\alpha\left(-y^\alpha\right) \tag{28}$$

Applying eqs. (16) and (21), we obtain the correction function:

$$u_{n+1}(x,y) = u_n(x,y) - \frac{1}{\Gamma(1+\alpha)} \cdot$$

$$\cdot\int_0^x \left(\frac{\partial^\alpha u_n(\tau,y)}{\partial\tau^\alpha} + \frac{\partial^\alpha}{\partial\tau^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^\alpha} Ł_\alpha \left\{ \frac{1}{2}\frac{\partial^\alpha u_n^2(\tau,y)}{\partial y^\alpha} - u_n(\tau,y)\left[1-u_n(\tau,y)\right] \right\} \right) \right] \right) (\mathrm{d}\tau)^\alpha \tag{29}$$

We can use the initial condition to select $u_0(x,y)=u(0,y)=1-E_\alpha(-y^\alpha)$ Using this selection into the correction functional (29) gives the following successive approximations:

$$u_0\left(x,y\right)=1-E_\alpha\left(-y^\alpha\right) \tag{30}$$

$$u_1\left(x,y\right)=u_0\left(x,y\right)-\frac{1}{\Gamma\left(1+\alpha\right)}\cdot$$

$$\cdot\int_0^x\left(\begin{array}{c}\dfrac{\partial^\alpha u_0\left(\tau,y\right)}{\partial\tau^\alpha}+\dfrac{\partial^\alpha}{\partial\xi^\alpha}\left[Ł_\alpha^{-1}\left(\dfrac{1}{s^\alpha}Ł_\alpha\left\{\dfrac{1}{2}\dfrac{\partial^\alpha u_0^2\left(\tau,y\right)}{\partial y^\alpha}-u_0\left(\tau,y\right)\left[1-u_0\left(\tau,y\right)\right]\right\}\right)\right]+\\+E_\alpha\left(\tau^\alpha-y^\alpha\right)\end{array}\right)\left(\mathrm{d}\tau\right)^\alpha=$$

$$=1-E_\alpha\left(-y^\alpha\right)-\frac{1}{\Gamma\left(1+\alpha\right)}\int_0^\infty\left[E_\alpha\left(\tau^\alpha-y^\alpha\right)\right]\left(\mathrm{d}\tau\right)^\alpha=$$

$$=1-E_\alpha\left(-y^\alpha\right)-E_\alpha\left(x^\alpha-y^\alpha\right)+E_\alpha\left(-y^\alpha\right)=$$

$$=1-E_\alpha\left(x^\alpha-y^\alpha\right) \tag{31}$$

$$u_2\left(x,y\right)=u_1\left(x,y\right)-\frac{1}{\Gamma\left(1+\alpha\right)}\cdot$$

$$\cdot\int_0^x\left(\begin{array}{c}\dfrac{\partial^\alpha u_1\left(\tau,y\right)}{\partial\tau^\alpha}+\dfrac{\partial^\alpha}{\partial\tau^\alpha}\left[Ł_\alpha^{-1}\left(\dfrac{1}{s^\alpha}Ł_\alpha\left\{\dfrac{1}{2}\dfrac{\partial^\alpha u_1^2\left(\tau,y\right)}{\partial y^\alpha}-u_1\left(\tau,y\right)\left[1-u_1\left(\tau,y\right)\right]\right\}\right)\right]+\\+E_\alpha\left(\tau^\alpha-y^\alpha\right)\end{array}\right)\left(\mathrm{d}\tau\right)^\alpha=$$

$$=1-E_\alpha\left(x^\alpha-y^\alpha\right) \tag{32}$$

$$\vdots$$

$$u_n\left(x,y\right)=1-E_\alpha\left(x^\alpha-y^\alpha\right) \tag{33}$$

Finally, the solution $u(x,y)$ is given:

$$u\left(x,y\right)=\lim_{n\to\infty}u_n\left(x,y\right)=1-E_\alpha\left(x^\alpha-y^\alpha\right) \tag{34}$$

Example 3. Consider the system of local fractional coupled KdV equations with local fractional derivative:

$$\begin{aligned}&\frac{\partial^\alpha u\left(x,y\right)}{\partial x^\alpha}+\frac{\partial^{3\alpha}u\left(x,y\right)}{\partial y^{3\alpha}}+2u\left(x,y\right)\frac{\partial^\alpha u\left(x,y\right)}{\partial y^\alpha}+2v\left(x,y\right)\frac{\partial^\alpha u\left(x,y\right)}{\partial y^\alpha}=0\\&\frac{\partial^\alpha v\left(x,y\right)}{\partial x^\alpha}+\frac{\partial^{3\alpha}v\left(x,y\right)}{\partial y^{3\alpha}}+2v\left(x,y\right)\frac{\partial^\alpha u\left(x,y\right)}{\partial y^\alpha}+2u\left(x,y\right)\frac{\partial^\alpha v\left(x,y\right)}{\partial y^\alpha}=0\end{aligned} \tag{35}$$

subject to the initial conditions:

$$\begin{aligned}u\left(0,y\right)&=E_\alpha\left(-y^\alpha\right)\\v\left(0,y\right)&=E_\alpha\left(-y^\alpha\right)\end{aligned} \tag{36}$$

Applying local fractional Laplace transform on eq. (35) and using the initial conditions (36), we have:

$$\begin{aligned}
&Ł_\alpha\{u(x,y)\}=\frac{1}{s^\alpha}E_\alpha(-y^\alpha)-\frac{1}{s^\alpha}Ł_\alpha\left\{\frac{\partial^{3\alpha}u(x,y)}{\partial y^{3\alpha}}+2u(x,y)\frac{\partial^\alpha u(x,y)}{\partial y^\alpha}+2v(x,y)\frac{\partial^\alpha u(x,y)}{\partial y^\alpha}\right\}\\
&Ł_\alpha\{v(x,y)\}=\frac{1}{s^\alpha}E_\alpha(-y^\alpha)-\frac{1}{s^\alpha}Ł_\alpha\left\{\frac{\partial^{3\alpha}v(x,y)}{\partial y^{3\alpha}}+2v(x,y)\frac{\partial^\alpha v(x,y)}{\partial y^\alpha}+2u(x,y)\frac{\partial^\alpha v(x,y)}{\partial y^\alpha}\right\}
\end{aligned}\tag{37}$$

Operating with the local fractional Laplace transform inverse on both sides of eq. (37) we obtain:

$$\begin{aligned}
&u(x,y)=E_\alpha(-y^\alpha)-Ł_\alpha^{-1}\left(\frac{1}{s^\alpha}Ł_\alpha\left\{\frac{\partial^{3\alpha}u(x,y)}{\partial y^{3\alpha}}+2u(x,y)\frac{\partial^\alpha u(x,y)}{\partial y^\alpha}+2v(x,y)\frac{\partial^\alpha u(x,y)}{\partial y^\alpha}\right\}\right)\\
&v(x,y)=E_\alpha(-y^\alpha)-Ł_\alpha^{-1}\left(\frac{1}{s^\alpha}Ł_\alpha\left\{\frac{\partial^{3\alpha}v(x,y)}{\partial y^{3\alpha}}+2v(x,y)\frac{\partial^\alpha v(x,y)}{\partial y^\alpha}+2u(x,y)\frac{\partial^\alpha v(x,y)}{\partial y^\alpha}\right\}\right)
\end{aligned}\tag{38}$$

Deriving both sides of eq. (38) with respect to x, we get:

$$\begin{aligned}
&\frac{\partial^\alpha u(x,y)}{\partial x^\alpha}+\frac{\partial^\alpha}{\partial x^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^\alpha}Ł_\alpha\left\{\frac{\partial^{3\alpha}u(x,y)}{\partial y^{3\alpha}}+2u(x,y)\frac{\partial^\alpha u(x,y)}{\partial y^\alpha}+2v(x,y)\frac{\partial^\alpha u(x,y)}{\partial y^\alpha}\right\}\right)\right]=0\\
&\frac{\partial^\alpha v(x,y)}{\partial x^\alpha}+\frac{\partial^\alpha}{\partial x^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^\alpha}Ł_\alpha\left\{\frac{\partial^{3\alpha}v(x,y)}{\partial y^{3\alpha}}+2v(x,y)\frac{\partial^\alpha v(x,y)}{\partial y^\alpha}+2u(x,y)\frac{\partial^\alpha v(x,y)}{\partial y^\alpha}\right\}\right)\right]=0
\end{aligned}\tag{39}$$

Making the correction function is given:

$$\begin{aligned}
&u_{m+1}(x,y)=u_m(x,y)-\frac{1}{\Gamma(1+\alpha)}\cdot\\
&\cdot\int_0^x\left\{\frac{\partial^\alpha u_m(\tau,y)}{\partial\tau^\alpha}+\frac{\partial^\alpha}{\partial\tau^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^\alpha}Ł_\alpha\left\{\begin{array}{l}\frac{\partial^{3\alpha}u_m(\tau,y)}{\partial y^{3\alpha}}+2u_m(\tau,y)\frac{\partial^{2\alpha}u_m(\tau,y)}{\partial y^\alpha}+\\+2v_m(x,y)\frac{\partial^\alpha u_m(\tau,y)}{\partial y^\alpha}\end{array}\right\}\right)\right]\right\}(\mathrm{d}\tau)^\alpha\\
&v_{m+1}(x,y)=v_m(x,y)-\frac{1}{\Gamma(1+\alpha)}\cdot\\
&\cdot\int_0^x\left\{\frac{\partial^\alpha v_m(\tau,y)}{\partial\tau^\alpha}+\frac{\partial^\alpha}{\partial\tau^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^\alpha}Ł_\alpha\left\{\begin{array}{l}\frac{\partial^{3\alpha}v_m(\tau,y)}{\partial y^{3\alpha}}+2v_m(\tau,y)\frac{\partial^{\alpha}v_m(\tau,y)}{\partial y^\alpha}+\\+2v_m(x,y)\frac{\partial^\alpha v_m(\tau,y)}{\partial y^\alpha}\end{array}\right\}\right)\right]\right\}(\mathrm{d}\tau)^\alpha
\end{aligned}\tag{40}$$

We can use the initial conditions to select $u_0(x,y)=-E_\alpha(-y^\alpha)$, $v_0(x,y)=-E_\alpha(-y^\alpha)$. Using this selection into the correction functional (40) gives the following successive approximations:

$$u_0(x,y) = -E_\alpha(-y^\alpha)$$

$$v_0(x,y) = -E_\alpha(-y^\alpha)$$

$$u_1(x,y) = u_0(x,y) - \frac{1}{\Gamma(1+\alpha)} \cdot$$

$$\cdot \int_0^x \left\{ \frac{\partial^\alpha u_0(\tau,y)}{\partial \tau^\alpha} + \frac{\partial^\alpha}{\partial \tau^\alpha} \left[Ⱡ_\alpha^{-1} \left(\frac{1}{s^\alpha} Ⱡ_\alpha \left\{ \begin{matrix} \dfrac{\partial^{3\alpha} u_0(\tau,y)}{\partial y^{3\alpha}} + 2u_0(\tau,y) \dfrac{\partial^{2\alpha} u_0(\tau,y)}{\partial y^\alpha} + \\ +2v_0(x,y) \dfrac{\partial^\alpha u_0(\tau,y)}{\partial y^\alpha} \end{matrix} \right\} \right) \right] \right\} (\mathrm{d}\tau)^\alpha$$

$$v_1(x,y) = v_0(x,y) - \frac{1}{\Gamma(1+\alpha)} \cdot$$

$$\cdot \int_0^x \left\{ \frac{\partial^\alpha v_0(\tau,y)}{\partial \tau^\alpha} + \frac{\partial^\alpha}{\partial \tau^\alpha} \left[Ⱡ_\alpha^{-1} \left(\frac{1}{s^\alpha} Ⱡ_\alpha \left\{ \begin{matrix} \dfrac{\partial^{3\alpha} v_0(\tau,y)}{\partial y^{3\alpha}} + 2v_0(\tau,y) \dfrac{\partial^{\alpha} v_0(\tau,y)}{\partial y^\alpha} + \\ +2v_0(x,y) \dfrac{\partial^\alpha v_0(\tau,y)}{\partial y^\alpha} \end{matrix} \right\} \right) \right] \right\} (\mathrm{d}\tau)^\alpha =$$

$$= E_\alpha(-y^\alpha) - \frac{1}{\Gamma(1+\alpha)} \int_0^x \left\{ 0 + \frac{\partial^\alpha}{\partial \tau^\alpha} \left[Ⱡ_\alpha^{-1} \left(\frac{1}{s^\alpha} Ⱡ_\alpha \left\{ \begin{matrix} -E_\alpha(-y^\alpha) - 2E_\alpha(-2y^\alpha) + \\ +2E_\alpha(-y^\alpha) \end{matrix} \right\} \right) \right] \right\} (\mathrm{d}\tau)^\alpha =$$

$$= -E_\alpha(-y^\alpha) - \frac{1}{\Gamma(1+\alpha)} \int_0^x \left\{ 0 + \frac{\partial^\alpha}{\partial \tau^\alpha} \left[Ⱡ_\alpha^{-1} \left(\frac{1}{s^\alpha} Ⱡ_\alpha \left\{ \begin{matrix} E_\alpha(-y^\alpha) - 2E_\alpha(-2y^\alpha) + \\ +2E_\alpha(-y^\alpha) \end{matrix} \right\} \right) \right] \right\} (\mathrm{d}\tau)^\alpha =$$

$$= E_\alpha(-y^\alpha) + \frac{1}{\Gamma(1+\alpha)} \int_0^x E_\alpha(-y^\alpha)(\mathrm{d}\tau)^\alpha =$$

$$= -E_\alpha(-y^\alpha) - \frac{1}{\Gamma(1+\alpha)} \int_0^x E_\alpha(-y^\alpha)(\mathrm{d}\tau)^\alpha =$$

$$= E_\alpha(-y^\alpha)\left[1 + \frac{x^\alpha}{\Gamma(1+\alpha)}\right] =$$

$$= -E_\alpha(-y^\alpha)\left[1 + \frac{x^\alpha}{\Gamma(1+\alpha)}\right]$$

$$u_2(x,y)=u_1(x,y)-\frac{1}{\Gamma(1+\alpha)}\cdot$$

$$\cdot\int_0^x\left\{\frac{\partial^\alpha u_1(\tau,y)}{\partial\tau^\alpha}+\frac{\partial^\alpha}{\partial\tau^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^\alpha}Ł_\alpha\left\{\begin{array}{c}\frac{\partial^{3\alpha}u_1(\tau,y)}{\partial y^{3\alpha}}+2u_1(\tau,y)\frac{\partial^{2\alpha}u_1(\tau,y)}{\partial y^\alpha}+\\+2v_1(x,y)\frac{\partial^\alpha u_1(\tau,y)}{\partial y^\alpha}\end{array}\right\}\right)\right]\right\}(\mathrm{d}\tau)^\alpha$$

$$v_2(x,y)=v_1(x,y)-\frac{1}{\Gamma(1+\alpha)}\cdot$$

$$\cdot\int_0^x\left\{\frac{\partial^\alpha v_1(\tau,y)}{\partial\tau^\alpha}+\frac{\partial^\alpha}{\partial\tau^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^\alpha}Ł_\alpha\left\{\begin{array}{c}\frac{\partial^{3\alpha}v_1(\tau,y)}{\partial y^{3\alpha}}+2v_1(\tau,y)\frac{\partial^\alpha v_1(\tau,y)}{\partial y^\alpha}+\\+2u_1(x,y)\frac{\partial^\alpha v_1(\tau,y)}{\partial y^\alpha}\end{array}\right\}\right)\right]\right\}(\mathrm{d}\tau)^\alpha=$$

$$=E_\alpha(-y^\alpha)+\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)-$$

$$-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left\{E_\alpha(-y^\alpha)+\frac{\partial^\alpha}{\partial\tau^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^\alpha}Ł_\alpha\left\{-E_\alpha(-y^\alpha)-\frac{\tau^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)\right\}\right)\right]\right\}(\mathrm{d}\tau)^\alpha=$$

$$=-E_\alpha(-y^\alpha)-\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)-\frac{1}{\Gamma(1+\alpha)}\cdot$$

$$\cdot\int_0^x\left\{-E_\alpha(-y^\alpha)+\frac{\partial^\alpha}{\partial\tau^\alpha}\left[Ł_\alpha^{-1}\left(\frac{1}{s^\alpha}Ł_\alpha\left\{E_\alpha(-y^\alpha)+\frac{\tau^\alpha}{\Gamma(1+\alpha)}E_\alpha(-2y^\alpha)\right\}\right)\right]\right\}(\mathrm{d}\tau)^\alpha=$$

$$=E_\alpha(-y^\alpha)+\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{\tau^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)(\mathrm{d}\tau)^\alpha=$$

$$=-E_\alpha(-y^\alpha)-\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{\tau^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)(\mathrm{d}\tau)^\alpha=$$

$$=E_\alpha(-y^\alpha)\left[1+\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right]=$$

$$=-E_\alpha(-y^\alpha)\left[1+\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right]=$$

$$\vdots$$

$$u_m(x,y)=E_\alpha(-y^\alpha)\sum_{k=0}^m\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}$$

$$v_m(x,y)=-E_\alpha(-y^\alpha)\sum_{k=0}^m\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}$$

Therefore, the series solutions can be written in the form:

$$\begin{aligned} u(x,y) &= \lim_{m\to\infty} u_m(x,y) = E_\alpha\left(x^\alpha - y^\alpha\right) \\ v(x,y) &= \lim_{m\to\infty} v_m(x,y) = -E_\alpha\left(x^\alpha - y^\alpha\right) \end{aligned} \tag{41}$$

Conclusions

In this work, local fractional variational iteration transform method has been successfully applied to finding the non-differentiable solution of non-linear gas dynamic and coupled KdV equations involving local fractional operator. The method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and non-linear local fractional PDE.

Acknowledgment

This work was supported by the China Government Young Talent Program.

References

[1] He, J. H., Approximate Analytical Solution for Seepage Flow with Fractional Derivatives in Porous Media, *Computer Methods in Applied Mechanics and Engineering*, *167* (1998) 1-2, pp. 57-68
[2] He, J. H., Variational Iteration Method a Kind of Non-Linear Analytical Technique: Some Examples, *International Journal of Non-Linear Mechanics*, *34* (1999), 4, pp. 699-708
[3] Momani, S., *et al.,* Application of He's Variational Iteration Method to Helmholtz Equation, *Chaos, Solitons & Fractals*, *27* (2006), 5, pp. 1119-1123
[4] Abdou, M. A., *et al.,* Variational Iteration Method for Solving Burger's and Coupled Burger's Equations, *Journal of Computational and Applied Mathematics*, *181* (2005), 2, pp. 245-251
[5] Abbas Bandy, S., *et al.,* Numerical Solution of Non-Linear Klein-Gordon Equations by Variational Iteration Method, *International Journal for Numerical Methods in Engineering*, *70* (2007), 7, pp. 876-881
[6] Mohyud-Din, S. T., *et al.,* Traveling Wave Solutions of Seventh-Order Generalized KdV Equations Using He's Polynomials, *International Journal of Nonlinear Sciences and Numerical Simulation*, *10* (2009), 2, pp. 227-234
[7] Marinca, V., *et al.,* Application of the Variational Iteration Method to Some Nonlinear One-Dimensional Oscillations, *Meccanica*, *43* (2008), 1, pp. 75-79
[8] Wazwaz, A. M., A Study on Linear and Nonlinear Schroedinger Equations by the Variational Iteration Method, *Chaos, Solitons & Fractals*, *37* (2008), 4, pp. 1136-1142
[9] Das, S., *et al.,* Approximate Solution of Fractional Diffusion Equation Revisited, *International Review of Chemical Engineering*, *4* (2012) 5, pp. 501-504
[10] Hristov, J., An Exercise with the He's Variation Iteration Method to a Fractional Bernoulli Equation Arising in a Transient Conduction with a Non-Linear Boundary Heat Flux, *International Review of Chemical Engineering*, *4* (2012), 5, pp. 489-497
[11] Wu, G. C., *et al.,* Fractional Variational Iteration Method and Its Application, *Physics Letters A: General, Atomic and Solid State Physics*, *374* (2010), 25, pp. 2506-2509
[12] He, J. H., A Short Remark on Fractional Variational Iteration Method, *Physics Letters A: General, Atomic and Solid State Physics*, *375* (2011), 38, pp. 3362-3364
[13] Baleanu, D., *et al.,* Improved (G'/G)-Expansion Method for the Time-Fractional Biological Population Model and Cahn-Hilliard Equation, *Journal of Computational and Nonlinear Dynamics, 10* (2015), 5, 051016
[14] Mustafa, I., *et al.,* An Approximate Solution of Fractional Cable Equation by HAM, Boundary Value Problem, *2014* (2014), 58, pp. 1-10
[15] Bulent, K., *et al.,* The First Integral Method for the Time Fractional Kaup-Boussinesq System with Time Dependent Coefficient, *Applied Mathematics and Computations, 254* (2015), C, pp. 70-74
[16] Yang, X. J., *et al.,* Fractal Heat Conduction Problem Solved by Local Fractional Variation Iteration Method, *Thermal Science, 17* (2013), 2, pp. 625-628

[17] Chen, L., *et al.,* Local Fractional Variational Iteration Method for Local Fractional Poisson Equations in Two Independent Variables, *Abstract and Applied Analysis*, *2014* (2014), ID484323
[18] Jafari, H., *et al.,* Approximate Solution for Nonlinear Gas Dynamic and Coupled KdV Equations Involving Local Fractional Operator, *Journal of Zankoy Sulaimani – Part A*, *18* (2016), 1, pp. 127-132
[19] Su, W. H., *et al.,* Damped Wave Equation and Dissipative Wave Equation in Fractal Strings within the Local Fractional Variational Iteration Method, *Fixed Point Theory and Applications*, *89* (2013), 1, pp. 1-11
[20] Jassim, H. K., New Approaches for Solving Fokker Planck Equation on Cantor Sets within Local Fractional Operators, *Journal of Mathematics*, *2015* (2015), ID684598
[21] Jafari, H., *et al.,* Local Fractional Variational Iteration Method for Nonlinear Partial Differential Equations within Local Fractional Operators, *Applications and Applied Mathematics*, *10* (2015), 2, pp. 1055-1065
[22] Jafari, H., Jassim, H. K., Local Fractional Adomian Decomposition Method for Solving Two Dimensional Heat conduction Equations within Local Fractional Operators, *Journal of Advance in Mathematics*, *9* (2014), 4, pp. 2574-2582
[23] Jafari, H., *et al.,* Application of the Local Fractional Adomian Decomposition and Series Expansion Methods for Solving Telegraph Equation on Cantor Sets Involving Local Fractional Derivative Operators, *Journal of Zankoy Sulaimani-Part A*, *17* (2015), June, pp.15-22
[24] Baleanu, D., *et al.,* Approximate Analytical Solutions of Goursat Problem within Local Fractional Operators, *Journal of Nonlinear Science and Applications*, *9* (2016), 6, pp. 4829-4837
[25] Yang, X. J., *et al.*, A New Numerical Technique for Solving the Local Fractional Diffusion Equation: Two-Dimensional Extended Differential Transform Approach, *Applied Mathematics and Computation, 274* (2016), Feb., pp. 143-151
[26] Jafari, H., *et al.,* On the Approximate Solutions of Local Fractional Differential Equations with Local Fractional Operator, *Entropy*, *18* (2016), 150, pp. 1-12
[27] Jassim, H. K., On Approximate Methods for Fractal Vehicular Traffic Flow, *TWMS Journal of Applied and Engineering Mathematics*, *7* (2017), 1, pp. 58-65
[28] Srivastava, H. M., *et al.,* Local Fractional Sumudu Transform with Application to IVPs on Cantor Set, *Abstract and Applied Analysis, 2014* (2014), ID620529
[29] Jassim, H. K., The Analytical Solutions for Volterra Integro-Differential Equations Involving Local fractional Operators by Yang-Laplace Transform, *Sahand Communications in Mathematical Analysis, 6* (2017), 1, pp. 69-76
[30] Jafari, H., *et al.,* Reduced Differential Transform Method for Partial Differential Equations within Local Fractional Derivative Operators, *Advances in Mechanical Engineering*, *8* (2016), 4, pp. 1-6
[31] Jassim, H. K., *et al.*, Local Fractional Laplace Variational Iteration Method for Solving Diffusion and Wave Equations on Cantor Sets within Local Fractional Operators, *Mathematical Problems in Engineering*, *2015* (2015), ID309870
[32] Jassim, H. K., The Approximate Solutions of Three-Dimensional Diffusion and Wave Equations within Local Fractional Derivative Operator, *Abstract and Applied Analysis*, *2016* (2016), ID2913539

Paper submitted: August 4, 2017
Paper revised: November 20, 2017
Paper accepted: November 24, 2017