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On Fractional Operators and Their Classifications

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Abstract: Fractional calculus dates its inception to a correspondence between Leibniz and L'Hopital in 1695, when Leibniz described "paradoxes" and predicted that "one day useful consequences will be drawn" from them. In today's world, the study of non-integer orders of differentiation has become a thriving field of research, not only in mathematics but also in other parts of science such as physics, biology, and engineering: many of the "useful consequences" predicted by Leibniz have been discovered. However, the field has grown so far that researchers cannot yet agree on what a "fractional derivative" can be. In this manuscript, we suggest and justify the idea of classification of fractional calculus into distinct classes of operators.

Keywords: fractional calculus; integral transforms; convergent series

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1. Background

Fractional calculus is a venerable branch of mathematics, first conceptualised in 1695 in a series of letters. L'Hopital posed the question to Leibniz of what would happen if the order of differentiation were taken to be $\frac{1}{2}$, and Leibniz replied [1]:

"It appears that one day useful consequences will be drawn from these paradoxes."

After these prophetic words, however, Leibniz did not propose a definition, leaving this task to the later scientists who followed him.

The concepts of fractional differentiation and fractional integration were examined further over the course of the 18th and 19th centuries. The topic attracted the attention of mathematical giants such as Riemann [2], Liouville [3], Abel [4], Laurent [5], and Hardy and Littlewood [6,7]. Detailed discussions of the history of fractional calculus may be found in [8–11]; here, we wish to focus on a few key points concerning the directions in which the field developed.

The "paradoxes" described by Leibniz were resolved by later authors, but this is not to say that the field of fractional calculus is now wholly free of open problems. One recurring issue through the centuries has been the existence of multiple conflicting definitions. In the mid-19th century, several different definitions of fractional calculus had already been proposed: Liouville had created one definition based on differentiating exponential functions and another based on an integral formula for inverse power functions, while Lacroix had created a different definition based on differentiating power functions. The definitions of Liouville and Lacroix are not equivalent, which led some critics to conclude that one must be "correct" and the other "wrong". De Morgan, however, wrote [12] that:

"Both these systems, then, may very possibly be parts of a more general system."

His words, like those of Leibniz 145 years earlier, were prophetic. Both Liouville's formula and Lacroix's are in fact special cases of what is now called the Riemann–Liouville definition of fractional calculus. This involves an arbitrary constant of integration c, which when set to zero yields Lacroix's formula and when set to $-\infty$ yields Liouville's.

This general Riemann–Liouville definition, for the fractional derivative and fractional integral of an arbitrary function, emerged in the late 19th century through a complex-analysis approach. Although the Riemann–Liouville formula is now used mostly in a real-analysis context, its original motivation came from generalising the Cauchy integral formula for repeated derivatives of a complex analytic function. Now, Riemann–Liouville is the most common way of defining fractional calculus. In this model, the fractional integral and fractional derivative of a function $f(x)$ are defined as follows:

$$
\begin{aligned}
{}^{RL}_{c}I^{\nu}_{x}f(x) &= \frac{1}{\Gamma(\nu)}\int_{c}^{x}(x-t)^{\nu-1}f(t)\,\mathrm{d}t, \qquad \mathrm{Re}(\nu)>0; \\
{}^{RL}_{c}D^{\nu}_{x}f(x) &= \frac{\mathrm{d}^n}{\mathrm{d}x^n}\,{}^{RL}_{c}I^{n-\nu}_{x}f(x), \quad n=\lfloor \mathrm{Re}(\nu)\rfloor+1, \qquad \mathrm{Re}(\nu)\geq 0.
\end{aligned}
$$

This definition is sufficiently general to cover the formulae both of Liouville and of Lacroix. However, it is still not the only proposed way of defining fractional calculus: multiple conflicting formulae persist to this day, confusing many newcomers to the field who expect to see a single definition of fractional derivatives just like there is a single definition of the first-order derivative. Fractional calculus may be called an "extension of meaning" [13], but there is more than one way to extend meaning. The Riemann–Liouville model can be used to describe processes with power-law behaviour, due to the power-function kernel in the definition of the integral transform, but there are many other types of behaviours that occur in nature and that cannot be described by simple power functions.

In the late 20th century, fractional calculus began to undergo a large increase in popularity and research output. The first international conference on fractional calculus was organised in 1974 in the USA; the same year also saw the publication of the first textbook [14] devoted to this field. Since then, fractional calculus has become a very active field of research, with several specialist journals on the topic. Applications have been discovered in many fields of science, as summarised in [15–18] and the references therein. In particular, the intermediate property of fractional-calculus operators is vital for the modelling of certain intermediate physical processes, e.g., in viscoelasticity [19,20]. Fractional calculus has also become a standard part of the graduate mathematics curriculum in some universities, with several textbooks [8,11,14,21–23] that can function as an introduction to the field for students and young researchers.

From the point of view of research, currently there are several differing perspectives and directions of exploration, which in some respects may be in opposition to each other. In the following section, we propose a possible way of resolving these issues.

2. The Question of Classification

In recent years, two trends have emerged in the consideration of fractional-calculus operators, motivated by a number of different considerations.

Firstly, there exists a desire to **explore and create new definitions and models** for fractional integral and differential operators. Dozens of definitions have been proposed in the 2010s alone, with a wide variety of types and properties [24–27]. One motivation here is the pure mathematician's desire to generalise: for example, to go beyond simple power functions and extend definitions to cover a whole host of different kernel functions. Another motivation is the applied scientist's need for models to describe accurately a wide variety of different systems: several definitions of fractional calculus have been inspired directly by real-world applications. The result of both types of research is to expand the field of fractional calculus. However, the question arises of how far the field can be stretched and still be called "fractional calculus", and the validity of some definitions has been debated.

Secondly, there exists a desire to **impose criteria and strict definitions** for what we call a "fractional derivative" or "fractional integral": which operators between functions should be named as such and which should not. The proposals range from strict requirements to mere suggestions, and multiple different criteria have been proposed [13,28–31]. The motivation here is to create a mathematical framework for fractional calculus, to know the boundaries of the field. Metric spaces and vector spaces, for example, have rigorous definitions and strict sets of criteria, so why not fractional integrals and fractional derivatives? The result of such a system would be to restrict the study of fractional calculus within certain boundaries. However, there is no consensus on where the boundaries should be drawn: opinions differ widely on what the criteria should be.

At first, these two ways of thinking seem very different. One seeks to expand the field without regard for boundaries, while the other seeks to restrict the field to within prescribed boundaries. However, as both points of view have some merit, we would like to seek a middle path, a way of satisfying both the desire for generalisation and diversification and the desire for rigorous classification.

The key lies in considering the valid motivations for both approaches. Mathematical structures have an aesthetic, intuitive logic, which guides our path to choosing appropriate criteria to define them and which often connects directly or indirectly with their physical applications. These real-world connections are of paramount importance: if one particular mathematical model emerges from some real data, then that model must be worth studying, and so we should not exclude it from consideration by imposing overly strict criteria.

The desire for generalisation and the desire for criteria, which seem opposed to each other, may both be satisfied by considering **broad classes of fractional-calculus operators**. We recall again the words of Augustus de Morgan, quoted above: if different definitions seem in contradiction, it is worth considering whether they may be unified as part of "a more general system". Ideally, such a system would be itself part of fractional calculus. Formally, then, we seek to define sets $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, etc. (we do not presume to know how many such sets will emerge), of operators between function spaces, such that each element of each of these sets may be interpreted as a "fractional operator" acting on functions and such that each set has some unifying properties which enable useful results to be proven for the entire class. We do not impose any requirements in general on how large or small these classes should be, or which function spaces they should act between, as we believe such a system should be able to cover many different families of operators.

Fractional calculus has been usefully interpreted in connection with many different branches of mathematics: for example, distributional calculus, functional calculus, spectral theory, Cauchy integrals, and Laplace transforms, as described and summarised in [32] (pp. 58–64). Our aim here is related but different: instead of embedding the whole of fractional calculus into other fields of analysis, we seek to create classifications within fractional calculus itself. Some recent studies [30,33–35] have proposed general classes of operators that are broad enough to cover many existing models of fractional calculus but still narrow enough to be rigorously analysed themselves. This approach is optimal for several reasons:

1. It satisfies the desire for generalisation. Any class of fractional-calculus operators will be more general than any one particular model, and the specific models can be studied as before within this framework or as special cases of the general class. If real-world applications give rise to a new model of fractional calculus, it may be able to fit into such a class, and then many of its properties would be known directly from general theorems about the class.
2. It also satisfies, to a certain degree, the desire for restrictions and criteria. Not all types of fractional calculus fall into one particular class, but each class can be studied in its own right; its defining attributes could be considered as "axioms" or criteria for that particular class. Thus, it is possible to study fractional calculus within the framework of certain prescribed conditions, without dismissing everything outside that framework as invalid.

In shaping the mathematical theory of fractional calculus, we should look beyond single specific formulae and create wider avenues of study. This will eliminate the need for many different research papers proving the same results in the same way for many different types of fractional calculus: instead, we can prove them just once for a whole class and then deduce the individual results as special cases. From the applications point of view, a particular collection of real data can be fitted to a particular model of fractional calculus which is already known as a special case of one of these broad classes.

At some point in the future, it may be possible to create a "most general" definition of fractional calculus by defining one single class $\mathscr{F}$ that covers all fractional operators and nothing else, with all the other classes of fractional derivatives and integrals as subsets. However, we believe that such a breakthrough is not imminent. We must wait to discover the full range of applications before we can decide where to draw the boundaries of the field, and at present, new applications of fractional calculus are still being discovered all the time. It would be hasty to restrict the field too far now and then discover after a few years that the restrictions exclude those fractional-calculus operators that are most useful in real-world modelling.

3. The Class of Analytic Kernels

To illustrate the ideas discussed in the previous section, we shall conduct a detailed analysis of one general class of fractional-calculus operators that was recently proposed in [35]. First we consider briefly some of the many models of fractional calculus that may be covered by this class.

- A model proposed by Atangana and Baleanu [25], which was defined more rigorously in [36] and whose applications have been discussed in [37–39], utilises an integral transform with a one-parameter Mittag-Leffler function ($E_\nu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\nu+1)}$ for $\mathrm{Re}(\nu) > 0$) in the kernel and an arbitrary normalisation function multiplier:

$$\begin{aligned}
{}^{AB}_{c}I^{\nu}_{x}f(x) &= \frac{1-\nu}{B(\nu)}f(x) + \frac{\nu}{B(\nu)}\,{}^{RL}_{c}I^{\nu}_{x}f(x); \\
{}^{ABRL}_{c}D^{\nu}_{x}f(x) &= \frac{B(\nu)}{1-\nu}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\int_c^x E_\nu\left(\tfrac{-\nu}{1-\nu}(x-t)^\nu\right)f(t)\,\mathrm{d}t; \\
{}^{ABC}_{c}D^{\nu}_{x}f(x) &= \frac{B(\nu)}{1-\nu}\int_c^x E_\nu\left(\tfrac{-\nu}{1-\nu}(x-t)^\nu\right)f'(t)\,\mathrm{d}t.
\end{aligned}$$

- A model due to Prabhakar [40], which was formally connected to fractional calculus in [41] and whose applications have been discussed in [42,43], utilises an integral transform with a three-parameter Mittag-Leffler function ($E^{\rho}_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{(\rho)_n z^n}{\Gamma(n\mu+\nu)}$ for $\mathrm{Re}(\mu), \mathrm{Re}(\nu) > 0$) in the kernel:

$$\begin{aligned}
{}^{P}_{c}I^{\mu,\nu,\rho,\omega}_{x}f(x) &= \int_c^x (x-t)^{\nu-1}E^{\rho}_{\mu,\nu}\left(\omega(t-x)^\mu\right)f(t)\,\mathrm{d}t; \\
{}^{P}_{c}D^{\mu,\nu,\rho,\omega}_{x}f(x) &= \frac{\mathrm{d}^n}{\mathrm{d}x^n}\,{}^{P}_{c}I^{\mu,n-\nu,-\rho,\omega}_{x}f(x), \quad n = \lfloor \mathrm{Re}(\nu)\rfloor + 1.
\end{aligned}$$

- A model known as tempered fractional calculus [44,45], utilises an integral transform with the product of a power function and an exponential function in the kernel:

$$\begin{aligned}
{}^{T}_{c}I^{(\alpha,\beta)}_{x}f(x) &= \frac{1}{\Gamma(\alpha)}\int_c^c (x-t)^{\alpha-1}e^{-\beta(x-t)}f(t)\,\mathrm{d}t; \\
{}^{T}_{c}D^{(\alpha,\beta)}_{x}f(x) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}+\beta\right)^n\left({}^{T}_{c}I^{(n-\alpha,\beta)}_{x}f(x)\right), \quad n = \lfloor \mathrm{Re}(\nu)\rfloor + 1.
\end{aligned}$$

- A model due to Srivastava et al. [26] utilises an integral transform with a Fox H-function in the kernel:

$$ {}^{SHJ}_{c}I^{\omega;m,n,p,q;\alpha,\beta}_{x}f(x) = \int_c^x (x-t)^{\alpha-1}H^{m,n}_{p,q}\left(\omega(x-t)^\beta\right)f(t)\,\mathrm{d}t, $$

where $H^{m,n}_{p,q}(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} \Theta(s) z^s \,\mathrm{d}s$ with $\mathcal{L}$ being a Mellin–Barnes contour from $-i\infty$ to $i\infty$ and $\Theta(s) = \frac{\prod_{j=1}^{m}\Gamma(c_j-d_js)\prod_{j=1}^{n}\Gamma(1-a_j+b_js)}{\prod_{j=n+1}^{p}\Gamma(a_j-b_js)\prod_{j=m+1}^{q}\Gamma(1-c_j+d_js)}$ with parameters satisfying the conditions stated in [26].

The definition presented in [35] is general enough to cover all of the above as special cases, while not so general as to lose its connection to fractional calculus. For this reason, we use it as an example of a broad class of fractional-calculus operators as discussed in the previous section. We may define a class $\mathscr{A}$ consisting of all operators given by the following general integral transform formula:

$$ {}^{A}_{c}I^{\alpha,\beta}_{x} f(x) = \int_c^x (x-t)^{\alpha-1} A\left((x-t)^{\beta}\right) f(t)\,\mathrm{d}t, \tag{1}$$

where c is a constant in the extended real line (often taken as zero or $-\infty$), α and β are complex parameters with positive real parts, and $A(z) = \sum_{k=0}^{\infty} a_k z^k$ is a general analytic function whose coefficients $a_k \in \mathbb{C}$ are permitted to depend on α and β. We may consider x as a real variable larger than c; function spaces for f are discussed below. Many properties of this newly-defined operator were already proved in [35]; here, as well as providing a brief summary of these, we shall extend the discussion by considering more properties and potential subclassifications.

Part of fractional calculus. The following series formula, proved in [35], expresses this integral transform directly in terms of the Riemann–Liouville fractional integral:

$$ {}^{A}_{c}I^{\alpha,\beta}_{x} f(x) = \sum_{k=0}^{\infty} a_k \Gamma(\beta k+\alpha)\, {}^{RL}_{c}I^{\beta k+\alpha}_{x} f(x). \tag{2}$$

Formally, we may write this series formula as a relation between functional operators:

$$ {}^{A}_{c}I^{\alpha,\beta}_{x} = A_{\Gamma}\left({}^{RL}_{c}I^{\beta}_{x}\right) {}^{RL}_{c}I^{\alpha}_{x}, \tag{3}$$

where A_{Γ} is the transformed analytic function defined by:

$$ A(z) = \sum_{k=0}^{\infty} a_k z^k \Rightarrow A_{\Gamma}(z) = \sum_{k=0}^{\infty} a_k \Gamma(\beta k+\alpha) z^k. \tag{4}$$

From the relation (2), it is clear that the general operator (1) can always be described using only the classical Riemann–Liouville fractional integral, which is indisputably part of fractional calculus. Thus, we contend that it makes sense to consider the general operator (1) as always a part of fractional calculus as well. It is already known [35] that the series formula (2) may be used to prove various useful properties, such as for example the product rule and chain rule [46,47], for the general operator (1) directly from the corresponding known result for Riemann–Liouville.

Generalisation of well-known models. It was verified in [35], or is clear from the definitions, that all four of the specific example models of fractional calculus mentioned above are special cases of the general definition (1). Of course, this class does not cover all possible types of fractional calculus: there are also many that are not special cases of (1). These include the Hadamard and Erdelyi–Kober definitions, and some definitions involving special functions applied to $1-\frac{t}{x}$ instead of $x-t$, like [27,48].

Now we have confirmed that it makes sense to use (1) as the definition of a class of fractional-calculus operators: not all of fractional calculus, not just one specific model, but a general class that covers many cases and can be analysed in its own right. We continue with a further analysis of this class, its properties, and subclasses.

Historical connections and integral transform. The transformation between A and A_Γ defined by Equation (4) has some historical significance. In one of his "notes" working on what we now call the Mittag-Leffler function, Gösta Mittag-Leffler himself [49] considered the following transformation:

$$F(x) = \sum_{n=0}^{\infty} k_n x^n \Rightarrow F_\beta(x) = \sum_{n=0}^{\infty} \frac{k_n}{\Gamma(\beta n + 1)} x^n. \tag{5}$$

After relabelling notation, it is clear that Mittag-Leffler's transformation of F to F_β is precisely the inverse of the transformation (4) from A to A_Γ in the case where $\alpha = 1$. Mittag-Leffler noted that using $F(x) = \frac{1}{1-x}$ yields $F_\beta(x) = E_\beta(x)$, which we now call the Mittag-Leffler function. Thus, the study of the general class (1) intimately involved with the transformation (4) has some historical justification.

Furthermore, Mittag-Leffler [49] found the following relation between the functions in (5):

$$F(x) = \int_0^\infty e^{-\omega} F_\beta(\omega^\beta x)\, \mathrm{d}\omega.$$

By a natural extension of this result to the case of general α, β, we obtain the following integral transform between A and A_Γ:

$$A_\Gamma(z) = \int_0^\infty e^{-\omega} \omega^{\alpha-1} A(\omega^\beta z)\, \mathrm{d}\omega. \tag{6}$$

Going back to the classics is often a useful endeavour, and indeed, Mittag-Leffler's 1905 paper provided us with an elegant integral formula (6) for transforming between the functions A and A_Γ, which are important in the analysis of the class (1) of fractional models.

Local and non-local operators. In classical fractional calculus such as the Riemann–Liouville model, the operators are non-local. Like integrals, fractional derivatives depend not just on the behaviour of a function near a single point, but also on its behaviour in a wider region. This non-locality is often useful in modelling physical processes that have memory effects.

For our general class, the fractional integrals are always non-local since they are defined by an integral from c to x. The fractional derivatives as discussed in [35] are also non-local, except in the very special case when they reduce to the standard differentiation operations $\frac{\mathrm{d}^n}{\mathrm{d}x^n}$. This reminds us that our class does not cover the entirety of what has been called fractional calculus: any operators with locality properties are not contained in this class and must be classified using some other class.

Possession or lack of a semigroup property. One important property of any fractional-calculus operator is whether or not it has a semigroup property in one (or more) of the parameters associated with the operator. For example, in the Riemann–Liouville model, fractional integrals have a semigroup property while fractional derivatives do not. It is natural to ask, is the mth derivative/integral of the nth derivative/integral always equal to the $(m+n)$th derivative/integral?

For the general class (1), it was proved in [35] that a semigroup property in both α and β is impossible, but a semigroup property in the first parameter α can be obtained under the following condition on the coefficients a_k for the analytic function A:

$$\sum_{m+n=k} a_n(\alpha_1, \beta) a_m(\alpha_2, \beta) B(\alpha_1 + n\beta, \alpha_2 + m\beta) = a_k(\alpha_1 + \alpha_2, \beta) \qquad \forall k \in \mathbb{Z}_0^+. \tag{7}$$

It is easy to see that this class is general enough to cover both some fractional models with a semigroup property (such as Riemann–Liouville and Prabhakar) and some without a semigroup property (such as Atangana–Baleanu). However, Equation (7) gives us an explicit condition to know whether a given special case possesses a semigroup property or not.

We note that a semigroup property is not always required by physical motivations: fractional models either with or without such properties can be used to describe real-world problems [50].

Singular and non-singular operators. Another property that has been subject to much discussion is the singularity or non-singularity of fractional-calculus operators. The classical Riemann–Liouville model is defined by a singular integral, due to the power function $(x-t)^{\nu-1}$ in the integrand, but the singularity is integrable provided that $\mathrm{Re}(\nu) > 0$. Some other models [24,25] have been promoted due to the non-singularity of their defining integrals.

Again, the class (1) is general enough to cover both some singular and some non-singular fractional-calculus operators. This time it is easy to find a condition for which is which. We write $v_0(A) \geq 0$ for the valency (multiplicity or ramification index) of the analytic function $A(z)$ at the point $z=0$, so that $A(z) = z^{v_0(A)}B(z)$ for some function B that is analytic and nonzero in a neighbourhood of $z=0$. Then, the general integral transform (1) is non-singular if:

$$\mathrm{Re}\left(\alpha + \beta v_0(A)\right) \geq 1$$

(the most usual case is $\alpha = 1, v_0(A) = 0$), and it has an integrable singularity if:

$$0 < \mathrm{Re}\left(\alpha + \beta v_0(A)\right) < 1.$$

(In the case where $\mathrm{Re}\left(\alpha + \beta v_0(A)\right) \leq 0$, we have a non-integrable singularity, and the integral (1) is not defined since the function cannot be integrated near $t = x$.)

Again, neither singularity nor non-singularity is always required by physical motivations. Both singular and non-singular fractional-calculus operators have discovered many applications to real-world problems [51].

Dual operators. The definition (1) is, for a left-sided fractional integral operator, the integration being performed from c to x. We can equally well define a right-sided fractional integral operator, for x contained in some fixed interval $[c,d]$, namely:

$$ {}^{A}_{x}I^{\alpha,\beta}_{d} f(x) = \int_x^d (t-x)^{\alpha-1} A\left((t-x)^{\beta}\right) f(t)\,\mathrm{d}t. \tag{8}$$

This modified operator has the property that it is the dual of the original left-sided fractional integral operator:

$$\int_c^d \left({}^{A}_{c}I^{\alpha,\beta}_{x} f(x)\right) g(x)\,\mathrm{d}x = \int_c^d f(x)\left({}^{A}_{x}I^{\alpha,\beta}_{d} g(x)\right)\,\mathrm{d}x.$$

This can be quickly proved using Fubini's theorem, and it is an analogue of the integration by parts rule for standard integrals and Riemann–Liouville fractional integrals.

Functional bounds. The operator ${}^{A}_{c}I^{\alpha,\beta}_{x}$ defined by (1) defines a map between function spaces, and it may be useful to consider bounds and properties of this functional map.

In [35] it was proved that ${}^{A}_{c}I^{\alpha,\beta}_{x}$ is bounded on the space $L^1[c, c+R]$, with

$$\left\|{}^{A}_{c}I^{\alpha,\beta}_{x} f(x)\right\|_{L^1} \leq R^{\mathrm{Re}(\alpha)} \sup_{|z|<R^{\mathrm{Re}(\beta)}} |A(z)| \left\|f(x)\right\|_{L^1}.$$

Using Young's inequality for convolutions, we can prove that the same operator is also bounded on any L^p space, with

$$\left\|{}^{A}_{c}I^{\alpha,\beta}_{x} f(x)\right\|_{L^p} \leq R^{\mathrm{Re}(\alpha)} \sup_{|z|<R^{\mathrm{Re}(\beta)}} |A(z)| \left\|f(x)\right\|_{L^p}$$

for all $p \in [1, \infty]$. This functional space bound strengthens the pure mathematical foundation for the general class of operators, and it may be useful in the future study of fractional differential equations using operators in this class.

Fractionally-iterated operators. Some fractional operators in the literature have arisen by means of iteration. The process here is to start from some standard operator K between functions, write a formula for the iterated operator K^n, and then generalise that formula to non-integer values of n. This idea is of course what gave rise to fractional calculus in the first place, with the starting operator being simply $K = \frac{\mathrm{d}}{\mathrm{d}x}$, but it is also possible to apply the same process from a starting operator K which is already fractional.

However, doing so does not always yield a new fractional operator. In some cases, the process does give rise to new types of fractional calculus [52,53], but this relies on the semigroup property not being valid for the starting operator D. For example, if K is the Riemann–Liouville fractional integral ${}^{RL}_{c}I^{\alpha}$, then $K^n = {}^{RL}_{c}I^{n\alpha}$, and so, the fractionally-iterated operator $K^{\nu} = {}^{RL}_{c}I^{\nu\alpha}$ is also a Riemann–Liouville fractional integral, not a new type of operator.

Some of the issues around fractional iteration were also discussed in ([54], §5).

4. Conclusions

Fractional calculus is currently in a stage of rapid and continuous expansion and development. Right now, several different fractional-calculus operators are being proposed, with many different behaviours such as singular or non-singular, semigroup law or none, etc. On the other hand, several classifications of fractional-calculus operators have been suggested, proposing a variety of possible conditions that might be imposed. Some models of fractional calculus are subject to debate, being acceptable under one classification system, but not another.

There are many different points of view and approaches being taken in the study of fractional calculus. In terms of real-world problems, it is important to remember that not everything is known: some systems and behaviours are not yet understood using fractional calculus. In our opinion, going to the extremes—e.g., creating operators without regard for applications, or imposing hard conditions for all potential fractional-calculus operators—will not lead to significant progress in the understanding of the still hidden flavours of fractional calculus and their applications.

Instead of imposing criteria, we suggest organising fractional-calculus operators into classes having different types of properties. One large class of operators, presented in detail in this manuscript, is one example of a class with real-world applications where both singular and non-singular operators, both with and without semigroup properties, may live together in the same class. We think the words "true" and "false" are too simplistic to describe the complex process of debates that is occurring nowadays.

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