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Natural Transform Decomposition Method for Solving Fractional-Order Partial Differential Equations with Proportional Delay

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Abstract: In the present article, fractional-order partial differential equations with proportional delay, including generalized Burger equations with proportional delay are solved by using Natural transform decomposition method. Natural transform decomposition method solutions for both fractional and integer orders are obtained in series form, showing higher convergence of the proposed method. Illustrative examples are considered to confirm the validity of the present method. Therefore, Natural transform decomposition method is considered to be one of the best analytical technique, to solve fractional-order linear and non-linear Partial deferential equations particularly fractional-order partial differential equations with proportional delay.

Keywords: natural transform decomposition method; fractional-order burger equations and partial differential equations with proportional delay; caputo operator

1. Introduction

A delay differential equations (DDE's) is one in which the derivative of the unknown function at a specific time is given as far as the values of function at the pastime. In the mathematical evaluation of a physical procedure, we commonly assume that, the characteristic of idea taken into account depends only on the present situation, a hypothesis which established for a dynamical system. However, there are a lot of physical situations which are not feasible under such hypothesis and may lead poor execution. In this circumstance it is smarter to think about that, the system conduct also relies upon the previous state. These schemes are called time-delay schemes [1,2].

Many physical procedure both natural and artificial, in economics, physics, chemistry, biology, medicine, and engineering involves time delay. A more realistic model must contain some past history of the system. Infect time delay occurs in almost every physical situation, responsible for results very closed to reality. Introduction of delay in the model, builds up its elements, and allows a detail investigation of this present reality wonders [3]. DDE's are considered to be very useful in biological as well as physical sciences for which the evaluation of the model does not depend only on the current situation with the scheme, yet in addition on the previous history, such as delayed epidemic model, delay modeling in psychology, exothermic and endothermic chemical reactions in chemical industry [4].

Generally it is tough experience in order to find an exact result of a wide range of differential equations. Nevertheless, numerous effective techniques has been developed to find the approximate result of partial differential equations (PDE's) with proportional delay, such as functional constraint's method [5], iterated pseudo spectral method [6], reduce differential transform method (RDTM) for PDE's by Abazari and Ganji [7], homotopy perturbation method for time fractional PDE's by Sakar et al. and F. Shakeri [4,8], variational iteration method (VIM) [9], predictor-corrector algorithm [10], Adams-Bashforth-Moulton algorithm [11], a numerical method for delayed differential equations in fractional order: based on the definition of GL [12], an extended predictor-corrector algorithm for differential equations in variable-order fractional delay [13], Optimal Periodic-Gain fractional delayed state feedback control for linear periodic time-delayed systems [14], spectral collocation methods [15], group analysis method for Burgers equation due to Tanthanuch [16], and the two-dimensional differential transform method the solution of specific class of PDE's [17].

Natural transform and Adomian decomposition method are two powerful methods that have been used to have developed Natural transform decomposition method. Many physical phenomena which are modeled by PDE's and fractional PDE's are solved by using NTDM, such as the analytical solution of couple system of non-linear PDE's are suggested in [18], the solution non-linear ODE,s are successfully presented in [19], non-linear PDE's [20], fractional unsteady flow of a poly tropic gas model [21], fractional telegraph equations [22], fractional Fokker-Plank equation and Schrodinger equation [23]. The rest of the article is structured as: In Section 2, we recall several basic properties and definition form Natural transform and fractional calculus. In Section 3, the idea of Natural transform decomposition method. In Section 4, we explain many problems to maintenance the accuracy and efficiency of the present method, while the last section is devoted conclusion.

2. Preliminaries

Definition 1. *The natural transform of $g(t)$ is define as [24,25]:*

$$\mathbb{N}^{+}[g(t)] = Q(s,u) = \frac{1}{u}\int_{0}^{\infty} e^{\frac{-st}{u}} g(t)dt; \quad s,u > 0$$

where s and u are the transform variables.

Definition 2. *The inverse natural transform of a function is defined by*

$$\mathbb{N}^{-}[Q(s,u)] = g(t) = \frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty} e^{\frac{st}{u}} Q(s,u)ds;$$

where s and u are the Natural transform variables and p is a real constant and the integral is taken along $s = p$ in the complex plane $s = x + iy$.

Definition 3. *Natural Transform of nth Derivative*

If $g^{n}(t)$ is the nth derivative of function $g(t)$ is given by,

$$\mathbb{N}[g^{n}(t)] = Q_n(s,u) = \frac{s^n}{u^n}Q(s,u) - \sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{u^{n-k}}g^{k}(0), \quad n{\geq}1.$$

Theorem 1. *If $H(s,u)$, $L(s,u)$ are the transform function respectively $h(t)$ and $l(t)$ is given by;*

$$\mathbb{N}[h * l] = uH(s,u)L(s,u)$$

where $h * l$ is convolution of two functions h and l.

Definition 4. *Riemann–Liouville fractional integral [26]*

$$I_x^{\gamma} g(x) = \begin{cases} g(x) & \text{if } \gamma = 0 \\ \frac{1}{\Gamma(\gamma)} \int_0^x (x-v)^{\gamma-1} g(v) dv & \text{if } \gamma > 0, \end{cases}$$

where Γ denotes the gamma function define by,

$$\Gamma(\omega) = \int_0^{\infty} e^{-x} x^{\omega-1} dx \qquad \omega \in \mathbb{C}.$$

Definition 5. *The Caputo operator of order γ for a fractional derivative is given by the following mathematical expression for $n \in \mathbb{N}$, $x > 0$, $g \in \mathbb{C}_t$, $t \geq -1$ [27,28].*

$$D^{\gamma} g(x) = \frac{\partial^{\gamma} g(x)}{\partial t^{\gamma}} = \begin{cases} I^{n-\gamma} \left[\frac{\partial^{\gamma} g(x)}{\partial t^{\gamma}} \right], & \text{if } (n-1 < \gamma \leq n), n \in \mathbb{N} \\ \frac{\partial^{\gamma} g(x)}{\partial t^{\gamma}}, \end{cases}$$

Definition 6. *The Mittag-Leffler function $E_{\gamma}(p)$ for $\gamma > 0$ is defined by the following subsequent series,*

$$E_{\gamma}(p) = \sum_{n=0}^{\infty} \frac{p^n}{\Gamma(\gamma n + 1)} \qquad \gamma > 0 \quad p \in \mathbb{C},$$

Theorem 2. *Here, we will study the convergence analysis as same manner in [29] of the NTDM applied to the fractional partial differential equations with proportional delay. Let us consider the Hilbert space H which may define by $H = L^2((\alpha, \beta) X[0, T])$ the set of applications:*

$$u : (\alpha, \beta) X[0, T] \rightarrow \quad with \quad \int_{(\alpha,\beta)X[0,T]} u^2(x, s) ds d\theta \quad < +\infty.$$

Now we consider the fractional partial differential equations with proportional delay in the above assumptions and let us denote

$$L(u) = \frac{\partial^{\gamma} u}{\partial t^{\gamma}},$$

then the fractional partial differential equations with proportional delay becomes, in an operator form

$$L(u) = \frac{\partial^2 v(x,t)}{\partial x^2} - v(\frac{x}{2}, \frac{t}{2}) \frac{\partial v(x, \frac{t}{2})}{\partial x}.$$

The NTDM is convergence, if the following two hypotheses are satisfied:

$$(H1)(L(u) - L(v), u - v) \geq k \|u - v\|^2; k > 0, \forall u, v \epsilon H.$$

$H(2)$ whatever may be $M > 0$, there exist a constant $C(M) > 0$ such that for $u, v \epsilon H$ with $\|u\| \leq M$, $\|v\| \leq M$ we have $(L(u) - L(v), u - v) \leq C(M) \|u - v\| \|w\|$ for every $w \epsilon H$.

3. Idea of Fractional Natural Transform Decomposition Method

In this section, the NTDM to find the general solution FPDE's with proportional delay.

$$D^{\gamma} v(x,t) + Lv(x,t) + Nv(\frac{x}{2}, \frac{t}{2}) = q(x,t), \quad x, t \geq 0, \quad m - 1 < \gamma < m, \tag{1}$$

where $D^{\gamma} = \frac{\partial^{\gamma}}{\partial t^{\gamma}}$ the Caputo Operator $\gamma, m \in \mathbb{N}$, where L and N are linear and non-linear functions, q is the origin function.

The initial condition is

$$v(x,0) = k(x), \tag{2}$$

Applying the Natural transform to Equation (1), we have

$$\mathbb{N}^{+}\left[D^{\gamma}v(x,t)\right] + \mathbb{N}^{+}\left[Lv(x,t) + Nv(\frac{x}{2},\frac{t}{2})\right] = \mathbb{N}^{+}\left[q(x,t)\right], \tag{3}$$

and using the differentiation property of Natural transform, we get

$$\frac{s^{\gamma}}{u^{\gamma}}\mathbb{N}^{+}\left[v(x,t)\right] - \frac{s^{\gamma-1}}{u^{\gamma}}v(x,0) = \mathbb{N}^{+}\left[q(x,t)\right] - \mathbb{N}^{+}\left[Lv(x,t) + Nv(\frac{x}{2},\frac{t}{2})\right],$$

$$\mathbb{N}^{+}\left[v(x,t)\right] = \frac{k(x)}{s} + \frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[q(x,t)\right] - \frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[Lv(x,t) + Nv(\frac{x}{2},\frac{t}{2})\right]. \tag{4}$$

The NTDM solution $v(x,t)$ is represented by the following infinite series

$$v(x,t) = \sum_{j=0}^{\infty} v_j(x,t), \tag{5}$$

and the non-linear terms (if any) in problem are define by the infinite series of Adomian polynomials,

$$Nv(\frac{x}{2},\frac{t}{2}) = \sum_{j=0}^{\infty} A_j, \tag{6}$$

$$A_j = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j}\left[N\sum_{j=0}^{\infty}(\lambda^j v_j)\right]\right]_{\lambda=0}, \qquad j = 0,1,2... \tag{7}$$

substitution Equation (5) and Equation (6) in Equation (4), we get

$$\mathbb{N}^{+}\left[\sum_{j=0}^{\infty} v(x,t)\right] = \frac{k(x)}{s} + \frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[q(x,t)\right] - \frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[L\sum_{j=0}^{\infty} v_j(x,t) + \sum_{j=0}^{\infty} A_j\right]. \tag{8}$$

Applying the linearity of the Natural transform,

$$\mathbb{N}^{+}\left[v_0(x,t)\right] = \frac{v(x,0)}{s} + \frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[q(x,t)\right],$$

$$\mathbb{N}^{+}\left[v_1(x,t)\right] = -\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[Lv_0(x,t) + A_0\right].$$

Generally, we can write

$$\mathbb{N}^{+}\left[v_{j+1}(x,t)\right] = -\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[Lv_j(x,t) + A_j\right], \quad j \geq 1. \tag{9}$$

Applying the inverse Natural transform, in Equation (9)

$$v_0(x,t) = k(x,t)$$

$$v_{j+1}(x,t) = -\mathbb{N}^{-}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[Lv_j(x,t) + A_j\right]\right]. \tag{10}$$

4. Results

Example 1. *Consider the proportional delay of generalized Burgers equation as given by [8]:*

$$\frac{\partial^\gamma v}{\partial t^\gamma} - \frac{\partial^2 v(x,t)}{\partial x^2} - v(\frac{x}{2},\frac{t}{2})\frac{\partial v(x,\frac{t}{2})}{\partial x} - \frac{1}{2}v(x,t) = 0, \tag{11}$$

with initial condition

$$v(x,0) = x \tag{12}$$

Taking Natural transform of Equation (11),

$$\frac{s^\gamma}{u^\gamma}\mathbb{N}^+\left[v(x,t)\right] - \frac{s^{\gamma-1}}{u^\gamma}v(x,0) = \mathbb{N}^+\left[\frac{\partial^2 v(x,t)}{\partial x^2} + v(\frac{x}{2},\frac{t}{2})\frac{\partial v(x,\frac{t}{2})}{\partial x} + \frac{1}{2}v(x,t)\right].$$

Applying inverse Natural transform

$$v(x,t) = \mathbb{N}^-\left[\frac{v(x,0)}{s} - \frac{u^\gamma}{s^\gamma}\mathbb{N}^+\left[\frac{\partial^2 v(x,t)}{\partial x^2} + v(\frac{x}{2},\frac{t}{2})\frac{\partial v(x,\frac{t}{2})}{\partial x} + \frac{1}{2}v(x,t)\right]\right].$$

Using ADM procedure, we get

$$v_0(x,t) = \mathbb{N}^-\left[\frac{v(x,0)}{s}\right] = \mathbb{N}^-\left[\frac{x}{s}\right]$$

$$v_0(x,t) = x \tag{13}$$

$$\sum_{j=0}^{\infty} v_{j+1}(x,t) = \mathbb{N}^-\left[\frac{u^\gamma}{s^\gamma}\mathbb{N}^+\left[\sum_{j=0}^{\infty}(v_{xx}(x,t))_j + \sum_{j=0}^{\infty}A_j(vv_x) + \frac{1}{2}\sum_{j=0}^{\infty}v_j(x,t)\right]\right], \qquad j = 0,1,2,\ldots$$

$$\begin{aligned} A_0(vv_x) &= v_0\frac{\partial v_0}{\partial x}, \\ A_1(vv_x) &= v_0\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_0}{\partial x}, \\ A_2(vv_x) &= v_0\frac{\partial v_2}{\partial x} + v_1\frac{\partial v_1}{\partial x} + v_2\frac{\partial v_0}{\partial x}. \end{aligned}$$

for $j = 0$

$$\begin{aligned} v_1(x,t) &= \mathbb{N}^-\left[\frac{u^\gamma}{s^\gamma}\mathbb{N}^+\left[\frac{\partial^2 v_0(x,t)}{\partial x^2} + v_0(\frac{x}{2},\frac{t}{2})\frac{\partial v_0(x,\frac{t}{2})}{\partial x} + \frac{1}{2}v_0(x,t)\right]\right], \\ v_1(x,t) &= \mathbb{N}^-\left[\frac{xu^\gamma}{s^{\gamma+1}}\right] = x\frac{t^\gamma}{\Gamma(\gamma+1)}. \end{aligned} \tag{14}$$

The subsequent terms are

$$
\begin{aligned}
&v_2(x,t) = \mathbb{N}^{-}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[\frac{\partial^2 v_1(x,t)}{\partial x^2} + v_0(\frac{x}{2},\frac{t}{2})\frac{\partial v_1(x,\frac{t}{2})}{\partial x} + v_1(\frac{x}{2},\frac{t}{2})\frac{\partial v_0(x,\frac{t}{2})}{\partial x} + \frac{1}{2}v_1(x,t)\right]\right],\\
&v_2(x,t) = \frac{x(2+2^{\gamma})t^{2\gamma}}{2^{\gamma}2\Gamma(2\gamma+1)}.\\
&v_3(x,t) = \mathbb{N}^{-}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[\frac{\partial^2 v_2(x,t)}{\partial x^2} + v_0(\frac{x}{2},\frac{t}{2})\frac{\partial v_2(x,\frac{t}{2})}{\partial x} + v_1(\frac{x}{2},\frac{t}{2})\frac{\partial v_1(x,\frac{t}{2})}{\partial x} + v_2(\frac{x}{2},\frac{t}{2})\frac{\partial v_0(x,\frac{t}{2})}{\partial x} + \frac{1}{2}v_2(x,t)\right]\right],\\
&v_3(x,t) = \frac{xt^{3\gamma}}{4\Gamma(3\gamma+1)}\left(1 + \frac{2}{2^{\gamma}} + \frac{2}{2^{2\gamma}} + \frac{2^2}{2^{3\gamma}} + \frac{2\Gamma(1+2\gamma)}{2^{\gamma}\Gamma(1+\gamma)^2}\right).\\
&v_4(x,t) = \mathbb{N}^{-}[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}[\frac{\partial^2 v_3(x,t)}{\partial x^2} + v_0(\frac{x}{2},\frac{t}{2})\frac{\partial v_3(x,\frac{t}{2})}{\partial x} + v_1(\frac{x}{2},\frac{t}{2})\frac{\partial v_2(x,\frac{t}{2})}{\partial x} + v_2(\frac{x}{2},\frac{t}{2})\frac{\partial v_1(x,\frac{t}{2})}{\partial x}\\
&+ v_3(\frac{x}{2},\frac{t}{2})\frac{\partial v_0(x,\frac{t}{2})}{\partial x} + \frac{1}{2}v_3(x,t)]],\\
&v_4(x,t) = \frac{xt^{4\gamma}}{8\Gamma(4\gamma+1)}[1 + \frac{2^9}{2^{6\gamma}} + \frac{2^8}{2^{5\gamma}} + \frac{3\times 2^7}{2^{3\gamma}} + \frac{2^7}{2^{2\gamma}} + \frac{2^7}{2^{\gamma}} + \frac{2^8}{2^{4\gamma}} + (\frac{2^8}{2^{5\gamma}} + \frac{2^7}{2^{2\gamma}})\frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+1)^2}\\
&+ (\frac{2^9}{2^{4\gamma}} + \frac{2^8}{2^{3\gamma}})\frac{\Gamma(3\gamma+1)}{\Gamma(\gamma+1)\Gamma(2\gamma+1)}].
\end{aligned}
\tag{15}
$$

The NTDM solution for example 1 is

$$v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + v_4(x,t)...$$

$$
\begin{aligned}
v(x,t) &= x + x\frac{t^{\gamma}}{\Gamma(\gamma+1)} + \frac{x(2+2^{\gamma})t^{2\gamma}}{2^{\gamma}2\Gamma(2\gamma+1)} + \frac{xt^{3\gamma}}{4\Gamma(3\gamma+1)}\left(1 + \frac{2}{2^{\gamma}} + \frac{2}{2^{2\gamma}} + \frac{2^2}{2^{3\gamma}} + \frac{2\Gamma(1+2\gamma)}{2^{\gamma}\Gamma(1+\gamma)^2}\right)\\
&+ \frac{xt^{4\gamma}}{8\Gamma(4\gamma+1)}[1 + \frac{2^9}{2^{6\gamma}} + \frac{2^8}{2^{5\gamma}} + \frac{3\times 2^7}{2^{3\gamma}} + \frac{2^7}{2^{2\gamma}} + \frac{2^7}{2^{\gamma}} + \frac{2^8}{2^{4\gamma}} + (\frac{2^8}{2^{5\gamma}} + \frac{2^7}{2^{2\gamma}})\frac{\Gamma(2\gamma+1)}{\Gamma(\gamma+1)^2}\\
&+ (\frac{2^9}{2^{4\gamma}} + \frac{2^8}{2^{3\gamma}})\frac{\Gamma(3\gamma+1)}{\Gamma(\gamma+1)\Gamma(2\gamma+1)}] + ...
\end{aligned}
$$

when $\gamma = 1$, then NTDM solution is

$$v(x,t) = x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + ...\right). \tag{16}$$

This result is calculated to the exact solution of Equation (11);

$$v(x,t) = xe^t$$

The NTDM solution for $\gamma = 0.70$ and 1 are reported in Table 1. The surface solution behavior of $v(x,t)$ for different values of $\gamma = 1, 0.80, 0.70$ and 0.50 are depicted in Figure 1, and the plots of the solution for at different time intervals $t \leq 1$ is depicted in Figure 1. The error plot of Example 1 show in Figure 2.

Table 1. Numerical results of Example 1 at $\gamma = 0.70, 1$ and Absolute Error.

x_1	t_1	$\gamma = 0.70$	NTDM ($\gamma = 1$)	HPM [8]	Exact	AE ($\gamma = 1$)
0.25	0.25	0.390443474	0.321004230	0.321004232	0.3210063542	2.12240×10^{-6}
	0.50	0.529812605	0.412109374	0.412109375	0.4121803178	7.09428×10^{-5}
	0.75	0.695464609	0.528686521	0.528686522	0.5292500042	5.63480×10^{-4}
	1.00	0.893414574	0.677083333	0.677083335	0.6795704570	2.48712×10^{-3}
0.50	0.25	0.780886948	0.642008463	0.642008465	0.6420127085	4.24500×10^{-6}
	0.50	1.059625210	0.824218749	0.824218750	0.8243606355	1.41885×10^{-4}
	0.75	1.390929217	1.057373039	1.057373040	1.0585000080	1.12696×10^{-3}
	1.00	1.786829148	1.354166666	1.354166670	1.3591409140	4.97424×10^{-3}

Table 1. *Cont.*

x_1	t_1	$\gamma = 0.70$	NTDM ($\gamma = 1$)	HPM [8]	Exact	AE ($\gamma = 1$)
0.75	0.25	1.171330424	0.963012695	0.963012698	0.9630190628	6.36750×10^{-6}
	0.50	1.589437815	1.236328118	1.236328120	1.2365409530	2.12828×10^{-4}
	0.75	2.086393828	1.586059568	1.586059570	1.5877500130	1.69044×10^{-3}
	1.00	2.680243723	2.031248999	2.031250000	2.0387113710	7.46137×10^{-3}

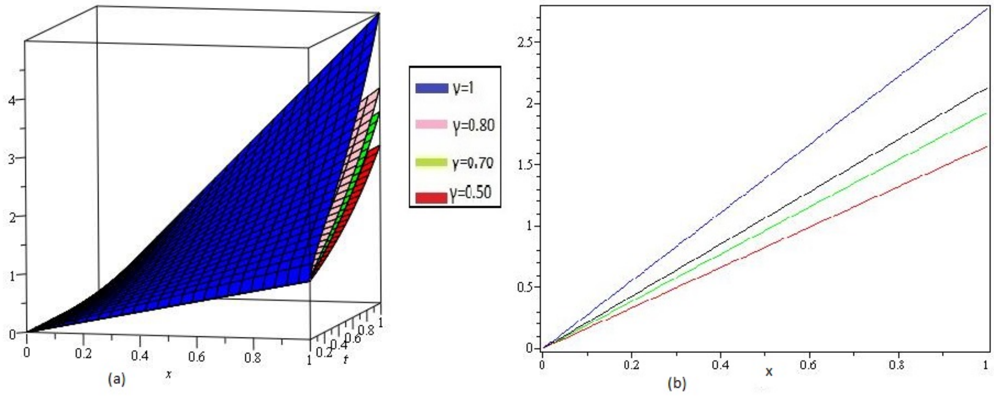

Figure 1. Numerical results of Example 1, for different value of γ and $t = 1$.

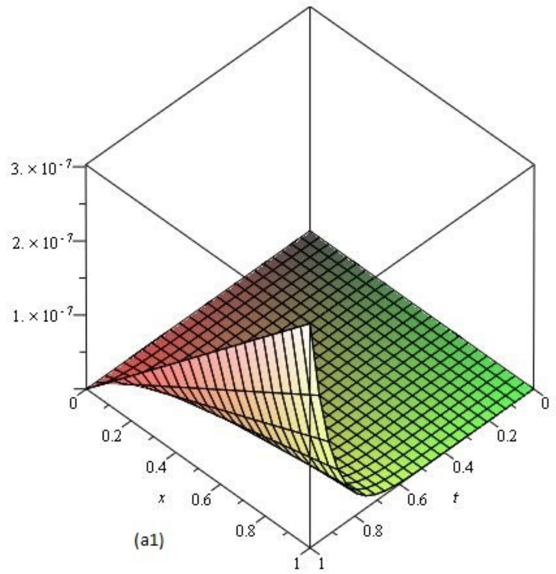

Figure 2. The error plot of Example 1, at $\gamma = 1$.

Example 2. *Consider the FPDE's with proportional delay as given in [8]:*

$$\frac{\partial^\gamma v(x,t)}{\partial t^\gamma} - v(x, \frac{t}{2})\frac{\partial^2 v(x, \frac{t}{2})}{\partial x^2} + v(x,t) = 0, \tag{17}$$

with the initial condition is

$$v(x,0) = x^2 \tag{18}$$

Taking Natural transform of Equation (17),

$$\frac{s^\gamma}{u^\gamma}\mathbb{N}^+\left[v(x,t)\right] - \frac{s^{\gamma-1}}{u^\gamma}v(x,0) = \mathbb{N}^+\left[v(x, \frac{t}{2})\frac{\partial^2 v(x, \frac{t}{2})}{\partial x^2} - v(x,t)\right].$$

Applying inverse Natural transform

$$v(x,t) = \mathbb{N}^{-}\left[\frac{v(x,0)}{s} - \frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[v(x,\frac{t}{2})\frac{\partial^2 v(x,\frac{t}{2})}{\partial x^2} - v(x,t)\right]\right].$$

Using ADM procedure, we get

$$v_0(x,t) = \mathbb{N}^{-}\left[\frac{v(x,0)}{s}\right] = \mathbb{N}^{-}\left[\frac{x^2}{s}\right]$$

$$v_0(x,t) = x^2 \tag{19}$$

$$\sum_{j=0}^{\infty} v_{j+1}(x,t) = \mathbb{N}^{-}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[\sum_{j=0}^{\infty} B_j(vv_{xx}) - \sum_{j=0}^{\infty} v_j(x,t)\right]\right], \qquad j = 0,1,2,...$$

$$\begin{aligned} B_0(vv_{xx}) &= v_0\frac{\partial^2 v_0}{\partial x^2}, \\ B_1(vv_{xx}) &= v_0\frac{\partial^2 v_1}{\partial x^2} + v_1\frac{\partial^2 v_0}{\partial x^2}, \\ B_2(vv_{xx}) &= v_0\frac{\partial^2 v_2}{\partial x^2} + v_1\frac{\partial^2 v_1}{\partial x^2} + v_2\frac{\partial^2 v_0}{\partial x^2}. \end{aligned}$$

for $j = 0$

$$\begin{aligned} v_1(x,t) &= \mathbb{N}^{-}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[v_0(x,\frac{t}{2})\frac{\partial^2 v_0(x,\frac{t}{2})}{\partial x^2} + v_0(x,t)\right]\right], \\ v_1(x,t) &= \mathbb{N}^{-}\left[\frac{x^2 u^{\gamma}}{s^{\gamma+1}}\right] = x^2\frac{t^{\gamma}}{\Gamma(\gamma+1)}. \end{aligned} \tag{20}$$

The subsequent terms are

$$\begin{aligned} v_2(x,t) &= \mathbb{N}^{-}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[v_0(x,\frac{t}{2})\frac{\partial^2 v_1(x,\frac{t}{2})}{\partial x^2} + v_1(x,\frac{t}{2})\frac{\partial^2 v_0(x,\frac{t}{2})}{\partial x^2} - v_1(x,t)\right]\right], \\ v_2(x,t) &= \frac{x^2(2-2^{\gamma})t^{2\gamma}}{2^{\gamma}\Gamma(2\gamma+1)} \\ v_3(x,t) &= \mathbb{N}^{-}\left[\frac{u^{\gamma}}{s^{\gamma}}\mathbb{N}^{+}\left[v_0(x,\frac{t}{2})\frac{\partial v_2(x,\frac{t}{2})}{\partial x} + v_1(x,\frac{t}{2})\frac{\partial v_1(x,\frac{t}{2})}{\partial x} + v_2(x,\frac{t}{2})\frac{\partial v_0(x,\frac{t}{2})}{\partial x} - v_2(x,t)\right]\right], \\ v_3(x,t) &= \frac{x^2 t^{3\gamma}}{\Gamma(3\gamma+1)}\left(1 - \frac{2}{2^{\gamma}} - \frac{2^2}{2^{2\gamma}} + \frac{2^4}{2^{3\gamma}} + \frac{2\Gamma(2\gamma+1)}{2^{\gamma}\Gamma(\gamma+1)^2}\right) \end{aligned} \tag{21}$$

The NTDM solution for Example 2 is

$$v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + v_4(x,t)...$$

$$v(x,t) = x^2 + x^2\frac{t^{\gamma}}{\Gamma(\gamma+1)} + \frac{x^2(2-2^{\gamma})t^{2\gamma}}{2^{\gamma}\Gamma(2\gamma+1)} + \frac{x^2 t^{3\gamma}}{\Gamma(3\gamma+1)}\left(1 - \frac{2}{2^{\gamma}} - \frac{2^2}{2^{2\gamma}} + \frac{2^4}{2^{3\gamma}} + \frac{2\Gamma(2\gamma+1)}{2^{\gamma}\Gamma(\gamma+1)^2}\right)....$$

when $\gamma = 1$*, then NTDM solution is*

$$v(x,t) = x^2\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + ...\right). \tag{22}$$

This result is calculated to the exact solution of Equation (17);

$$v(x,t) = x^2 e^t$$

The NTDM solution for $\gamma = 0.70$ and 1 are reported in Table 2. The surface solution behavior of $v(x,t)$ for different values of $\gamma = 1, 0.80, 0.70$ and 0.50 are depicted in Figure 3, and the plots of the solution for at different time intervals $t \leq 1$ is depicted in Figure 3. The error plot of Example 2 show in Figure 4.

Table 2. Numerical results of Example 2 at $\gamma = 0.70$ and $\gamma = 1$.

x_1	t_1	$\gamma = 0.70$	$\gamma = 1$	Exact	Absolute Error
0.25	0.25	0.0976593036	0.08025158853	0.08025158856	3.00000×10^{-11}
	0.50	0.1330574978	0.10304507290	0.10304507940	6.50000×10^{-9}
	0.75	0.1765973237	0.13231233190	0.13231250110	1.69200×10^{-7}
	1.00	0.2314929791	0.16989087310	0.16989261420	1.74110×10^{-6}
0.50	0.25	0.3906372147	0.32100635410	0.32100635420	1.00000×10^{-10}
	0.50	0.5322299913	0.41218029200	0.41218031780	2.58000×10^{-8}
	0.75	0.7063892947	0.52924932750	0.52925000420	6.76700×10^{-7}
	1.00	0.9259719163	0.67956349210	0.67957045700	6.96490×10^{-6}
1	0.25	1.5625488590	1.28402541600	1.28402541700	1.00000×10^{-9}
	0.50	2.1289199640	1.64872116800	1.64872127100	1.03000×10^{-7}
	0.75	2.8255571770	2.11699731000	2.11700001700	2.70700×10^{-6}
	1.00	3.7038876660	2.71825396900	2.71828182800	2.78590×10^{-5}

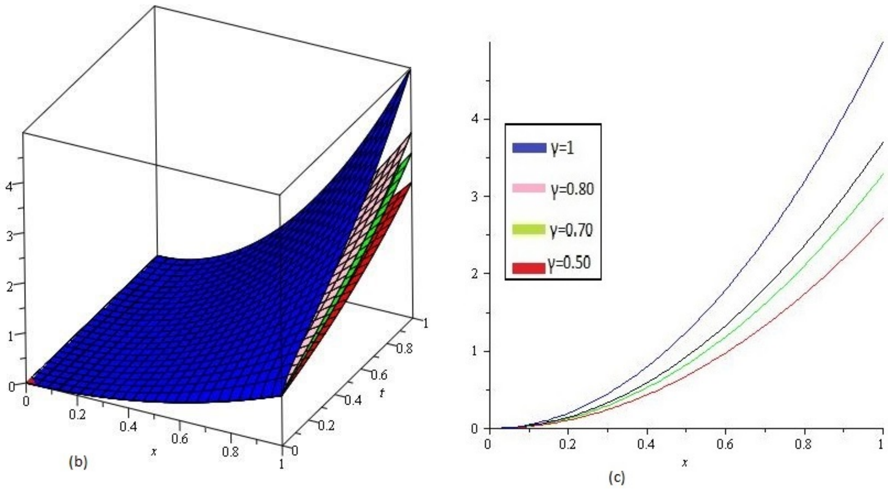

Figure 3. Numerical results of $v(x,t)$ for Example 2, for different value of γ and $t = 1$.

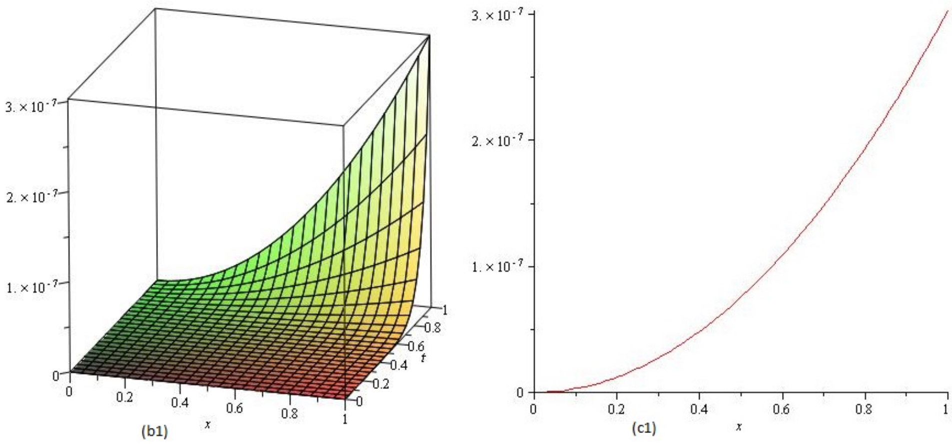

Figure 4. The error plot of Example 2, at $\gamma = 1$ and $t = 1$.

Example 3. *Consider the TFPDE with proportional delay as given by [8]:*

$$\frac{\partial^\gamma v(x,t)}{\partial t^\gamma} - \frac{\partial^2 v(\frac{x}{2},\frac{t}{2})}{\partial x^2}\frac{\partial v(\frac{x}{2},\frac{t}{2})}{\partial x} + \frac{1}{8}v(x,t) + v(x,t) = 0, \tag{23}$$

with the initial condition is

$$v(x,0) = x^2 \tag{24}$$

Taking Natural transform of Equation (23),

$$\frac{s^\gamma}{u^\gamma}\mathbb{N}^+\left[v(x,t)\right] - \frac{s^{\gamma-1}}{u^\gamma}v(x,0) = \mathbb{N}^+\left[\frac{\partial^2 v(\frac{x}{2},\frac{t}{2})}{\partial x^2}\frac{\partial v(\frac{x}{2},\frac{t}{2})}{\partial x} - \frac{1}{8}v(x,t) - v(x,t)\right].$$

Applying inverse Natural transform

$$v(x,t) = \mathbb{N}^-\left[\frac{v(x,0)}{s} - \frac{u^\gamma}{s^\gamma}\mathbb{N}^+\left[\frac{\partial^2 v(\frac{x}{2},\frac{t}{2})}{\partial x^2}\frac{\partial v(\frac{x}{2},\frac{t}{2})}{\partial x} - \frac{1}{8}v(x,t) - v(x,t)\right]\right].$$

Using ADM procedure, we get

$$v_0(x,t) = \mathbb{N}^-\left[\frac{v(x,0)}{s}\right] = \mathbb{N}^-\left[\frac{x^2}{s}\right]$$

$$v_0(x,t) = x^2 \tag{25}$$

$$\sum_{j=0}^{\infty} v_{j+1}(x,t) = \mathbb{N}^-\left[\frac{u^\gamma}{s^\gamma}\mathbb{N}^+\left[\sum_{j=0}^{\infty} C_j(v_{xx}v_x) - \frac{1}{8}\sum_{j=0}^{\infty} v_j(x,t) - \sum_{j=0}^{\infty} v_j(x,t)\right]\right], \qquad j = 0,1,2,\ldots$$

$$\begin{aligned} C_0(v_{xx}v_x) &= \frac{\partial^2 v_0}{\partial x^2}\frac{\partial v_0}{\partial x}, \\ C_1(v_{xx}v_x) &= \frac{\partial^2 v_0}{\partial x^2}\frac{\partial v_1}{\partial x} + \frac{\partial^2 v_1}{\partial x^2}\frac{\partial v_0}{\partial x}, \\ C_2(v_{xx}v_x) &= \frac{\partial^2 v_0}{\partial x^2}\frac{\partial v_2}{\partial x} + \frac{\partial^2 v_1}{\partial x^2}\frac{\partial v_1}{\partial x} + \frac{\partial^2 v_2}{\partial x^2}\frac{\partial v_0}{\partial x}. \end{aligned}$$

for $j = 0$

$$\begin{aligned} v_1(x,t) &= \mathbb{N}^-\left[\frac{u^\gamma}{s^\gamma}\mathbb{N}^+\left[\frac{\partial^2 v_0(\frac{x}{2},\frac{t}{2})}{\partial x^2}\frac{\partial v_0(\frac{x}{2},\frac{t}{2})}{\partial x} - \frac{1}{8}v_0(x,t) - v_0(x,t)\right]\right], \\ v_1(x,t) &= \mathbb{N}^-\left[\frac{-x^2 u^\gamma}{s^{\gamma+1}}\right] = -x^2\frac{t^\gamma}{\Gamma(\gamma+1)}. \end{aligned} \tag{26}$$

The subsequent terms are

$$\begin{aligned} v_2(x,t) &= \mathbb{N}^-\left[\frac{u^\gamma}{s^\gamma}\mathbb{N}^+\left[\frac{\partial^2 v_0(\frac{x}{2},\frac{t}{2})}{\partial x^2}\frac{\partial v_1(\frac{x}{2},\frac{t}{2})}{\partial x} + \frac{\partial^2 v_1(\frac{x}{2},\frac{t}{2})}{\partial x^2}\frac{\partial v_0(\frac{x}{2},\frac{t}{2})}{\partial x} - \frac{1}{8}v_1(x,t) - v_1(x,t)\right]\right], \\ v_2(x,t) &= \frac{x(2^{1-\gamma} + 2^2 x + 1)t^{2\gamma}}{2\Gamma(2\gamma+1)} \\ v_3(x,t) &= \mathbb{N}^-\left[\frac{u^\gamma}{s^\gamma}\mathbb{N}^+\left[\frac{\partial^2 v_0(\frac{x}{2},\frac{t}{2})}{\partial x^2}\frac{\partial v_1(\frac{x}{2},\frac{t}{2})}{\partial x} + \frac{\partial^2 v_1(\frac{x}{2},\frac{t}{2})}{\partial x^2}\frac{\partial v_0(\frac{x}{2},\frac{t}{2})}{\partial x} - \frac{1}{8}v_1(x,t) - v_1(x,t)\right]\right], \\ v_3(x,t) &= \frac{t^{3\gamma}}{2\Gamma(3\gamma+1)}\left(-1 - 2x^2 - 2^4 + \frac{1}{2^\gamma} + \frac{1}{2^{2\gamma}} + \frac{2^{-3}}{2^\gamma} + \frac{2^{-2}}{2^{3\gamma}} + x\frac{2^{-1}}{2^{2\gamma}}\frac{2\Gamma(1+2\gamma)}{2^\gamma\Gamma(1+\gamma)^2}\right) \end{aligned} \tag{27}$$

The NTDM solution for example 3 is

$$v(x,t) = v_0(x,t) + v_1(x,t) + v_2(x,t) + v_3(x,t) + v_4(x,t)...$$

$$v(x,t) = x^2 - x^2\frac{t^\gamma}{\Gamma(\gamma+1)} + \frac{x(2^{1-\gamma}+2^2x+1)t^{2\gamma}}{2\Gamma(2\gamma+1)}$$
$$+ \frac{t^{3\gamma}}{2\Gamma(3\gamma+1)}\left(-1-2x^2-2^4+\frac{1}{2^\gamma}+\frac{1}{2^{2\gamma}}+\frac{2^{-3}}{2^\gamma}+\frac{2^{-2}}{2^{3\gamma}}+x\frac{2^{-1}}{2^{2\gamma}}\frac{2\Gamma(1+2\gamma)}{2^\gamma\Gamma(1+\gamma)^2}\right)+...$$

when $\gamma = 1$, then NTDM solution is

$$v(x,t) = x^2\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+...\right). \tag{28}$$

This result is calculated to the exact solution of Equation (23);

$$v(x,t) = x^2e^{-t}$$

The NTDM solution for $\gamma = 0.70$ and 1 are reported in Table 3. The surface solution behavior of $v(x,t)$ for different values of $\gamma = 1$, 0.80, 0.70 and 0.50 are depicted in Figure 5, and the plot of the solution for at different time intervals $t \leq 1$ is depicted in Figure 5. The error plot of Example 3 show in Figure 6.

Table 3. Numerical results of Example 3 at $\gamma = 0.70$ and $\gamma = 1$.

x	t	$\gamma = 0.70$	$\gamma = 1$	Exact	Absolute Error
0.25	0.25	0.0976593036	0.04867504891	0.04867504894	3.00000×10^{-11}
	0.50	0.1330574978	0.03790816051	0.03790816623	5.72000×10^{-9}
	0.75	0.1765973237	0.02952276638	0.02952290954	1.43160×10^{-7}
	1.00	0.2314929791	0.02299107144	0.02299246508	1.39364×10^{-6}
0.50	0.25	0.3906372147	0.19470019570	0.19470019580	1.00000×10^{-10}
	0.50	0.5322299913	0.15163264200	0.15163266490	2.29000×10^{-8}
	0.75	0.7063892947	0.11809106550	0.11809163820	5.72700×10^{-7}
	1.00	0.9259719163	0.09196428572	0.09196986030	5.57458×10^{-6}
1	0.25	1.5625488590	0.77880078270	0.77880078310	4.00000×10^{-10}
	0.50	2.1289199640	0.60653056800	0.60653065970	9.17000×10^{-8}
	0.75	2.8255571770	0.47236426220	0.47236655270	2.29050×10^{-6}
	1.00	3.7038876660	0.36785714290	0.36787944120	2.22983×10^{-5}

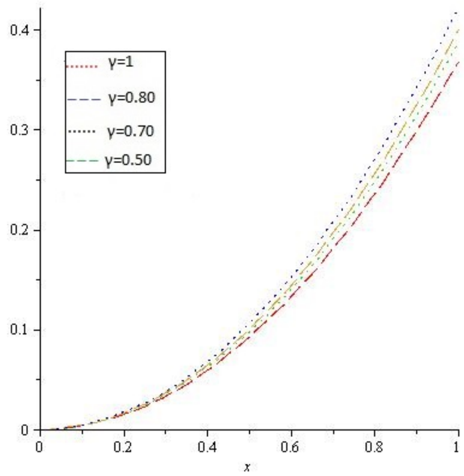

Figure 5. Numerical results of $v(x,t)$ for Example 3, for different value of γ.

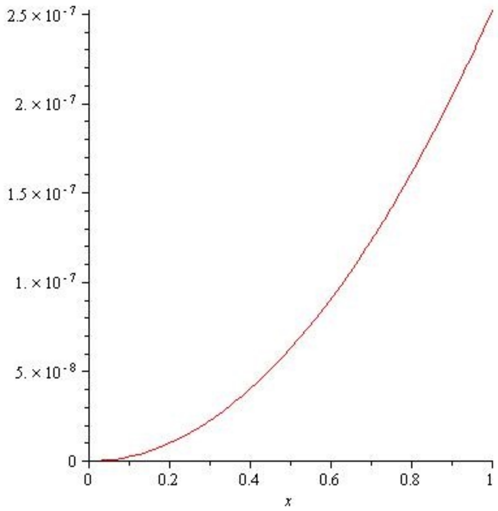

Figure 6. The error plot of Example 3, at $\gamma = 1$.

5. Conclusions

In this paper, the analytical solutions of fractional-order partial differential equations with Proportional delays are determined, using NTDM. The NDTM solutions are determined for both integer and fractional order problems. The proposed solutions agreed excellently with homotopy perturbation method (HPM) [8], homotopy perturbation transform method (HPTM) [30] and reduced differential transform method (RDTM) [7]. The NTDM solutions have shown the highest concurrence with the exact solutions of the problems. Moreover, the validity and applicability of the proposed method is confirmed with the help of some numerical examples. On the basis of NTDM solutions, it is observed that the solutions at different fractional-order, is approaches to integer-order solution of the problems. On the basis of the above narration, NTDM can be used to find the analytical solution of other non-linear FPDEs, which are frequently used in science and engineering. NTDM, solutions for fractional-order problems, will prove the better understanding of the real world problems represented by FPDEs.

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