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Family of odd point non-stationary subdivision schemes and their applications

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Abstract

The $(2s - 1)$-point non-stationary binary subdivision schemes (SSs) for curve design are introduced for any integer $s \geq 2$. The Lagrange polynomials are used to construct a new family of schemes that can reproduce polynomials of degree $(2s - 2)$. The usefulness of the schemes is illustrated in the examples. Moreover, the new schemes are the non-stationary counterparts of the stationary schemes of (Daniel and Shunmugaraj in 3rd International Conference on Geometric Modeling and Imaging, pp. 3–8, 2008; Hassan and Dodgson in Curve and Surface Fitting: Sant-Malo 2002, pp. 199–208, 2003; Hormann and Sabin in Comput. Aided Geom. Des. 25:41–52, 2008; Mustafa et al. in Lobachevskii J. Math. 30(2):138–145, 2009; Siddiqi and Ahmad in Appl. Math. Lett. 20:707–711, 2007; Siddiqi and Rehan in Appl. Math. Comput. 216:970–982, 2010; Siddiqi and Rehan in Eur. J. Sci. Res. 32(4):553–561, 2009). Furthermore, it is concluded that the basic shapes in terms of limiting curves produced by the proposed schemes with fewer initial control points have less tendency to depart from their tangent as well as their osculating plane compared to the limiting curves produced by existing non-stationary subdivision schemes.

MSC: Primary 41A05; 41A25; 41A30; secondary 65D17; 65D10; 40A05; 52B55

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1 Introduction

The importance of SSs cannot be denied because they play a vital role in computer-aided geometric designing (CAGD), geometric modeling, computer graphics, medical image processing, scientific visualization, reverse engineering, robotics, etc. Nowadays, SSs can be distinguished in various types: they can range from uniform to non-uniform; from binary to an arbitrary arity; from interpolatory to approximating; from stationary to non-stationary. It seems that stationary SSs have interesting features, but reconstruction of special types of limit curves of various shapes, including polynomial functions, conic sections such as circles, ellipses, and spiral curves, could not be accomplished without the non-stationary SSs.

In literature, several articles have been published during the last couple of decades. In 2003, Jena et al. [13] constructed a 4-point interpolating non-stationary SS generating limit curves of C^1-continuity. In 2007, Beccari et al. [2] derived a non-stationary binary 4-point uniform tension controlled interpolating SS reproducing conics, and they also

proposed another ternary 4-point C^2-continuous interpolating non-stationary SS with tension control [3] in the same year. In 2009, Daniel and Shunmugaraj [6] introduced a 6-point binary interpolating non-stationary SS that is C^2 limit curve. Conti and Romani [4] presented and investigated a 6-point interpolatory non-stationary subdivision SS capable of reproducing important curves in 2010. In 2007, Daniel and Shunmugaraj [7] introduced some 4-point ternary interpolating non-stationary schemes. In 2013, Li et al. [16] developed a new technique to establish a non-stationary SS that can generate functions in a finite-dimensional subspace of exponential polynomials. Mustafa et al. [19] introduced a subdivision-regularization framework for preventing over-fitting of data by a model in 2013. In 2016, Salam et al. [23] presented two non-stationary forms of Chaikin perturbation SS, and Tan et al. [27] derived a 3-point approximating non-stationary SS. For more recent work on SSs, one may be referred to [1, 15, 17, 18, 21].

Another aim of this research is to discuss and compare the limit curves of our proposed SSs. Also, we have measured curvature and torsion to find the rate at which limiting curves approach to deviate from their tangents. It is observed that the limit curves of our approximating schemes are near to the initial control polygons and for a certain range of parameter limit curves pass through the initial polygons. Moreover, the proposed SSs are non-stationary counterparts of the stationary SSs of [5, 11, 12, 22, 24–26].

The plan of this article is as follows: Sect. 2 is devoted to the introduction of some basic identities and definitions. We also derived some lemmas that are applied for construction of $(2s-1)$-point non-stationary. Construction and smoothness of the proposed SSs are discussed in Sect. 3. Section 4 deals with various properties such as the shape of the limit curve, curvature, and torsion, while Sect. 5 is concerned with the conclusion.

2 Preliminaries and definition

A general form of univariate binary subdivision scheme S which maps a polygon $\alpha^j = \{\alpha_i^j\}_{i\in\mathbb{Z}}$ to a refined polygon $\alpha^{j+1} = \{\alpha_i^{j+1}\}_{i\in\mathbb{Z}}$ is defined by

$$q_{2i+\gamma}^{j+1} = \sum_{k=0}^{s} \alpha_{i+\gamma}^j q_{i+k}^j, \quad \gamma = 0,1, \tag{1}$$

where $s > 0$, $\mathbb{Z}$ is the set of integers. The set of coefficients $\{\alpha_{i,\gamma}^j, \gamma = 0,1\}_{k=0}^s$ is called subdivision mask. This scheme is formally denoted by $q^{j+1} = Sq^j$.

A necessary condition for uniform convergence of the subdivision scheme (1) is

$$\sum_{k=0}^{s} \alpha_{i,\gamma}^j \cong 1, \quad \gamma = 0,1. \tag{2}$$

For the given s, we define Lagrange fundamental polynomials of degree $2s-2$ and $2s-3$ corresponding to nodes $\{t\}_{-(s-1)}^{s-1}$ and $\{t\}_{-(s-1)}^{s-2}$, respectively, and they are

$$L_t^{2s-2}(x) = \prod_{j=-(s-1), t\neq j}^{s-1} \frac{(x-j)}{(t-j)}, \tag{3}$$

where $t = -(s-1), -(s-2), \ldots, (s-1)$, and

$$L_t^{2s-3}(x) = \prod_{j=-(s-2), t \neq j}^{s-1} \frac{(x-j)}{(t-j)}, \tag{4}$$

where $t = -(s-2), -(s-3), \ldots, (s-1)$.

Lemma 1 ([20]) *If $t = -(s-1), \ldots, (s-1)$, then the following implication holds*:

$$\prod_{j=-(s-1), j \neq t}^{s-1} (t-j) = (-1)^{s-t-1}(s+t-1)!(s-t-1)!. \tag{5}$$

Lemma 2 ([20]) *If $t = -(s-2), \ldots, (s-1)$, then following implication holds*:

$$\prod_{j=-(s-2), j \neq t}^{s-1} (t-j) = (-1)^{s-t-1}(s+t-2)!(s-t-1)!. \tag{6}$$

Lemma 3 ([20]) *If $L_t^{2s-2}(x)$ is a Lagrange fundamental polynomial of degree $(2s-2)$ corresponding to nodes $\{t\}_{-(s-1)}^{s-1}$ defined by* (3), *then*

$$L_t^{2s-2}\left(\frac{1}{4}\right) = \frac{(-1)^t(4s-3)!}{2^{6s-6}(2s-2)!(1-4t)(s-t-1)!(s+t-1)!}, \tag{7}$$

where $t = -(s-1), \ldots, (s-1)$.

Lemma 4 ([20]) *If $L_t^{2s-3}(x)$ is a Lagrange fundamental polynomial of degree $(2s-3)$ defined by* (4) *corresponding to the nodes $\{t\}_{-(s-2)}^{s-1}$, then we get*

$$L_t^{2s-3}\left(\frac{1}{4}\right) = \frac{(-1)^t(4s-5)(4s-7)!}{2^{6s-10}(1-4t)(2s-4)!(s+t-2)!(s-t-1)!}, \tag{8}$$

where $t = -(s-2), \ldots, (s-1)$.

Lemma 5 *If $t = -(s-2), \ldots, (s-1)$, then the following implication holds*:

$$\begin{aligned}
\beta_t^k &= \frac{\sin\left(\frac{\theta}{2^{k+1}}\left(\prod_{j=-(s-2), j \neq t}^{s-1}\left(\frac{1}{4}-j\right)\right)\right)}{\sin\left(\frac{\theta}{2^{k+1}}\left(\prod_{j=-(s-2), j \neq t}^{s-1}(t-j)\right)\right)} \\
&= \frac{\sin\left(\frac{\theta}{2^{k+1}}\left(\frac{(-1)^{s-1}(4s-5)(4s-7)!}{2^{6s-10}(2s-4)!(1-4t)}\right)\right)}{\sin\left(\frac{\theta}{2^{k+1}}(-1)^{s-t-1}((s-t-1)!(s+t-2)!)\right)}.
\end{aligned} \tag{9}$$

Proof Since

$$\prod_{j=-(s-2)}^{s-1}\left(\frac{1}{4}-j\right) = \left(\frac{1}{4}\right)^{2s-2}\Big\{(4s-7)(4s-11)\cdots(5)(1)(-3)\cdots(-4s+13)$$
$$\times (-4s+9)(-4s+5)\Big\},$$

then

$$\prod_{j=-(s-2),j\neq t}^{s-1}\left(\frac{1}{4}-j\right) = \frac{1}{4^{2s-3}(1-4t)}\prod_{t=-s+2}^{s-1}(1-4t).$$

This implies

$$\prod_{j=-(s-2),j\neq t}^{s-1}\left(\frac{1}{4}-j\right) = \frac{(-1)^{s-1}}{4^{2s-3}(1-4t)}\bigg\{(4s-7)$$

$$\times \frac{(4s-8)}{(4s-8)}\frac{(4s-9)}{(4s-9)}\frac{(4s-10)}{(4s-10)}(4n-11)$$

$$\times \frac{(4s-12)}{(4s-12)}\cdots\left(\frac{8}{8}\right)\left(\frac{7}{7}\right)\left(\frac{6}{6}\right)(5)\left(\frac{4}{4}\right)\left(\frac{3}{3}\right)$$

$$\times \left(\frac{2}{2}\right)(1)\left(\frac{2}{2}\right)(3)\left(\frac{4}{4}\right)\left(\frac{5}{5}\right)\left(\frac{6}{6}\right)(7)\cdots(4s-13)$$

$$\times \frac{(4s-12)}{(4s-12)}\frac{(4s-11)}{(4s-11)}\frac{(4s-10)}{(4s-10)}(4s-9)$$

$$\times \frac{(4s-8)}{(4s-8)}\frac{(4s-7)}{(4s-7)}\frac{(4s-6)}{(4s-6)}(4s-5)\frac{(4s-4)}{(4s-4)}\bigg\}.$$

This leads to

$$\prod_{j=-(s-2),j\neq t}^{s-1}\left(\frac{1}{4}-j\right) = \frac{(-1)^{s-1}(4s-5)(4s-7)!}{2^{6s-10}(1-4t)(2s-4)!}. \tag{10}$$

Using (6) and (10), we get (9). This completes the proof.   □

Theorem 6 *If $L_t^{2s-2}(x)$ and $L_t^{2s-3}(x)$ are Lagrange fundamental polynomials of degree $(2s-2)$ and $(2s-3)$ corresponding to the nodes $\{t\}_{-(s-1)}^{s-1}$ and $\{t\}_{-(s-2)}^{s-1}$, respectively, then the following implication holds*:

$$\lambda_t^k = \frac{\sin(\frac{\theta}{2^{k+1}}(L_t^{2s-2}(\frac{1}{4})-L_t^{2s-3}(\frac{1}{4})))}{\sin(\frac{\theta}{2^{k+1}}(L_{-s+1}^{2s-2}(\frac{1}{4})))}$$

$$= \frac{\sin(\frac{\theta}{2^{k+1}}(\frac{(-1)^t(4s-4)!}{2^{6s-6}(2s-2)!(s-t-1)!(s+t-1)!}))}{\sin(\frac{\theta}{2^{k+1}}(\frac{(-1)^{-s+1}(4s-3)!}{2^{6s-6}(4s-3)(2s-2)!(2s-2)!}))}, \tag{11}$$

where $t = -(s-2),\ldots,(s-1)$.

Proof Since

$$L_t^{2s-2}\left(\frac{1}{4}\right)-L_t^{2s-3}\left(\frac{1}{4}\right) = \frac{(-1)^t(4s-4)!}{2^{6s-6}(2s-2)!(s-t-1)!(s+n-1)!} \tag{12}$$

and substituting $t = -(s-1)$ in (7), we have

$$L_{-(s-1)}^{2s-2}\left(\frac{1}{4}\right) = \frac{(-1)^{-s+1}(4s-3)!}{2^{6s-6}(4s-3)(2s-2)!(2s-2)!}. \tag{13}$$

Using (12) and (13), we get (11), which completes the proof.   □

3 The $(2s-1)$-point non-stationary SS

In this section, we present general explicit formulae to construct the mask of a $(2s-1)$-point non-stationary binary approximating subdivision scheme.

Now, for $s \geq 2$, the mask of $(2s-1)$-point binary approximating schemes SS can be generated by

$$
\begin{cases}
p_{2i}^{k+1} = \sum_{t=-(s-1)}^{s-1} \eta_{-t}^k p_{i+t}^k, \\
p_{2i+1}^{k+1} = \sum_{t=-(s-1)}^{s-1} \eta_t^k p_{i+t}^k,
\end{cases}
\tag{14}
$$

can be generated by

$$
\begin{cases}
\eta_{-s+1}^k = \dfrac{\sin(\frac{\theta\mu}{2^{k+1}})}{\sin(\frac{\theta}{2^{k+1}})}, \quad \mu < 1, \\
\eta_t^k = \beta_t^k + \lambda_t^k \mu, \quad t = -s+2,\ldots,s-1,
\end{cases}
\tag{15}
$$

where

$$
\beta_t^k = \frac{\sin(\frac{\theta}{2^{k+1}} \frac{(-1)^{s-1}(4s-5)(4s-7)!}{2^{6s-10}(1-4t)(2s-4)!})}{\sin(\frac{\theta}{2^{k+1}}(-1)^{s-t-1}(s-t-1)!(s+t-2)!)},
$$

$$
\lambda_m^k = \frac{\sin(\frac{\theta}{2^{k+1}} \frac{(-1)^t(4s-4)!}{2^{6s-6}(2s-2)!(s-t-1)!(s+t-1)!})}{\sin(\frac{\theta}{2^{k+1}} \frac{(-1)^{-s+1}(4s-4)!}{2^{6s-4}((2s-2)!)^2})},
$$

and μ and θ are free parameters.

Here we see that some of the schemes are special cases of the scheme proposed above. Some of new non-stationary schemes are also given below.

- Substituting $s = 2$ in (14) and (15), we get a new 3-point symmetric binary approximating SS with free parameters μ and θ:

$$
\begin{aligned}
p_{2i}^{k+1} &= \eta_1^k p_{i-1}^k + \eta_0^k p_i^k + \eta_{-1}^k p_{i+1}^k, \\
p_{2i+1}^{k+1} &= \eta_{-1}^k p_{i-1}^k + \eta_0^k p_i^k + \eta_1^k p_{i+1}^k,
\end{aligned}
\tag{16}
$$

where

$$
\eta_{-1}^k = \frac{\sin(\frac{\theta\mu}{2^{k+1}})}{\sin(\frac{\theta}{2^{k+1}})},
$$

$$
\eta_0^k = \frac{\sin(\frac{1}{4}\frac{3\theta}{2^{k+1}})}{\sin(\frac{\theta}{2^{k+1}})} - \frac{\sin(\frac{1}{16}\frac{3\theta}{2^{k+1}})}{\sin(\frac{1}{32}\frac{3\theta}{2^{k+1}})}\mu,
$$

$$
\eta_1^k = \frac{\sin(\frac{1}{4}\frac{\theta}{2^{k+1}})}{\sin(\frac{\theta}{2^{k+1}})} + \mu.
$$

- Substituting $s = 3$ in (14) and (15), we get a new 5-point symmetric binary approximating SS with free parameters μ and θ:

$$
\begin{aligned}
p_{2i}^{k+1} &= \eta_2^k p_{i-2}^k + \eta_1^k p_{i-1}^k + \eta_0^k p_i^k + \eta_{-1}^k p_{i+1}^k + \eta_{-2}^k p_{i+2}^k, \\
p_{2i+1}^{k+1} &= \eta_{-2}^k p_{i-2}^k + \eta_{-1}^k p_{i-1}^k + \eta_0^k p_i^k + \eta_1^k p_{i+1}^k + \eta_2^k p_{i+2}^k,
\end{aligned}
\tag{17}
$$

where

$$\eta_{-2}^k = \frac{\sin(\frac{\theta\mu}{2^{k+1}})}{\sin(\frac{\theta}{2^{k+1}})},$$

$$\eta_{-1}^k = -\frac{\sin(\frac{1}{64}\frac{21\theta}{2^{k+1}})}{\sin(\frac{6\theta}{2^{k+1}})} - \frac{\sin(\frac{1}{512}\frac{35\theta}{2^{k+1}})}{\sin(\frac{1}{2048}\frac{35\theta}{2^{k+1}})}\mu,$$

$$\eta_0^k = \frac{\sin(\frac{1}{64}\frac{105\theta}{2^{k+1}})}{\sin(\frac{2\theta}{2^{k+1}})} + \frac{\sin(\frac{1}{1024}\frac{105\theta}{2^{k+1}})}{\sin(\frac{1}{2048}\frac{35\theta}{2^{k+1}})}\mu,$$

$$\eta_1^k = \frac{\sin(\frac{1}{64}\frac{35\theta}{2^{k+1}})}{\sin(\frac{2\theta}{2^{k+1}})} - \frac{\sin(\frac{1}{512}\frac{35\theta}{2^{k+1}})}{\sin(\frac{1}{2048}\frac{35\theta}{2^{k+1}})}\mu,$$

$$\eta_2^k = -\frac{\sin(\frac{1}{64}\frac{15\theta}{2^{k+1}})}{\sin(\frac{6\theta}{2^{k+1}})} + \mu.$$

3.1 Convergence and smoothness of the proposed SS

In this section, we use asymptotic equivalence method to find the smoothness of the normalized SSs (16) and (17).

Definition 1 Two binary SSs $\{S_{\alpha_j}\}$ and $\{S_{\beta_j}\}$ are asymptotically equivalent if

$$\sum_{j=1}^{\infty} \|S_{\alpha_j} - S_{\beta_j}\| < \infty,$$

where $\|S_{\alpha_j}\|_{\infty} = \max\{\sum_{i\in\mathbb{Z}} |\alpha_{2i}^{(j)}|, |\alpha_{2i+1}^{(j)}|\}$.

Theorem 7 ([9]) *Consider* $\{S_{\alpha_j}\}$ *is a non-stationary SS and* $\{S_{\beta_j}\}$ *is a stationary SS. Let* $\{S_{\alpha_j}\}$ *and* $\{S_{\beta_j}\}$ *be the two asymptotically equivalent SSs having finite masks of the same support. If* $\{S_{\beta_j}\}$ *is* C^m *and* $\sum_{j=0}^{\infty} 2^{mj} \|S_{\alpha_j} - S_{\beta_j}\| < \infty$, *then the non-stationary SS* $\{S_{a_j}\}$ *is* C^m.

Some estimates of stencils η_i^j, $i = 0, 1$, and λ_i^j, $i = -1, 0, 1, 2$, are required to find smoothness of the proposed schemes which are given in the following lemmas.

Lemma 8 *For some* $k \geq 0$ *and* $0 < \theta < \frac{\pi}{2}$,

(a) $\quad \mu \leq \eta_{-1}^k \leq \dfrac{\mu}{\cos(\frac{\theta}{2^{k+1}})}$;

(b) $\quad \left(\dfrac{3}{4} - 2\mu\right) \leq \eta_0^k \leq \left(\dfrac{3}{4\cos(\frac{\theta}{2^{k+1}})} - 2\cos\left(\dfrac{1}{32}\dfrac{3\theta}{2^{k+1}}\right)\mu\right)$;

(c) $\quad \left(\dfrac{1}{4} + \mu\right) \leq \eta_1^k \leq \left(\dfrac{1}{4\cos(\frac{\theta}{2^{k+1}})} + \mu\right)$.

Proof We give the proof of (a). Note that

$$\eta_{-1}^k = \frac{\sin(\frac{\theta\mu}{2^{k+1}})}{\sin(\frac{\theta}{2^{k+1}})} \geq \frac{\frac{\theta\mu}{2^{k+1}}}{\frac{\theta}{2^{k+1}}} = \mu$$

and

$$\eta_{-1}^{k} \leq \frac{\sin(\frac{\theta\mu}{2^{k+1}})}{\sin(\frac{\theta}{2^{k+1}})} \leq \frac{\frac{\theta\mu}{2^{k+1}}}{\frac{\theta}{2^{k+1}}\cos(\frac{\theta}{2^{k+1}})} \leq \frac{\mu}{\cos(\frac{\theta}{2^{k+1}})}.$$

This completes the proof of (a) the proofs of (b) and (c) are obtained in a similar way. $\square$

By Lemma 8, we get the following lemma.

Lemma 9

(a) $\left|\eta_{-1}^{k} - \mu\right| \leq C_0 \dfrac{1}{2^{3k}}$,

(b) $\left|\eta_{0}^{k} - \left(\dfrac{3}{4} - 2\mu\right)\right| \leq C_1 \dfrac{1}{2^{3k}}$,

(c) $\left|\eta_{1}^{k} - \left(\dfrac{1}{4} + \mu\right)\right| \leq C_2 \dfrac{1}{2^{3k}}$,

where C_0, C_1, and C_2 are some constants independent of k.

Proof We give the proof of (a). By (a) of Lemma 8, we have

$$\begin{aligned}
\left|\eta_{-1}^{k} - \mu\right| &\leq \mu\left(\frac{1 - \cos(\frac{\theta}{2^{k+1}})}{\cos(\frac{\theta}{2^{k+1}})}\right) \\
&\leq \frac{2\mu}{\cos(\frac{\theta}{2^{k+1}})}\sin^2\left(\frac{\theta}{2^{k+2}}\right) \\
&\leq \frac{2\mu\theta^2}{16\cos(\theta)}\frac{1}{2^{3k}} \leq C_0\frac{1}{2^{3k}}.
\end{aligned}$$

This proves (a). The proofs of (b) and (c) are similar. $\square$

Remark 1 The proposed 3-point SS (16) is a non-stationary counterpart of the following stationary SS:

$$\begin{cases} p_{2i}^{k+1} = (\frac{1}{4} + \mu)p_{i-1}^{k} + (\frac{3}{4} - 2\mu)p_i^{k} + \mu p_{i+1}^{k}, \\ p_{2i+1}^{k+1} = \mu p_{i-1}^{k} + (\frac{3}{4} - 2\mu)p_i^{k} + (\frac{1}{4} + \mu)p_{i+1}^{k}, \end{cases} \tag{18}$$

as the stencils of the normalized SS (16) converge to the stencils of (18): $\eta_{-1}^{k} \to \mu$, $\eta_0^{k} \to (\frac{3}{4} - 2\mu)$, and $\eta_1^{k} \to (\frac{1}{4} + \mu)$ as $k \to \infty$. The proofs of these convergences follow from Lemma 9.

Remark 2 Similarly, for $n = 2$ and $\mu = -3\omega$, $\mu = \frac{1}{16}$, $\mu = \frac{-3}{32}$, $\mu = \frac{1}{24} + \frac{1}{4}w$, $\mu = \frac{1}{32}$, and $\mu = \frac{-3}{32} + w$, in (14) and (15), we get non-stationary counterparts of the stationary 3-point schemes of [5, 11, 12, 22, 24, 26] respectively.

Lemma 10 *The Laurent polynomial $\alpha(z)$ of scheme* (18) *can be written as*

$$\alpha(z) = \left\{ \mu + \left(\frac{1}{4} + \mu\right)z + \left(\frac{3}{4} - 2\mu\right)z^2 + \left(\frac{3}{4} - 2\mu\right)z^3 \right.$$
$$\left. + \left(\frac{1}{4} + \mu\right)z^4 + \mu z^5 \right\},$$

and a subdivision scheme S_α corresponding to the symbol $\alpha(z)$ is C^2 for $\mu \in (0, \frac{1}{8})$ and C^3 for $\mu = \frac{1}{16}$.

Proof Consider

$$c(z) = \frac{8\alpha(z)}{(1+z)^3} = \left(8\mu + (2 - 16\mu)z + 8\mu z^2\right).$$

Note that, for $\mu \in (0, \frac{1}{8})$,

$$\left\| \frac{1}{2} S_c \right\| = \frac{1}{2} \max\left\{ \sum_{j \in \mathbb{Z}} |c_{2j}|, \sum_{j \in \mathbb{Z}} |c_{2j+1}| \right\}$$
$$= \frac{1}{2} \max\left\{ 2|8\mu|, |2 - 16\mu| \right\} < 1.$$

So the scheme is C^2-continuous for $\mu \in (0, \frac{1}{8})$. In order to prove C^3 smoothness, we put $\mu = \frac{1}{16}$ in $c(z)$, that is,

$$c(z) = \frac{1}{2}\left(1 + 2z + z^2\right).$$

If

$$d(z) = \frac{2c(z)}{(1+z)} = (1 + z),$$

then $\left\| \frac{1}{2} S_d \right\| = \frac{1}{2} \max\{\sum_{j \in \mathbb{Z}} |d_{2j}|, \sum_{j \in \mathbb{Z}} |d_{2j+1}|\} = \max\{\frac{1}{2}, \frac{1}{2}\} < 1$. Hence by ([10], Corollary 4.11) the stationary SS S_α is C^3. □

Theorem 11 *The stationary SSs* (16) *and* (18) *are asymptotically equivalent, that is,*

$$\sum_{k=0}^{\infty} \|S_{\alpha^k} - S_\alpha\|_\infty < \infty.$$

Proof From (16) and (18), we have

$$\sum_{k=0}^{\infty} \|S_{\alpha^k} - S_\alpha\|_\infty = \sum_{k=0}^{\infty} \left\{ \left|\eta_{-1}^k - \mu\right| + \left|\eta_0^k - \left(\frac{3}{4} - 2\mu\right)\right| \right.$$
$$\left. + \left|\eta_1^k - \left(\frac{1}{4} + \mu\right)\right| \right\}.$$

From (a) of Lemma 9, it follows that

$$\sum_{k=0}^{\infty} \left| \eta_{-1}^k - \mu \right| \leq \sum_{k=0}^{\infty} C_0 \frac{1}{2^{3k}} < \infty.$$

Similarly, from (a) and (b) of Lemma 9, we have

$$\sum_{k=0}^{\infty} \left| \eta_0^k - \left(\frac{3}{4} - 2\mu \right) \right| < \infty, \qquad \sum_{k=0}^{\infty} \left| \eta_1^k - \left(\frac{1}{4} + \mu \right) \right| < \infty.$$

Hence

$$\sum_{k=0}^{\infty} \| S_{\alpha^k} - S_\alpha \|_\infty < \infty. \qquad \qquad \square$$

Theorem 12 *The non-stationary SS* (16) *is* C^2 *for* $\mu \in (0, \frac{1}{8})$ *and* C^3 *for* $\mu = \frac{1}{16}$.

Proof Since S_α is C^3 by Lemma 10 and also SSs (17) and (18) are asymptotically equivalent by Theorem 11, now by ([10], Theorem 8(a)), it is sufficient to prove that

$$\sum_{k=0}^{\infty} 2^{3k} \| S_{\alpha^k} - S_\alpha \|_\infty < \infty,$$

where

$$\| S_{\alpha^k} - S_\alpha \|_\infty = \max \left\{ \sum_{j \in \mathbb{Z}} \left| \alpha_{2j}^k - \alpha_{2j} \right|, \sum_{j \in \mathbb{Z}} \left| \alpha_{2j+1}^k - \alpha_{2j+1} \right| \right\}$$

$$= \left| \eta_{-1}^k - \mu \right| + \left| \eta_0^k - \left(\frac{3}{4} - 2\mu \right) \right| + \left| \eta_1^k - \left(\frac{1}{4} + \mu \right) \right|.$$

This implies

$$\sum_{k=0}^{\infty} 2^{3k} \| S_{\alpha^k} - S_\alpha \|_\infty = \sum_{k=0}^{\infty} 2^{3k} \left| \eta_{-1}^k - \mu \right|$$

$$+ \sum_{k=0}^{\infty} 2^{3k} \left| \eta_0^k - \left(\frac{3}{4} - 2\mu \right) \right|$$

$$+ \sum_{k=0}^{\infty} 2^{3k} \left| \eta_1^k - \left(\frac{1}{4} + \mu \right) \right|.$$

Since by (a), (b), and (c) of Lemma 9, we have

$$\sum_{k=0}^{\infty} 2^{3k} \left| \eta_{-1}^k - \mu \right| < \infty, \qquad \sum_{k=0}^{\infty} 2^{3k} \left| \eta_0^k - \left(\frac{3}{4} - 2\mu \right) \right| < \infty$$

and

$$\sum_{k=0}^{\infty} 2^{3k} \left| \eta_1^k - \left(\frac{1}{4} + \mu \right) \right| < \infty,$$

it follows that

$$\sum_{k=0}^{\infty} 2^{3k} \|S_{\alpha^k} - S_\alpha\|_\infty < \infty.$$

$\square$

Here we will discuss the convergence and smoothness of 5-point SS (17). The proofs of lemmas given below are similar to the proofs of Lemmas 8 and 9.

Lemma 13 *For some $k \geq 0$ and $0 < \theta < \frac{\pi}{2}$,*

(a) $\mu \leq \eta_{-2}^k \leq \dfrac{\mu}{\cos(\frac{\theta}{2^{k+1}})}$;

(b) $-\dfrac{7}{128 \cos(\frac{6\theta}{2^{k+1}})} - \dfrac{4\mu}{\cos(\frac{1}{2048} \frac{35\theta}{2^{k+1}})} \leq \eta_{-1}^k \leq -\dfrac{7}{128} - 4\mu$;

(c) $\dfrac{105}{128} + 6\mu \leq \eta_0^k \leq \dfrac{105}{128 \cos(\frac{2\theta}{2^{k+1}})} + \dfrac{6\mu}{\cos(\frac{1}{2048} \frac{35\theta}{2^{k+1}})}$;

(d) $\dfrac{35}{128} - 4\mu \leq \eta_1^k \leq \dfrac{105}{128} - 4\mu \cos\left(\dfrac{1}{512} \dfrac{35\theta}{2^{k+1}}\right)$;

(e) $-\dfrac{5}{128 \cos(\frac{6\theta}{2^{k+1}})} + \mu \leq \eta_2^k \leq -\dfrac{5}{128} + \mu$.

By Lemma 13, we get the following lemma.

Lemma 14

(a) $\left|\eta_{-2}^k - \mu\right| \leq D_0 \dfrac{1}{2^{4k}}$,

(b) $\left|\eta_{-1}^k - \left(-\dfrac{7}{128} - 4\mu\right)\right| \leq D_1 \dfrac{1}{2^{4k}}$,

(c) $\left|\eta_0^k - \left(\dfrac{105}{128} + 6\mu\right)\right| \leq D_2 \dfrac{1}{2^{4k}}$,

(d) $\left|\eta_1^k - \left(\dfrac{35}{128} - 4\mu\right)\right| \leq D_3 \dfrac{1}{2^{4k}}$,

(e) $\left|\eta_2^k - \left(-\dfrac{5}{128} + \mu\right)\right| \leq D_4 \dfrac{1}{2^{4k}}$,

where D_0, D_1, D_2, D_3, and D_4 are some constants independent of k.

Remark 3 For $\mu = u$, the proposed 5-point SS (17) is a non-stationary counterpart of the following stationary SS:

$$\begin{cases} p_{2i}^{k+1} = (-\frac{5}{128} + \mu)p_{i-2}^k + (\frac{35}{128} - 4\mu)p_{i-1}^k, +(\frac{105}{128} + 6\mu)p_i^k \\ \qquad + (-\frac{7}{128} - 4\mu)p_{i+1}^k + \mu p_{i+2}^k, \\ p_{2i+1}^{k+1} = \mu p_{i-2}^k + (-\frac{7}{128} - 4\mu)p_{i-1}^k + (\frac{105}{128} + 6\mu)p_i^k \\ \qquad + (\frac{35}{128} - 4\mu)p_{i+1}^k + (-\frac{5}{128} + \mu)p_{i+2}^k, \end{cases} \qquad (19)$$

because the stencils of SS (17) converge to the stencils of SS (19): $\eta_{-2}^k \to \mu$, $\eta_{-1}^k \to (-\frac{7}{128} - 4\mu)$, $\eta_o^k \to (\frac{105}{128} + 6\mu)$, $\eta_1^k \to (\frac{35}{128} - 4\mu)$, and $\eta_2^k \to (-\frac{5}{128} + \mu)$ as $k \to \infty$. The proofs of these convergences follow from Lemma 14.

Remark 4 Similarly, for $\mu = \frac{35}{2048}$, in (17), we get a non-stationary counterpart of the stationary SS of [24].

Lemma 15 *The Laurent polynomial $\alpha(z)$ of SS (19) can be written as*

$$
\alpha(z) = \mu + \left(-\frac{5}{128} + \mu\right)z + \left(-\frac{7}{128} - 4\mu\right)z^2
$$
$$
+ \left(\frac{35}{128} - 4\mu\right)z^3 + \left(\frac{105}{128} + 6\mu\right)z^4
$$
$$
+ \left(\frac{105}{128} + 6\mu\right)z^5 + \left(\frac{35}{128} - 4\mu\right)z^6
$$
$$
+ \left(-\frac{7}{128} - 4\mu\right)z^7 + \left(-\frac{5}{128} + \mu\right)z^8 + \mu z^9,
$$

and subdivision scheme S_α corresponding to the symbol $\alpha(z)$ is C^4 for $\mu \in (-\frac{9}{512}, -\frac{5}{512})$.

Proof Consider

$$
e(z) = \frac{32\alpha(z)}{(1+z)^5}
$$
$$
= 32\left\{\mu - \left(\frac{5}{128} + 4\mu\right)z + \left(\frac{18}{128} + 6\mu\right)z^2 \right.
$$
$$
\left. - \left(\frac{5}{128} + 4\mu\right)z^3 + \mu z^4\right\}.
$$

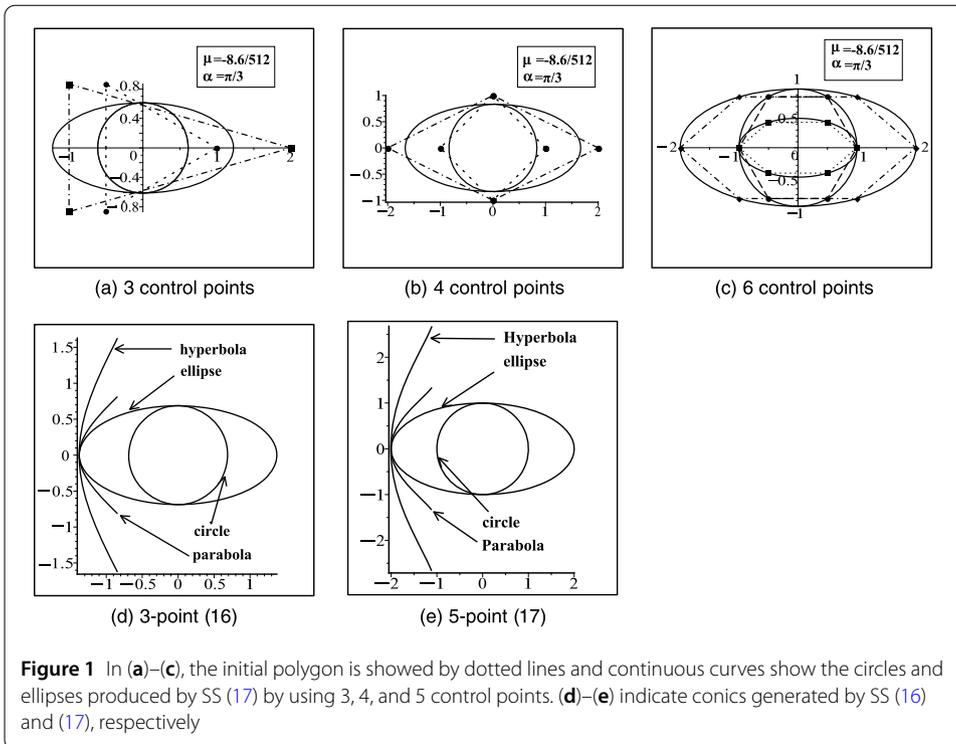Note that, for $\mu \in (-\frac{9}{512}, -\frac{5}{512})$,

$$
\left\|\frac{1}{2}S_e\right\| = \frac{1}{2}\max\left\{\sum_{j\in\mathbb{Z}}|e_{2j}|, \sum_{j\in\mathbb{Z}}|e_{2j+1}|\right\}
$$
$$
= \frac{1}{2}\max\left\{|32\mu| + 64\left|\frac{18}{128} + 6\mu\right|, \right.
$$
$$
\left. 64\left|-\frac{5}{128} - 4\mu\right|\right\} < 1.
$$

So by ([10], Corollary 4.11) the SS S_α is C^4. □

In the following result, we show that SSs (17) and (19) are asymptotically equivalent.

Theorem 16 *SSs (17) and (19) are asymptotically equivalent, that is,*

$$
\sum_{k=0}^{\infty} \|S_{\alpha^k} - S_\alpha\|_\infty < \infty.
$$

Figure 1 In (**a**)–(**c**), the initial polygon is showed by dotted lines and continuous curves show the circles and ellipses produced by SS (17) by using 3, 4, and 5 control points. (**d**)–(**e**) indicate conics generated by SS (16) and (17), respectively
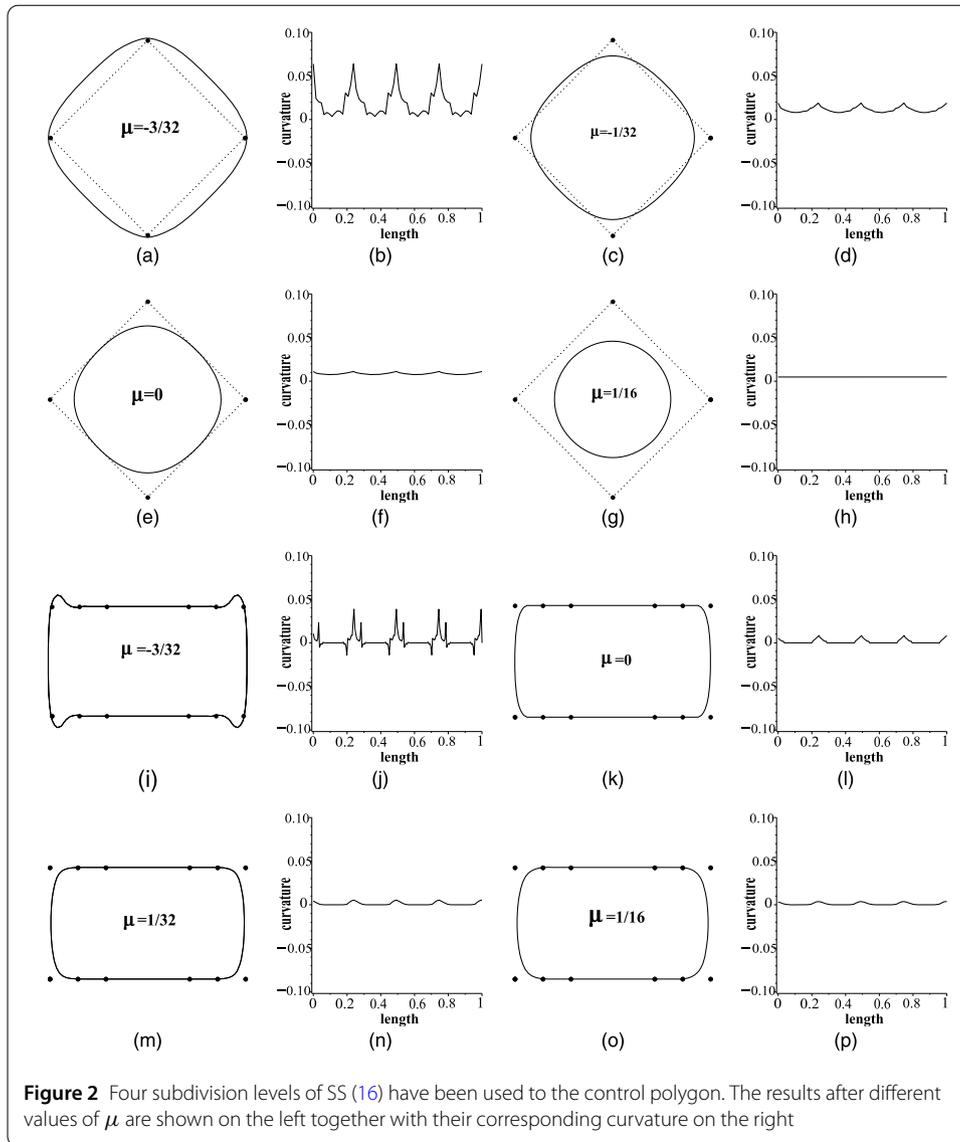
Proof From (17) and (19), we have

$$\sum_{k=0}^{\infty} \|S_{\alpha^k} - S_\alpha\|_\infty = \sum_{k=0}^{\infty} \left\{ \left| \eta_{-2}^k - \mu \right| \right.$$
$$+ \left| \eta_{-1}^k - \left(-\frac{7}{128} - 4\mu \right) \right| + \left| \eta_0^k - \left(\frac{105}{128} + 6\mu \right) \right|$$
$$\left. + \left| \eta_1^k - \left(\frac{35}{128} - 4\mu \right) \right| + \left| \eta_2^k - \left(-\frac{5}{128} + \mu \right) \right| \right\}.$$

From (a) of Lemma 14, it follows that

$$\sum_{k=0}^{\infty} \left| \eta_{-1}^k - \mu \right| \le \sum_{k=0}^{\infty} D_0 \frac{1}{2^{4k}} < \infty.$$

Similarly, from (b), (c), (d), and (e) of Lemma 14, we have

$$\sum_{k=0}^{\infty} \left| \eta_{-1}^k - \left(-\frac{7}{128} - 4\mu \right) \right| < \infty,$$

$$\sum_{k=0}^{\infty} \left| \eta_0^k - \left(\frac{105}{128} + 6\mu \right) \right| < \infty,$$

$$\sum_{k=0}^{\infty} \left| \eta_1^k - \left(\frac{35}{128} - 4\mu \right) \right| < \infty,$$

Figure 2 Four subdivision levels of SS (16) have been used to the control polygon. The results after different values of μ are shown on the left together with their corresponding curvature on the right
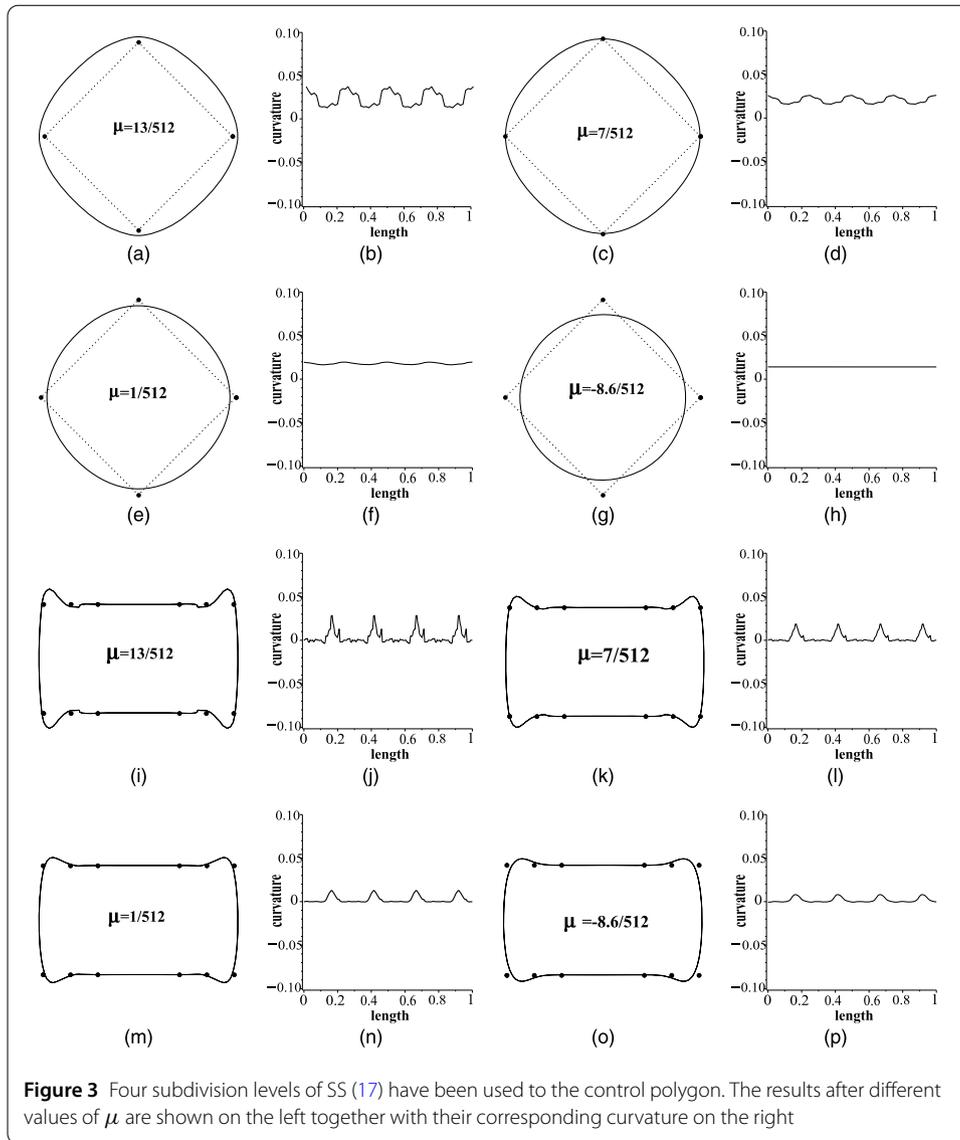
and

$$\sum_{k=0}^{\infty} \left| \eta_2^k - \left(-\frac{5}{128} + \mu \right) \right| < \infty.$$

Hence

$$\sum_{k=0}^{\infty} \| S_{\alpha^k} - S_{\alpha} \|_{\infty} < \infty. \qquad \square$$

Theorem 17 *The non-stationary SS* (17) *is C^4 for $\mu \in (-\frac{9}{512}, -\frac{5}{512})$.*

The proof of the above theorem is similar to the proof of Theorem 12.

Figure 3 Four subdivision levels of SS (17) have been used to the control polygon. The results after different values of μ are shown on the left together with their corresponding curvature on the right
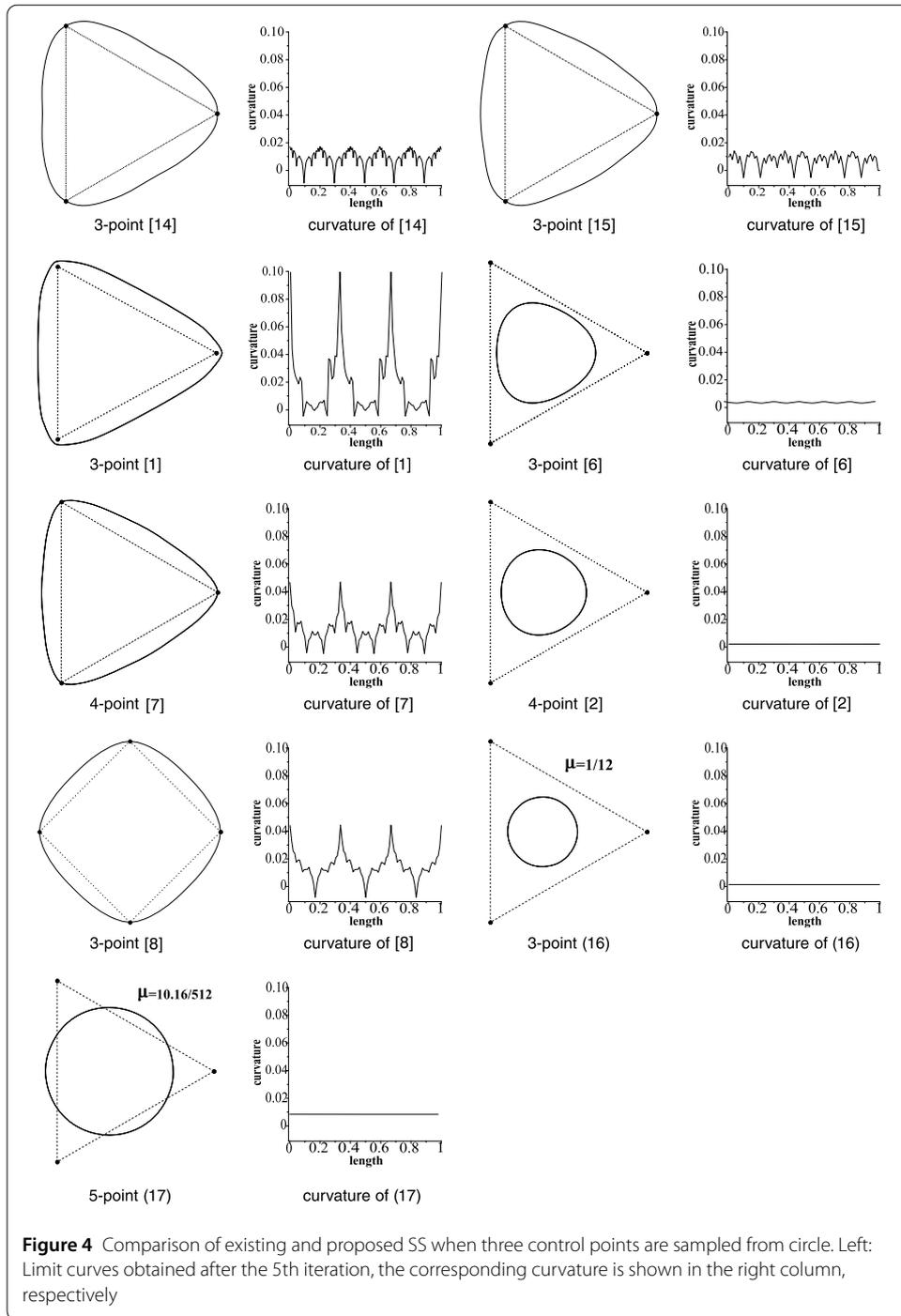
4 Results and comparisons

Here we discuss visual quality of limit curves obtained by the proposed SSs. Then we have measured the curvature and torsion which compare the accuracy of various shapes achieved by non-stationary SSs.
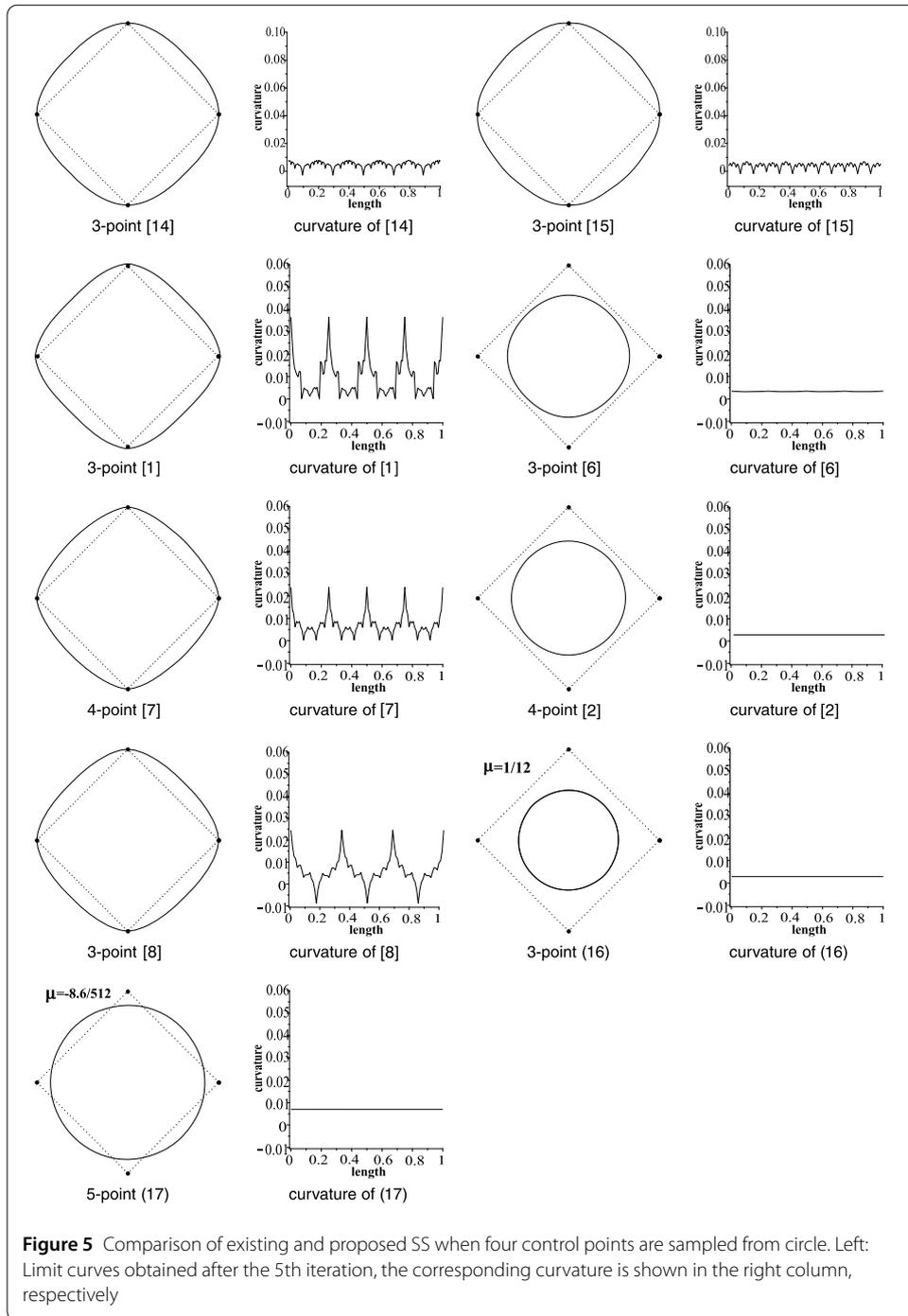
In Fig. 1, basic shapes (circle, ellipse, parabola, and hyperbola) are obtained from the proposed SSs (16) and (17) as limiting curves by taking 3, 4, and 6 equidistance control points.

As SSs (16) and (17) are parametric, therefore it is natural to see the result of values of such variables on the deviation of a limiting curve from its tangent. In Fig. 2 and Fig. 3, we show the rate at which limiting curves tend to depart from their tangents. In Figs. 2(g) and 3(o), it is noticed that the limiting curves, obtained by SSs (16) and (17) at parametric values $\mu = 1/16$ and $-8.6/512$ respectively, have very little tendency to depart from their tangents compared to the other parametric values.

Figure 4 Comparison of existing and proposed SS when three control points are sampled from circle. Left: Limit curves obtained after the 5th iteration, the corresponding curvature is shown in the right column, respectively
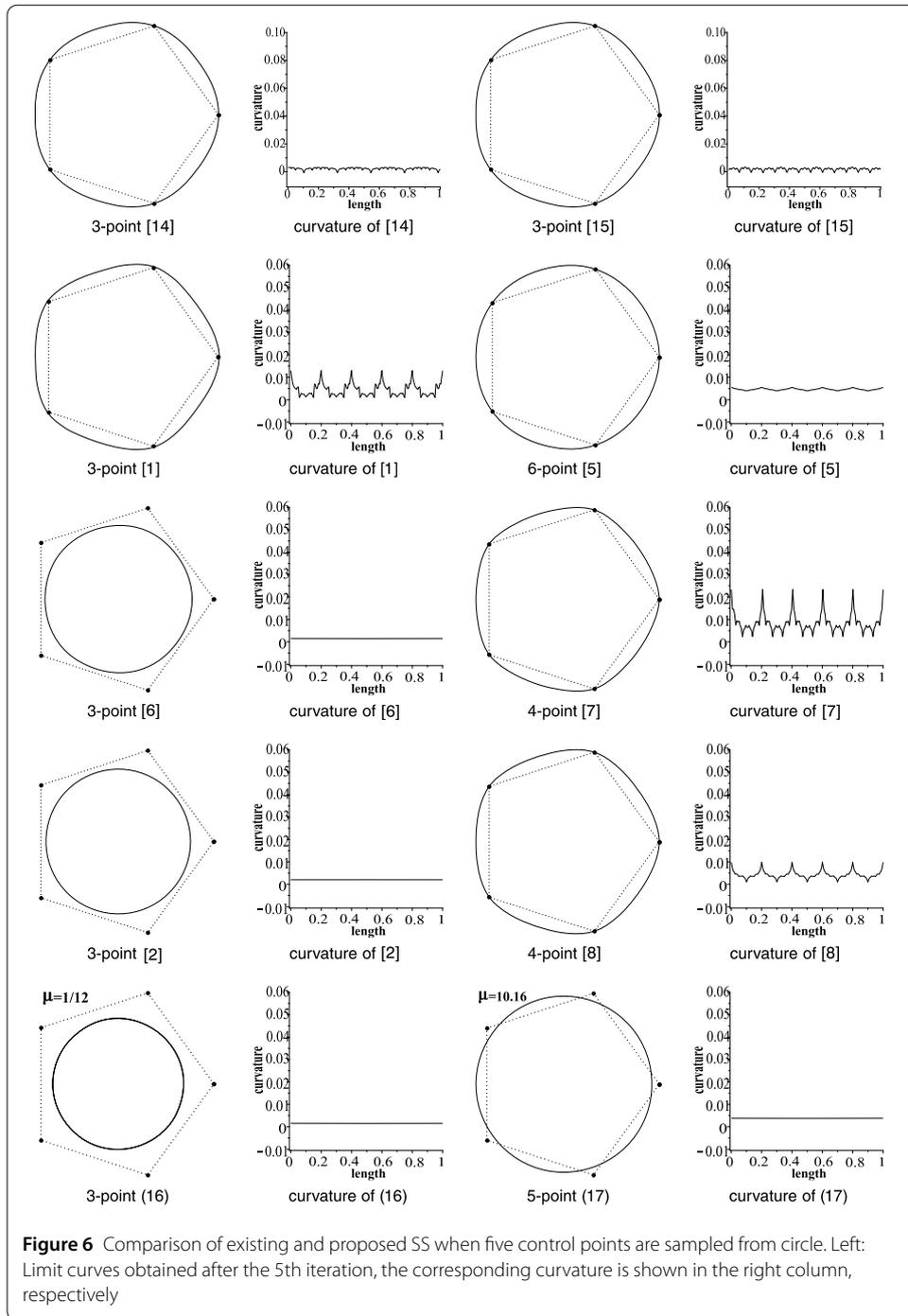
Comparison of curvature/torsion of schemes (16) and (17) with the well-known schemes of [2, 3, 5–8, 13, 14] is given in Figs. 4–8. The following general characteristics of the SSs are observed from these figures:

- Proposed SSs (16) and (17) need at least three control points to produce a circle.
- The SS of [6] needs at least five control points to create a closed curve.
- Figure 4 is achieved by applying three control points. It is showed that the limiting circle obtained by SS (16) has less tendency to depart from its tangent compared to the limiting circle obtained by the existing SS [2, 3, 5, 7, 8, 13, 14].

Figure 5 Comparison of existing and proposed SS when four control points are sampled from circle. Left: Limit curves obtained after the 5th iteration, the corresponding curvature is shown in the right column, respectively

- In Fig. 5, four control points are sampled to achieve a limiting circle. It is showed that the limiting circle obtained by SSs (16), (17), and [13] has less tendency to depart from its tangent compared to the limiting curves produced by [2, 3, 5, 7, 8, 14].
- In Fig. 6, five control points are applied to generate a limiting circle. It is observed that the limiting curves obtained by SSs (16), (17), [7], and [13] have less tendency to depart from their tangents compared to the limiting curves produced by [2, 3, 5, 8, 14].
- Figures 7 and 8 indicate that curvature and torsion behavior of limiting curves are closely related.

Figure 6 Comparison of existing and proposed SS when five control points are sampled from circle. Left: Limit curves obtained after the 5th iteration, the corresponding curvature is shown in the right column, respectively

From the above discussion it is observed that the basic shapes generated by proposed SSs with fewer initial data have less tendency to depart from their tangents as compared to the limiting curves produced by the existing non-stationary subdivision schemes.

5 Conclusions

We offered and analyzed families of $(2s − 1)$-point non-stationary subdivision schemes. It is observed that the basic shapes generated by the proposed SSs with fewer initial data in the initial control polygon have less tendency to depart from their tangents as compared
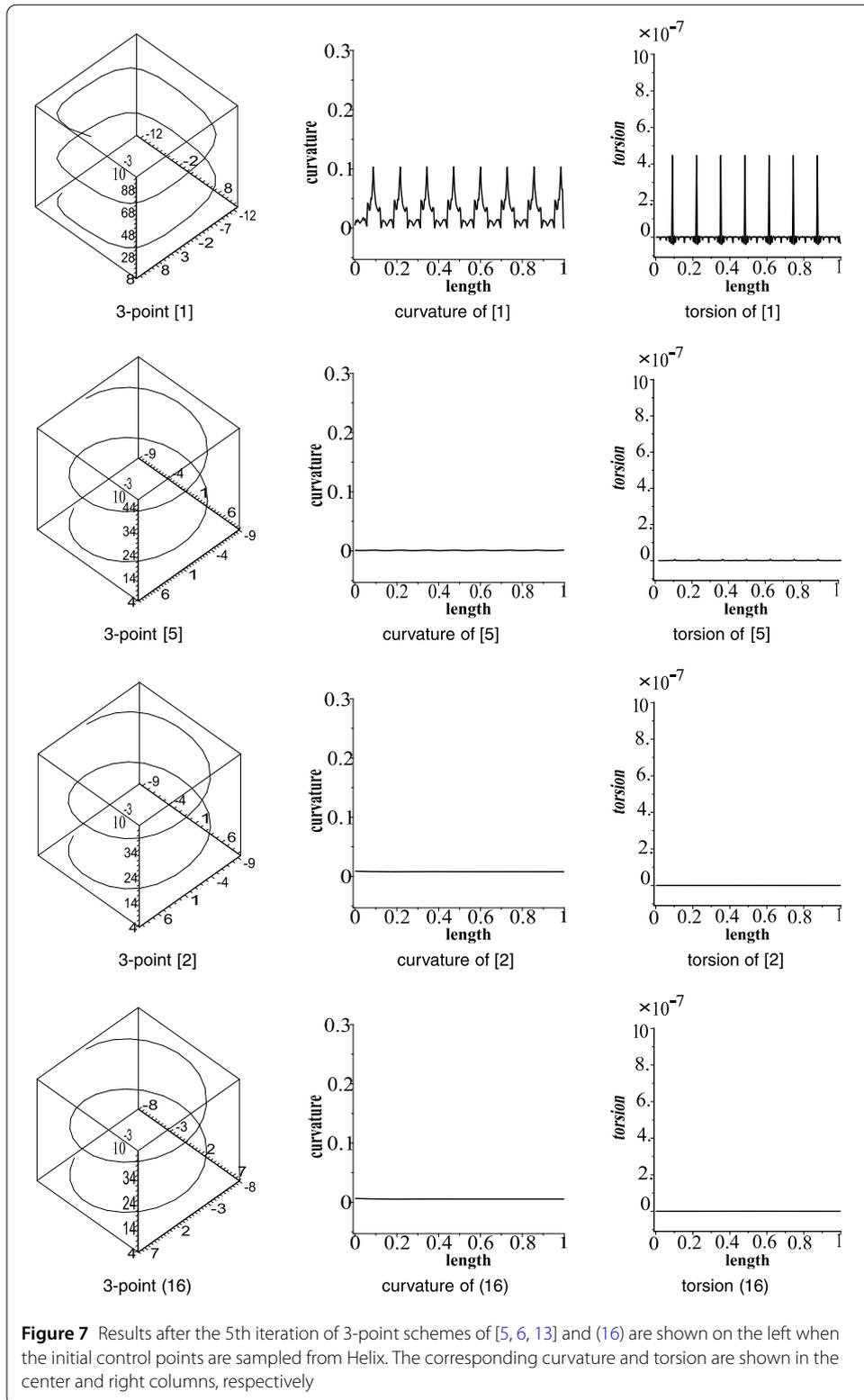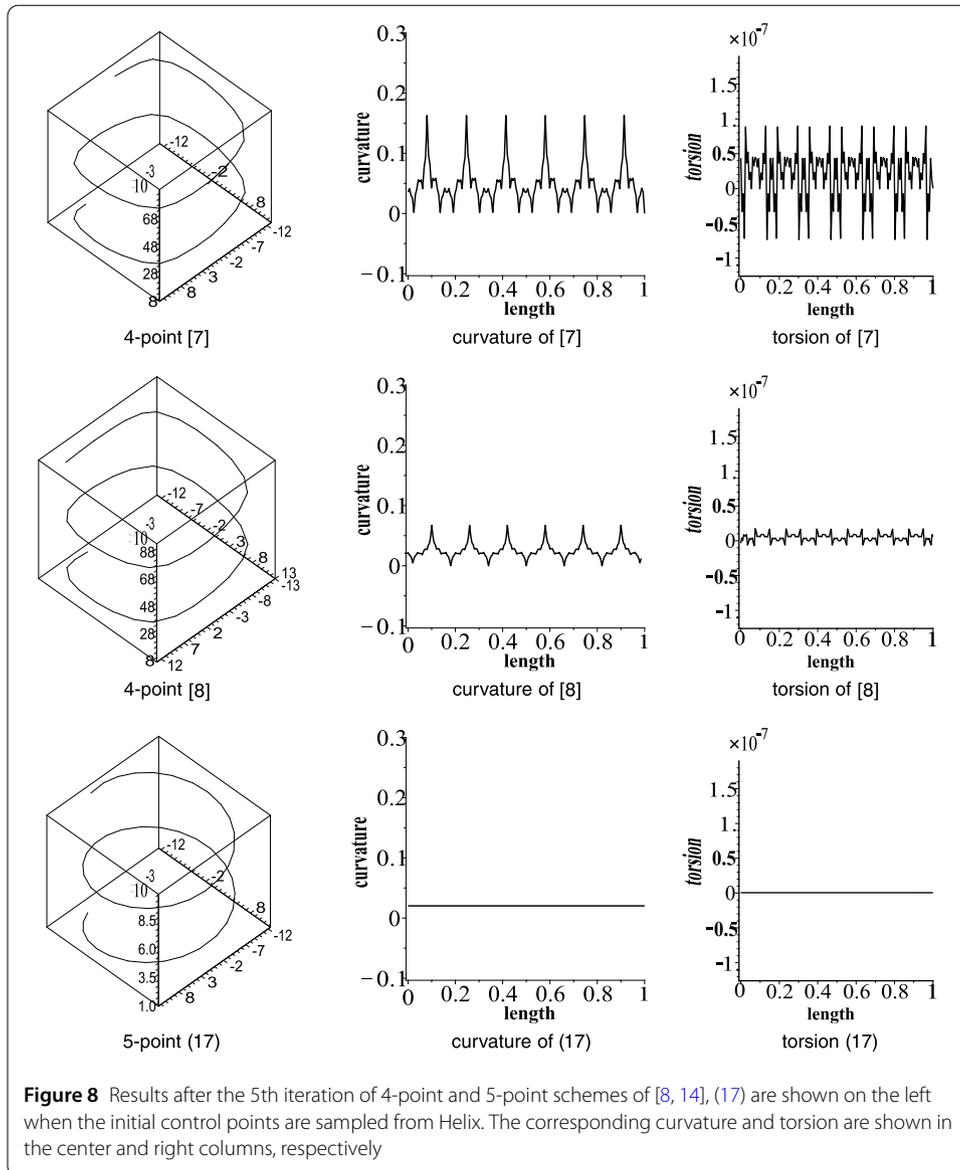
Figure 7 Results after the 5th iteration of 3-point schemes of [5, 6, 13] and (16) are shown on the left when the initial control points are sampled from Helix. The corresponding curvature and torsion are shown in the center and right columns, respectively

to the limiting curves obtained by the existing non-stationary SSs of [2, 3, 5–8, 13, 14]. It is noticed that SS (17) performed interpolatory role (see Fig. 1(c)). In addition, the proposed SSs are the non-stationary counterparts of the stationary SSs of [5, 11, 12, 22, 24–26].

Figure 8 Results after the 5th iteration of 4-point and 5-point schemes of [8, 14], (17) are shown on the left when the initial control points are sampled from Helix. The corresponding curvature and torsion are shown in the center and right columns, respectively

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Publisher's Note

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