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Existence and stability analysis to a coupled system of implicit type impulsive boundary value problems of fractional-order differential equations

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Abstract

In this paper, we study a coupled system of implicit impulsive boundary value problems (IBVPs) of fractional differential equations (FODEs). We use the Schaefer fixed point and Banach contraction theorems to obtain conditions for the existence and uniqueness of positive solutions. We discuss Hyers–Ulam (HU) type stability of the concerned solutions and provide an example for illustration of the obtained results.

Keywords: Coupled system; Arbitrary order differential equations; Impulsive conditions; Hyers–Ulam stability

1 Introduction

The fractional calculus is one of the most emerging areas of investigation. The fractional differential operators are used to model many physical phenomena in a much better form as compared to ordinary differential operators, which are local. Results derived by FDEs are much better and more accurate. For applications and details on fractional calculus, we refer the readers to [1–7]. Our work is concerned with implicit-type coupled systems of FODEs with impulsive conditions. The IFODEs are of high worth. Such equations arise in management sciences, business mathematics and other managerial sciences, and so on. Some physical phenomena have sudden changes and discontinuous jumps. To model such problems, we impose impulsive conditions on the differential equations at discontinuity points. For applications and recent work, we refer the readers to [8–29]. Coupled systems of FODEs have been studied extensively in the last few decades because in applied sciences, we deal with many physical problems that can be modeled via these systems. We would like to refer the readers to [30–36] and references therein.

Since in many situations, such as nonlinear analysis and optimization, finding the exact solution of differential equations is almost difficult or impossible, we consider approximate solutions. It is important to note that only stable approximate solutions are acceptable. Various approaches of stability analysis are adopted for this purpose. The HU-type stability concept has been considered in the numerous literature. The said stability analysis is an easy and simple way in this regard. This type concept of stability was formulated for the first time by Ulam [37], and then the next year it was elaborated by Hyers [38].

In the beginning, this concept was applied to ordinary differential equations and then extended to FODEs. We refer the readers to [39–44]. Very recently, Ali et al. [45], studied the Ulam-type stability for coupled systems of nonlinear implicit fractional differential equations.

Motivated by the aforesaid work, in this paper, we investigate the following coupled system with impulsive and $(m+2)$-point boundary conditions:

$$\begin{cases} {}_{0}^{C}\mathrm{D}_{t_j}^{\alpha}\xi(t) = \Phi(t,\mu(t),{}_{0}^{C}\mathrm{D}_{t_j}^{\alpha}\xi(t)), & t \in [0,1], t \neq t_j, j = 1,2,\ldots,m, \\ {}_{0}^{C}\mathrm{D}_{t_i}^{\beta}\mu(t) = \Psi(t,\xi(t),{}_{0}^{C}\mathrm{D}_{t_i}^{\beta}\mu(t)), & t \in [0,1], t \neq t_i, i = 1,2,\ldots,n, \\ \xi(0) = h(\xi), \quad \xi(1) = g(\xi) \quad \text{and} \quad \mu(0) = \kappa(\mu), \quad \mu(1) = f(\mu), \\ \Delta\xi(t_j) = I_j(\xi(t_j)), \quad \Delta\xi'(t_j) = \bar{I}_j(\xi(t_j)), \quad j = 1,2,\ldots,m, \\ \Delta\mu(t_i) = I_i(\mu(t_i)), \quad \Delta\mu'(t_i) = \bar{I}_i(\mu(t_i)), \quad i = 1,2,\ldots,n, \end{cases} \tag{1}$$

where $1 < \alpha, \beta \leq 2$, Φ, $\Psi : [0,1] \times \mathrm{R} \times \mathrm{R} \to \mathrm{R}$, and $g, h; f, \kappa : C(\mathrm{J},\mathrm{R}) \to \mathrm{R}$ are continuous functions defined as

$$g(\xi) = \sum_{j=1}^{\mathbf{p}} \lambda_j \xi(\xi_j), \qquad h(\xi) = \sum_{j=1}^{\mathbf{p}} \lambda_j \xi(\eta_j),$$
$$f(\mu) = \sum_{i=1}^{\mathbf{q}} \delta_i \mu(\xi_i), \qquad \kappa(\mu) = \sum_{i=1}^{\mathbf{q}} \delta_i \mu(\eta_i),$$

$\xi_i, \eta_i, \xi_j, \eta_j \in (0,1)$ for $i = 1,2,\ldots,\mathbf{q}$, $j = 1,2,\ldots,\mathbf{p}$, and

$$\Delta\xi(t_j) = \xi(t_j^+) - \xi(t_j^-),$$
$$\Delta\xi'(t_j) = \xi'(t_j^+) - \xi'(t_j^-),$$
$$\Delta\mu(t_i) = \mu(t_i^+) - \mu(t_i^-),$$
$$\Delta\mu'(t_i) = \mu'(t_i^+) - \mu'(t_i^-).$$

The notations $\xi(t_j^+)$, $\mu(t_i^+)$ are right limits, and $\xi(t_j^-)$, $\mu(t_i^-)$ are left limits; $I_j, \bar{I}_j, I_i, \bar{I}_i : \mathrm{R} \to \mathrm{R}$ are continuous functions; and D_{0+}^{α}, D_{0+}^{β} are the Caputo-type fractional differential operators of order α and β, respectively.

For system (1), we discuss necessary and sufficient conditions for the existence and uniqueness of a positive solution by using the Schaefer fixed point and Banach contraction theorems. Further, we investigate various kinds of HU and GHU stability.

2 Background materials and some auxiliary results

In this section, we give some basic definitions and results, which are used in the proof of our results.

We define the spaces of all piecewise continuous functions

$$\begin{aligned} \mathrm{B}_1 = PC(\mathrm{J},\mathrm{R}) = \bigl\{\xi : \mathrm{J} \to \mathrm{R} : j = 0,1,2,3,\ldots,m, \xi\bigl(t_j^+\bigr), \xi\bigl(t_j^-\bigr) \text{ and } \xi'\bigl(t_j^+\bigr), \xi'\bigl(t_j^-\bigr) \\ \text{exist for } j = 0,1,2,3,\ldots,m\bigr\}, \\ \mathrm{B}_2 = PC(\mathrm{J},\mathrm{R}) = \bigl\{\mu : \mathrm{J} \to \mathrm{R} : i = 0,1,2,3,\ldots,n, \mu\bigl(t_i^+\bigr), \mu\bigl(t_i^-\bigr) \text{ and } \mu'\bigl(t_i^+\bigr), \mu'\bigl(t_i^-\bigr) \\ \text{exist for } i = 0,1,2,3,\ldots,n\bigr\}. \end{aligned}$$

Clearly, B_1 and B_2 are Banach spaces under the norms $\|\xi\|_{\mathrm{B}_1} = \max_{t\in\mathrm{J}} |\xi(t)|$ and $\|\mu\|_{\mathrm{B}_2} = \max_{t\in\mathrm{J}} |\mu(t)|$, respectively. Their product $\mathbf{B} = \mathrm{B}_1 \times \mathrm{B}_2$ is also a Banach space with norm $\|(\xi,\mu)\|_{\mathbf{B}} = \|\xi\|_{\mathrm{B}_1} + \|\mu\|_{\mathrm{B}_2}$.

Definition 1 ([1]) The Caputo fractional derivative of a function $\xi : (0,\infty) \to \mathrm{R}$ is defined by

$${}_0^C\mathrm{D}_t^\alpha \xi(t) = \int_0^t \frac{(t-s)^{l-\alpha-1}}{\Gamma(l-\alpha)} \xi^{(l)}(s)\,ds,$$

where $l = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of a real number α.

Definition 2 ([4]) The Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}_+$ of a function $\xi \in C((0,\infty),\mathrm{R})$ is defined as

$${}_0\mathrm{I}_t^\alpha \xi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi(s)\,ds,$$

where $\alpha > 0$, and Γ is the gamma function, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 1 ([46]) *For $\alpha > 0$, we have*

$${}_0\mathrm{I}_t^\alpha \bigl[{}_0^C\mathrm{D}_t^\alpha \xi(t)\bigr] = \xi(t) - \sum_{i=0}^{l-1} \frac{\xi^{(i)}(0)}{i!} t^i, \quad \textit{where } l = [\alpha] + 1.$$

Lemma 2 ([46]) *For $\alpha > 0$, the differential equation ${}^C\mathrm{D}_t^\alpha \xi(t) = x(t)$ has the following solution*:

$$\xi(t) = {}_0\mathrm{I}_t^\alpha x(t) + \sum_{i=0}^{l-1} \frac{\xi^{(i)}(0)}{i!} t^i,$$

where $l = [\alpha] + 1$.

Theorem 1 (Schaefer's fixed point theorem [47]) *Let $\mathfrak{B}$ be a Banach space, and let $\mathcal{T} : \mathfrak{B} \to \mathfrak{B}$ be a completely continuous operator. If the set $\mathrm{W} = \{\xi \in \mathfrak{B} : \xi = \eta\mathcal{T}\xi, 0 < \eta < 1\}$ is bounded, then $\mathcal{T}$ has a fixed point in $\mathfrak{B}$.*

Definition 3 ([48]) The coupled system (1) is said to be HU stable if there exists $\mathbf{K}_{\alpha,\beta} = \max\{\mathbf{K}_\alpha, \mathbf{K}_\beta\} > 0$ such that, for $\epsilon = \max\{\epsilon_\alpha, \epsilon_\beta\} > 0$ and for every solution $(\xi,\mu) \in \mathbf{B}$ of the inequality

$$\begin{cases} |{}_0^C\mathrm{D}_{t_j}^{\alpha}\xi(t) - \Phi(t,\mu(t),{}_0^C\mathrm{D}_{t_j}^{\alpha}\xi(t))| \le \epsilon_\alpha, & t\in \mathrm{J}, \\ |\Delta\xi(t_j) - I_j(\xi(t_j))| \le \epsilon_\alpha, & j = 1,2,\dots,m, \\ |\Delta\xi'(t_j) - \bar{I}_j(\xi(t_j))| \le \epsilon_\alpha, & j = 1,2,\dots,m; \\ |{}_0^C\mathrm{D}_{t_i}^{\beta}\mu(t) - \Psi(t,\xi(t),{}_0^C\mathrm{D}_{t_i}^{\beta}\mu(t))| \le \epsilon_\beta, & t\in \mathrm{J}, \\ |\Delta\mu(t_i) - I_i(\mu(t_i))| \le \epsilon_\beta, & i = 1,2,\dots,n, \\ |\Delta\mu'(t_i) - \bar{I}_i(\mu(t_i))| \le \epsilon_\beta, & i = 1,2,\dots,n, \end{cases} \tag{2}$$

there exists a unique solution $(\vartheta,\sigma)\in \mathbf{B}$ with

$$\big|(\xi,\mu)(t) - (\vartheta,\sigma)(t)\big| \le \mathbf{K}_{\alpha,\beta}\epsilon, \quad t\in \mathrm{J}. \tag{3}$$

Definition 4 ([48]) The coupled system (1) is said to be GHU stable if there exists $\varphi \in \mathcal{C}(\mathrm{R}^+,\mathrm{R}^+)$ with $\varphi(0) = 0$ such that, for any approximate solution $(\xi,\mu)\in \mathbf{B}$ of inequality (2), there exists a unique solution $(\vartheta,\sigma)\in \mathbf{B}$ of (1) satisfying

$$\big|(\xi,\mu)(t) - (\vartheta,\sigma)(t)\big| \le \varphi(\epsilon), \quad t\in \mathrm{J}. \tag{4}$$

Denote $\Phi_{\alpha,\beta} = \max\{\Phi_\alpha,\Phi_\beta\}\in \mathcal{C}(\mathrm{J},\mathrm{R}) > 0$ and $\mathbf{K}_{\Phi_\alpha,\Phi_\beta} = \max\{\mathbf{K}_{\Phi_\alpha},\mathbf{K}_{\Phi_\alpha}\} > 0$.

Definition 5 ([48]) The coupled system (1) is said to be HU-Rassias stable with respect to $\Phi_{\alpha,\beta}$ if there exists a constant $\mathbf{K}_{\Phi_\alpha,\Phi_\beta}$ such that, for some $\epsilon > 0$ and for any approximate solution $(\xi,\mu)\in \mathbf{B}$ of the inequalities

$$\begin{cases} |{}_0^C\mathrm{D}_{t_j}^{\alpha}\xi(t) - \Phi(t,\mu(t),{}_0^C\mathrm{D}_{t_j}^{\alpha}\xi(t))| \le \Phi_\alpha(t)\epsilon_\alpha, & t\in \mathrm{J}, \\ |{}_0^C\mathrm{D}_{t_i}^{\beta}\mu(t) - \Psi(t,\xi(t),{}_0^C\mathrm{D}_{t_i}^{\beta}\mu(t))| \le \Phi_\beta(t)\epsilon_\beta, & t\in \mathrm{J}, \end{cases} \tag{5}$$

there exists a unique solution $(\vartheta,\sigma)\in \mathbf{B}$ with

$$\big|(\xi,\mu)(t) - (\vartheta,\sigma)(t)\big| \le \mathbf{K}_{\Phi_\alpha,\Phi_\beta}\Phi_{\alpha,\beta}\epsilon, \quad t\in \mathrm{J}. \tag{6}$$

Definition 6 ([48]) The coupled system (1) is said to be GHU-Rassias stable with respect to $\Phi_{\alpha,\beta}$ if there exists a constant $\mathbf{K}_{\Phi_\alpha,\Phi_\beta}$ such that, for any approximate solution $(\xi,\mu)\in \mathbf{B}$ of inequality (5), there exists a unique solution $(\vartheta,\sigma)\in \mathbf{B}$ of (1) satisfying

$$\big|(\xi,\mu)(t) - (\vartheta,\sigma)(t)\big| \le \mathbf{K}_{\Phi_\alpha,\Phi_\beta}\Phi_{\alpha,\beta}(t), \quad t\in \mathrm{J}. \tag{7}$$

Remark 1 We say that $(\xi,\mu)\in \mathbf{B}$ is a solution of the system of inequalities (2) if there exist functions $\Theta,\theta\in \mathcal{C}(\mathrm{J},\mathrm{R})$ depending upon ξ, μ, respectively, such that

(i) $|\Theta(t)| \le \epsilon_\alpha$, $|\theta(t)| \le \epsilon_\beta$, $t\in \mathrm{J}$;

(ii) and

$$\begin{cases} {}^{C}_{0}\mathrm{D}^{\alpha}_{t_j}\xi(t) = \Phi(t,\mu(t),{}^{C}_{0}\mathrm{D}^{\alpha}_{t_j}\xi(t)) + \Theta(t), \quad t \in \mathrm{J}, \\ \Delta\xi(t_j) = I_j(\xi(t_j)) + \Theta_j, \\ \Delta\xi'(t_j) = \bar{I}_j(\xi(t_j)) + \Theta_j, \\ {}^{C}_{0}\mathrm{D}^{\beta}_{t_i}\mu(t) = \Psi(t,\xi(t),{}^{C}_{0}\mathrm{D}^{\beta}_{t_i}\mu(t)) + \theta(t), \quad t \in \mathrm{J}, \\ \Delta\mu(t_i) = I_i(\mu(t_i)) + \theta_i, \\ \Delta\mu'(t_i) = \bar{I}_i(\mu(t_i)) + \theta_i. \end{cases}$$

3 Main results

In this section, we present our main results.

Theorem 2 *The solution* $(\xi,\mu) \in \mathbf{B}$ *of the coupled system*

$$\begin{cases} {}^{C}_{0}\mathrm{D}^{\alpha}_{t_j}\xi(t) = \omega(t), \quad t \in [0,1], t \neq t_j, j = 1,2,\ldots,m, \\ {}^{C}_{0}\mathrm{D}^{\beta}_{t_i}\mu(t) = \zeta(t), \quad t \in [0,1], t \neq t_i, i = 1,2,\ldots,n, \\ \xi(0) = h(\xi), \qquad \xi(1) = g(\xi) \quad \text{and} \quad \mu(0) = \kappa(\mu), \qquad \mu(1) = f(\mu), \\ \Delta\xi(t_j) = I_j(\xi(t_j)), \qquad \Delta\xi'(t_j) = \bar{I}_j(\xi(t_j)), \quad j = 1,2,\ldots,m, \\ \Delta\mu(t_i) = I_i(\mu(t_i)), \qquad \Delta\mu'(t_i) = \bar{I}_i(\mu(t_i)), \quad i = 1,2,\ldots,n, \end{cases} \tag{8}$$

is given by the integral equations

$$\begin{cases} \xi(t) = tg(\xi) + (1-t)h(\xi) + \sum_{j=1}^{k}(t-t_j)\bar{I}_j(\xi(t_j)) - \sum_{j=1}^{k} t(1-t_j)\bar{I}_j\xi(t_j) \\ \qquad + \sum_{j=1}^{k} I_j(\xi(t_j)) - \sum_{j=1}^{k} tI_j\xi(t_j) + \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t}(t-s)^{\alpha-1}\omega(s)\,ds \\ \qquad + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\omega(s)\,ds \\ \qquad + \frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(t-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}\omega(s)\,ds \\ \qquad - \frac{t}{\Gamma(\alpha)}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\omega(s)\,ds \\ \qquad - \frac{t}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}\omega(s)\,ds, \\ \quad k = 1,2,\ldots,m, \\ \mu(t) = tf(\mu) + (1-t)\kappa(\mu) + \sum_{i=1}^{k}(t-t_i)\bar{I}_i(\mu(t_i)) - \sum_{i=1}^{k} t(1-t_i)\bar{I}_i\mu(t_i) \\ \qquad + \sum_{i=1}^{k} I_i(\mu(t_i)) - \sum_{i=1}^{k} tI_i\mu(t_i) + \frac{1}{\Gamma(\beta)}\int_{t_i}^{t}(t-s)^{\beta-1}\zeta(s)\,ds \\ \qquad + \frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\zeta(s)\,ds\mu(t_i) \\ \qquad + \frac{1}{\Gamma(\beta-1)}\sum_{i=1}^{k}(t-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-2}\zeta(s)\,ds \\ \qquad - \frac{t}{\Gamma(\beta)}\sum_{i=1}^{k+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\zeta(s)\,ds\mu(t_i) \\ \qquad - \frac{t}{\Gamma(\beta-1)}\sum_{i=1}^{k}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-2}\zeta(s)\,ds, \\ \quad k = 1,2,\ldots,n. \end{cases} \tag{9}$$

Proof The proof can be obtained as in [14, 34]. □

Corollary 1 *In view of Theorem 2, our coupled system (1) has the following solution:*

$$\begin{cases} \xi(t) = tg(\xi) + (1-t)h(\xi) + \sum_{j=1}^{k}(t-t_j)\bar{I}_j(\xi(t_j)) - \sum_{j=1}^{k} t(1-t_j)\bar{I}_j\xi(t_j) \\ \quad + \sum_{j=1}^{k} I_j(\xi(t_j)) - \sum_{j=1}^{k} tI_j\xi(t_j) \\ \quad + \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t}(t-s)^{\alpha-1}\Phi(s,\mu(s),{}_0^C\mathrm{D}_{t_i}^{\beta}\xi(s))\,ds \\ \quad + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\Phi(s,\mu(s),{}_0^C\mathrm{D}_{t_i}^{\beta}\xi(s))\,ds \\ \quad + \frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(t-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}\Phi(s,\mu(s),{}_0^C\mathrm{D}_{t_i}^{\beta}\xi(s))\,ds \\ \quad - \frac{t}{\Gamma(\alpha)}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\Phi(s,\mu(s),{}_0^C\mathrm{D}_{t_i}^{\beta}\xi(s))\,ds \\ \quad - \frac{t}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}\Phi(s,\mu(s),{}_0^C\mathrm{D}_{t_i}^{\beta}\xi(s))\,ds, \\ \quad k = 1,2,\ldots,m, \\ \mu(t) = tf(\mu) + (1-t)\kappa(\mu) + \sum_{i=1}^{k}(t-t_i)\bar{I}_i(\mu(t_i)) - \sum_{i=1}^{k} t(1-t_i)\bar{I}_i\mu(t_i) \\ \quad + \sum_{i=1}^{k} I_i(\mu(t_i)) - \sum_{i=1}^{k} tI_i\mu(t_i) \\ \quad + \frac{1}{\Gamma(\beta)}\int_{t_i}^{t}(t-s)^{\beta-1}\Psi(s,\xi(s),{}_0^C\mathrm{D}_{t_i}^{\beta}\mu(s))\,ds \\ \quad + \frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\Psi(s,\xi(s),{}_0^C\mathrm{D}_{t_i}^{\beta}\mu(s))\,ds \\ \quad + \frac{1}{\Gamma(\beta-1)}\sum_{i=1}^{k}(t-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-2}\Psi(s,\xi(s),{}_0^C\mathrm{D}_{t_i}^{\beta}\mu(s))\,ds \\ \quad - \frac{t}{\Gamma(\beta)}\sum_{i=1}^{k+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\Psi(s,\xi(s),{}_0^C\mathrm{D}_{t_i}^{\beta}\mu(s))\,ds \\ \quad - \frac{t}{\Gamma(\beta-1)}\sum_{i=1}^{k}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-2}\Psi(s,\xi(s),{}_0^C\mathrm{D}_{t_i}^{\beta}\mu(s))\,ds, \\ \quad k = 1,2,\ldots,n. \end{cases} \tag{10}$$

For simplicity, we use use the notations $u_{\mu,\xi}(t) = \Phi(t,\mu(t),{}_0^C\mathrm{D}_{t_j}^{\beta}\xi(t))$ and $v_{\xi,\mu}(t) = \Phi(t,\xi(t),{}_0^C\mathrm{D}_{t_i}^{\beta}\mu(t))$. To convert the considered problem into a fixed point problem, we define the operator $T:\mathbf{B}\to\mathbf{B}$ by $T(\xi,\mu)(t) = \binom{T_\alpha(\mu,\omega)(t)}{T_\beta(\xi,\zeta)(t)}$ such that

$$\begin{aligned} T_\alpha(\xi,\mu)(t) &= tg(\xi) + (1-t)h(\xi) + \sum_{j=1}^{k}(t-t_j)\bar{I}_j\bigl(\xi(t_j)\bigr) \\ &\quad - \sum_{j=1}^{k} t(1-t_j)\bar{I}_j\xi(t_j) + \sum_{j=1}^{k} I_j\bigl(\xi(t_j)\bigr) \\ &\quad - \sum_{j=1}^{k} tI_j\xi(t_j) + \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t}(t-s)^{\alpha-1}u_{\mu,\xi}(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}u_{\mu,\xi}(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(t-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}u_{\mu,\xi}(s)\,ds \\ &\quad - \frac{t}{\Gamma(\alpha)}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}u_{\mu,\xi}(s)\,ds \\ &\quad - \frac{t}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}u_{\mu,\xi}(s)\,ds, \end{aligned}$$

$$\begin{aligned}
T_\beta(\xi,\mu)(t) &= tf(\mu) + (1-t)\kappa(\mu) + \sum_{i=1}^{k}(t-t_i)\bar{I}_i\big(\mu(t_i)\big) - \sum_{i=1}^{k} t(1-t_i)\bar{I}_i\mu(t_i) \\
&\quad + \sum_{i=1}^{k} I_i\big(\mu(t_i)\big) - \sum_{i=1}^{k} tI_i\mu(t_i) + \frac{1}{\Gamma(\beta)}\int_{t_i}^{t}(t-s)^{\beta-1}\nu_{\xi,\mu}(s)\,ds \\
&\quad + \frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\nu_{\xi,\mu}(s)\,ds \\
&\quad + \frac{1}{\Gamma(\beta-1)}\sum_{i=1}^{k}(t-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-2}\nu_{\xi,\mu}(s)\,ds \\
&\quad - \frac{t}{\Gamma(\beta)}\sum_{i=1}^{k+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\nu_{\xi,\mu}(s)\,ds \\
&\quad - \frac{t}{\Gamma(\beta-1)}\sum_{i=1}^{k}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-2}\nu_{\xi,\mu}(s)\,ds.
\end{aligned}$$

We obtain our results under the following assumptions:

(H_1) for any $\xi,\mu \in C([0,1],\mathrm{R})$, there exist $K_g, K_h, K_f, K_\kappa > 0$ such that

$$\begin{aligned}
&\big\|g(\xi)-g(\mu)\big\|_{PC} \le K_g\|\xi-\mu\|_{PC}, \qquad \big\|f(\xi)-f(\mu)\big\|_{PC} \le K_f\|\xi-\mu\|_{PC}, \\
&\big\|h(\xi)-h(\mu)\big\|_{PC} \le K_h\|\xi-\mu\|_{PC}, \qquad \big\|\kappa(\xi)-\kappa(\mu)\big\|_{PC} \le K_\kappa\|\xi-\mu\|_{PC};
\end{aligned}$$

(H_2) for all $\xi,\bar{\xi},\mu,\bar{\mu} \in \mathrm{R}$ and $t \in [0,1]$ there exist $L_{\Phi 1} > 0$, $0 < L_{\Phi 2} < 1$, $L_{\Psi 1} > 0$, and $0 < L_{\Psi 2} < 1$ such that

$$\begin{aligned}
&\big|\Phi(t,\xi,\mu) - \Phi(t,\bar{\xi},\bar{\mu})\big| \le L_{\Phi 1}|\xi-\bar{\xi}| + L_{\Phi 2}|\mu-\bar{\mu}|, \\
&\big|\Psi(t,\xi,\mu) - \Psi(t,\bar{\xi},\bar{\mu})\big| \le L_{\Psi 1}|\xi-\bar{\xi}| + L_{\Psi 2}|\mu-\bar{\mu}|;
\end{aligned}$$

(H_3) there exist constants A_1, A_2, A_3 and $A_4 > 0$ such that, for $\xi,\bar{\xi},\mu,\bar{\mu} \in \mathrm{R}$,

$$\begin{aligned}
&\big|I_j(\xi) - I_j(\bar{\xi})\big| \le A_1|\xi-\bar{\xi}|, \qquad \big|\bar{I}_j(\xi) - \bar{I}_j(\bar{\xi}\big| \le A_2|\xi-\bar{\xi}|, \quad j = 1,2,\ldots,m, \\
&\big|I_i(\mu) - I_i(\bar{\mu})\big| \le A_3|\mu-\bar{\mu}|, \qquad \big|\bar{I}_i(\mu) - \bar{I}_i(\bar{\mu}\big| \le A_4|\mu-\bar{\mu}|, \quad i = 1,2,\ldots,n;
\end{aligned}$$

(H_4) there exist constants $\Bbbk_1, \mathcal{N}_1, \Bbbk_2, \mathcal{N}_2, \Bbbk_3, \mathcal{N}_3, \Bbbk_4, \mathcal{N}_4 > 0$ such that

$$\begin{aligned}
&\big|I_j(\xi_j)\big| \le \Bbbk_1|\xi| + \mathcal{N}_1, \qquad \big|\bar{I}_j(\xi_j)\big| \le \Bbbk_2|\xi| + \mathcal{N}_2, \quad j = 1,2,\ldots,m, \\
&\big|I_i(\mu_i)\big| \le \Bbbk_3|\mu| + \mathcal{N}_3
\end{aligned}$$

and $|\bar{I}_i(\mu_i)| \le \Bbbk_4|\mu| + \mathcal{N}_4$, $i = 1,2,\ldots,n$;

(H_5) there exist constants $\Bbbk_5$, $\Bbbk_6$, $\Bbbk_7$, $\Bbbk_8$ such that

$$\begin{aligned}
&\big|g(\xi)\big| \le \Bbbk_5, \qquad \big|h(\xi)\big| \le \Bbbk_6 \quad \text{for all } \xi \in C\big([0,1],\mathrm{R}\big), \\
&\big|f(\mu)\big| \le \Bbbk_7\big|\kappa(\mu)\big| \le \Bbbk_8
\end{aligned}$$

for all $\mu \in C([0,1],\mathrm{R})$;

(H_6) there exist some functions p_1, q_1, r_1 and $p_2, q_2, r_2 \in C(\mathrm{J}, \mathrm{R}^+)$ such that, for $t \in \mathrm{J}$ and $(\mu, \xi) \in \mathbf{B}$, we have

$$\left|\Phi\left(t, \mu(t), {}_0^C\mathrm{D}_{t_j}^\alpha \xi(t)\right)\right| \le p_1(t) + q_1(t)|\mu| + r_1(t)\left|{}_0^C\mathrm{D}_{t_j}^\alpha \xi(t)\right|$$

with ${p_1}^* = \sup_{t\in\mathrm{J}} |p_1(t)|$, ${q_1}^* = \sup_{t\in\mathrm{J}} |q_1(t)|$, and ${r_1}^* = \sup_{t\in\mathrm{J}} |r_1(t)| < 1$ and

$$\left|\Psi\left(t, \xi(t), {}_0^C\mathrm{D}_{t_j}^\alpha \mu(t)\right)\right| \le p_2(t) + q_2(t)|\mu| + r_2(t)\left|{}_0^C\mathrm{D}_{t_j}^\alpha \xi(t)\right|,$$

with ${p_2}^* = \sup_{t\in\mathrm{J}} |p_2(t)|$, ${q_2}^* = \sup_{t\in\mathrm{J}} |q_2(t)|$, and ${r_2}^* = \sup_{t\in\mathrm{J}} |r_2(t)| < 1$.

Theorem 3 *If assumptions* (H_1), (H_2), (H_3) *and the inequality*

$$\aleph = \max(\aleph_1, \aleph_2) < 1 \tag{11}$$

are satisfied, where

$$\aleph_1 = \left[K_g + K_h + 2m(A_1 + A_2) + \frac{2L_{\Phi_1}}{1 - L_{\Phi_2}}\left(\frac{1+m}{\Gamma(\alpha+1)} + \frac{m}{\Gamma(\alpha)}\right)\right]$$

and

$$\aleph_2 = \left[K_f + K_\kappa + 2n(A_3 + A_4) + \frac{2L_{\Psi_1}}{1 - L_{\Psi_2}}\left(\frac{1+n}{\Gamma(\beta+1)} + \frac{n}{\Gamma(\beta)}\right)\right],$$

then the coupled system (1) *has a unique solution.*

Proof Take $(\xi, \mu), (\bar{\xi}, \bar{\mu}) \in \mathbf{B}$ and consider

$$\begin{aligned}
&\left|T_\alpha(\xi, \mu)(t) - T_\alpha(\bar{\xi}, \bar{\mu})(t)\right| \\
&\quad = \Bigg| t\left(g(\xi) - g(\bar{\xi})\right) + (1-t)\left(h(\xi) - h(\bar{\xi})\right) \\
&\qquad + \sum_{j=1}^{k}(t - t_j)\bar{I}_j\left(\xi(t_j) - \bar{\xi}(t_j)\right) - \sum_{j=1}^{k} t(1 - t_j)\bar{I}_j\left(\xi(t_j) - \bar{\xi}(t_j)\right) + \sum_{j=1}^{k} I_j\left(\xi(t_j) - \bar{\xi}(t_j)\right) \\
&\qquad - \sum_{j=1}^{k} tI_j\left(\xi(t_j) - \bar{\xi}(t_j)\right) + \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t}(t-s)^{\alpha-1}\left(u_{\mu,\xi}(s) - \bar{u}_{\mu,\xi}(s)\right)ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j - s)^{\alpha-1}\left(u_{\mu,\xi}(s) - \bar{u}_{\mu,\xi}(s)\right)ds \\
&\qquad + \frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(t - t_j)\int_{t_{j-1}}^{t_j}(t_j - s)^{\alpha-2}\left(u_{\mu,\xi}(s) - \bar{u}_{\mu,\xi}(s)\right)ds \\
&\qquad - \frac{t}{\Gamma(\alpha)}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(t_j - s)^{\alpha-1}\left(u_{\mu,\xi}(s) - \bar{u}_{\mu,\xi}(s)\right)ds
\end{aligned}$$

$$- \frac{t}{\Gamma(\alpha-1)} \sum_{j=1}^{k}(1-t_j) \int_{t_{j-1}}^{t_j} (t_j-s)^{\alpha-2}\bigl(u_{\mu,\xi}(s)-\bar{u}_{\mu,\xi}(s)\bigr)\,ds\Bigg|, \tag{12}$$

which further means that

$$\begin{aligned}
&\bigl|T_\alpha(\xi,\mu)(t)-T_\alpha(\bar{\xi},\bar{\mu})(t)\bigr| \\
&\quad \le |t|\bigl|g(\xi)-g(\bar{\xi})\bigr| + |1-t|\bigl|h(\xi)-h(\bar{\xi})\bigr| + \sum_{j=1}^{k}|t-t_j| \\
&\qquad \times \bar{I}_j\bigl|\xi(t_j)-\bar{\xi}(t_j)\bigr| + \sum_{j=1}^{k}|t||1-t_j|\bigl|\bar{I}_j\xi(t_j)-\bar{I}_j\bar{\xi}(t_j)\bigr| + \sum_{j=1}^{k}\bigl|I_j(\xi(t_j)-I_j\bar{\xi}(t_j)\bigr| \\
&\qquad + \sum_{j=1}^{k}|t|\bigl|I_j\xi(t_j)-I_j\bar{\xi}(t_j)\bigr| + \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t}(t-s)^{\alpha-1}\bigl|u_{\mu,\xi}(s)-\bar{u}_{\mu,\xi}(s)\bigr|\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\bigl|u_{\mu,\xi}(s)-\bar{u}_{\mu,\xi}(s)\bigr|\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}|t-t_j|\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}\bigl|u_{\mu,\xi}(s)-\bar{u}_{\mu,\xi}(s)\bigr|\,ds \\
&\qquad + \frac{t}{\Gamma(\alpha)}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\bigl|u_{\mu,\xi}(s)-\bar{u}_{\mu,\xi}(s)\bigr|\,ds \\
&\qquad + \frac{t}{\Gamma(\alpha-1)}\sum_{j=1}^{k}|1-t_j|\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}|u_{\mu,\xi}(s)-\bar{u}_{\mu,\xi}(s)|\,ds.
\end{aligned} \tag{13}$$

By assumption (H_2) we have

$$\begin{aligned}
\bigl|u_{\mu,\xi}(t)-\bar{u}_{\mu,\xi}(t)\bigr| &= \bigl|\Phi\bigl(t,\mu(t),u_{\mu,\xi}(t)\bigr)-\Phi\bigl(t,\bar{\mu}(t),\bar{u}_{\mu,\xi}(t)\bigr)\bigr| \\
&\le L_{\Phi 1}\bigl|\mu(t)-\bar{\mu}(t)\bigr| + L_{\Phi 2}\bigl|u_{\mu,\xi}(t)-\bar{u}_{\mu,\xi}(t)\bigr| \\
&= \frac{L_{\Phi 1}}{1-L_{\Phi 2}}\bigl|\mu(t)-\bar{\mu}(t)\bigr|.
\end{aligned} \tag{14}$$

By assumptions (H_1) and (H_3) and inequality (14), taking the maximum over the interval J, from inequality (13) we have

$$\begin{aligned}
&\bigl\|T_\alpha(\xi,\mu)-T_\alpha(\bar{\xi},\bar{\mu})\bigr\|_{\mathrm{B}_1} \\
&\quad \le K_g\|\xi-\bar{\xi}\|_{\mathrm{B}_1} + K_h\|\xi-\bar{\xi}\|_{\mathrm{B}_1} + mA_2\|\xi-\bar{\xi}\|_{\mathbf{B}_1} \\
&\qquad + mA_2\|\xi-\bar{\xi}\|_{\mathbf{B}_1} + mA_1\|\xi-\bar{\xi}\|_{\mathbf{B}_1} + mA_1\|\xi-\bar{\xi}\|_{\mathbf{B}_1} + \frac{L_{\Phi_1}}{(1-L_{\Phi_2})\Gamma(\alpha+1)} \\
&\qquad \times \|\mu-\bar{\mu}\|_{\mathbf{B}_1} + \frac{L_{\Phi_1}m}{(1-L_{\Phi_2})\Gamma(\alpha+1)}\|\mu-\bar{\mu}\|_{\mathbf{B}_1} + \frac{L_{\Phi_1}m}{(1-L_{\Phi_2})\Gamma(\alpha)}\|\mu-\bar{\mu}\|_{\mathbf{B}_1} \\
&\qquad + \frac{L_{\Phi_1}(m+1)}{(1-L_{\Phi_2})\Gamma(\alpha+1)}\|\mu-\bar{\mu}\|_{\mathbf{B}_1} + \frac{L_{\Phi_1}m}{(1-L_{\Phi_2})\Gamma(\alpha)}\|\mu-\bar{\mu}\|_{\mathbf{B}_1} \\
&\quad \le \aleph_1\bigl(\|\xi-\bar{\xi}\|_{\mathbf{B}_1} + \|\mu-\bar{\mu}\|_{\mathbf{B}_1}\bigr),
\end{aligned}$$

where

$$\aleph_1 = \left[K_g + K_h + 2m(A_1 + A_2) + \frac{2L_{\Phi_1}}{1 - L_{\Phi_2}}\left(\frac{1+m}{\Gamma(\alpha+1)} + \frac{m}{\Gamma(\alpha)}\right)\right].$$

Similarly, we have

$$\left\| T_\beta(\xi,\mu) - T_\beta(\bar{\xi},\bar{\mu})\right\|_{\mathbf{B}_2} \le \aleph_2\left(\|\xi - \bar{\xi}\|_{\mathbf{B}_2} + \|\mu - \bar{\mu}\|_{\mathbf{B}_2}\right),$$

where

$$\aleph_2 = \left[K_f + K_\kappa + 2n(A_3 + A_4) + \frac{2L_{\Psi_1}}{1 - L_{\Psi_2}}\left(\frac{1+n}{\Gamma(\beta+1)} + \frac{n}{\Gamma(\beta)}\right)\right],$$

from which we have

$$\left\| T(\xi,\mu) - T(\bar{\xi},\bar{\mu})\right\|_{\mathbf{B}} \le \aleph\left[\left\|(\xi,\mu) - (\bar{\xi},\bar{\mu})\right\|_{\mathbf{B}}\right],$$

where $\aleph = \max\{\aleph_1, \aleph_2\}$. Hence T is a contraction, and therefore, by the Banach contraction principle, T has a unique fixed point. □

Theorem 4 *If assumptions* (H_1)–(H_6) *hold, then the coupled system* (1) *has at least one solution.*

Proof Here we use the Schaefer fixed point theorem. We need to show that the operator T has at least one fixed point. There are several steps involved in this method.

Step 1: We will show that the operator T is continuous. Take a sequence $(\xi_n, \mu_n) \to (\xi, \mu) \in \mathbf{B}$. For any $t \in \mathrm{J}$, we consider

$$\begin{aligned}
&\left|T_\alpha(\xi_n,\mu_n)(t) - T_\alpha(\xi,\mu)(t)\right| \\
&\quad\le |t|\left|g(\xi_n) - g(\xi)\right| + |1-t|\left|h(\xi_n) - h(\xi)\right| \\
&\qquad + \sum_{j=1}^{k}|t - t_j|\left|\bar{I}_j\left(\xi_n(t_j)\right) - \bar{I}_j\left(\xi(t_j)\right)\right| + \sum_{j=1}^{k}|t||1 - t_j|\left|\bar{I}_j\left(\xi_n(t_j)\right) - \bar{I}_j\left(\xi(t_j)\right)\right| \\
&\qquad + \sum_{j=1}^{k}\left|I_j\left(\xi_n(t_j)\right) - I_j\left(\xi(t_j)\right)\right| + \sum_{j=1}^{k}|t|\left|I_j(\xi_n(t_j) - I_j\left(\xi(t_j)\right)\right| \\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t}(t-s)^{\alpha-1}\left|u_{\mu,\xi,n}(s) - u_{\mu,\xi}(s)\right|\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\left|u_{\mu,\xi,n}(s) - u_{\mu,\xi}(s)\right|\,ds \\
&\qquad + \frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(t - t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}\left|u_{\mu,\xi,n}(s) - u_{\mu,\xi}(s)\right|\,ds \\
&\qquad + \frac{|t|}{\Gamma(\alpha)}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\left|u_{\mu,\xi,n}(s) - u_{\mu,\xi}(s)\right|\,ds
\end{aligned}$$

$$+\frac{|t|}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}\big|u_{\mu,\xi,n}(s)-u_{\mu,\xi}(s)\big|\,ds. \tag{15}$$

By assumption (H_2) we have

$$\begin{aligned}\big|u_{\mu,\xi,n}(t)-u_{\mu,\xi}(t)\big| &= \big|\Phi\big(t,\mu_n(t),u_{\mu,\xi,n}(t)\big)-\Phi\big(t,\mu(t),u_{\mu,\xi}(t)\big)\big| \\ &\leq L_{\Phi 1}\big|\mu_n(t)-\mu(t)\big|+L_{\Phi 2}\big|u_{\mu,\xi,n}(t)-u_{\mu,\xi}(t)\big| \\ &= \frac{L_{\Phi 1}}{1-L_{\Phi 2}}\big|\mu_n(t)-\mu(t)\big|. \end{aligned} \tag{16}$$

Since $\mu_n \to \mu$ as $n\to\infty$, we have that, for each $t\in \mathrm{J}$, $u_{\mu,\xi,n}(t)\to u_{\mu,\xi}(t)$ as $n\to\infty$. Also, for each $t\in \mathrm{J}$, $\xi_n(t)\to\xi(t)$ as $n\to\infty$. Since every convergent sequence is bounded, there exists a constant $\mathbf{b}$ such that $|u_{\mu,\xi,n}(t)|\leq \mathbf{b}$ and $|u_{\mu,\xi}(t)|\leq\mathbf{b}$ for each $t\in\mathrm{J}$. We have

$$\begin{aligned}(t-s)^{\alpha-1}\big|u_{\mu,\xi,n}(s)-u_{\mu,\xi}(s)\big| &\leq (t-s)^{\alpha-1}\big(\big|u_{\mu,\xi,n}(s)\big|+\big|u_{\mu,\xi}(s)\big|\big) \\ &\leq 2\mathbf{b}(t-s)^{\alpha-1}, \\ (t_j-s)^{\alpha-1}\big|u_{\mu,\xi,n}(s)-u_{\mu,\xi}(s)\big| &\leq (t_j-s)^{\alpha-1}\big(\big|u_{\mu,\xi,n}(s)\big|+\big|u_{\mu,\xi}(s)\big|\big) \\ &\leq 2\mathbf{b}(t_j-s)^{\alpha-1}, \\ (t_j-s)^{\alpha-2}\big|u_{\mu,\xi,n}(s)-u_{\mu,\xi}(s)\big| &\leq (t_j-s)^{\alpha-2}\big(\big|u_{\mu,\xi,n}(s)\big|+\big|u_{\mu,\xi}(s)\big|\big) \\ &\leq 2\mathbf{b}(t_j-s)^{\alpha-2}. \end{aligned}$$

Clearly, the functions $s\to 2\mathbf{b}(t-s)^{\alpha-1}$, $s\to 2\mathbf{b}(t_j-s)^{\alpha-1}$, and $s\to 2\mathbf{b}(t_j-s)^{\alpha-2}$ are integrable on the interval [0, t]. Thus, by assumptions (H_1)–(H_3), inequality (16), and the Lebesgue dominated convergence theorem, the right-hand side of inequality (15) goes to zero, that is,

$$\big|T_\alpha(\xi_n,\mu_n)(t)-T_\alpha(\xi,\mu)(t)\big|\to 0 \quad \text{as } n\to\infty,$$

and thus

$$\big\|T_\alpha(\xi_n,\mu_n)-T_\alpha(\xi,\mu)\big\|\to 0 \quad \text{as } n\to\infty.$$

This implies that the operator T_α is continuous. Similarly, we can show that the operator T_β is continuous, so that the operator $T=\binom{T_\alpha}{T_\beta}$ is continuous.

Step 2: We define the set $\Omega_\varrho=\{(\xi,\mu)\in\mathbf{B}:|(\xi,\mu)|\leq\varrho \text{ with } |\xi|\leq\varrho_1 \text{ and } |\mu|\leq\varrho_2\}$, where $\max\{\varrho_1,\varrho_2\}=\varrho$. For $t\in\mathrm{J}$, we consider

$$\begin{aligned}\big|T_\alpha(\xi,\mu)\big| &\leq |t|\big|g(\xi)\big|+|1-t|\big|h(\xi)\big|+\sum_{j=1}^{k}|t-t_j|\bar{I}_j\big|\xi(t_j)\big| \\ &\quad+\sum_{j=1}^{k}|t||1-t_j|\big|\bar{I}_j\xi(t_j)\big|+\sum_{j=1}^{k}\big|I_j\big(\xi(t_j)\big)\big|+\sum_{j=1}^{k}|t|\big|I_j\xi(t_j)\big| \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t}(t-s)^{\alpha-1}\big|u_{\mu,\xi}(s)\big|\,ds+\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}|u_{\mu,\xi}(s)|}{\Gamma(\alpha)}\,ds\end{aligned}$$

$$+\sum_{j=1}^{k}|t-t_j|\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}|u_{\mu,\xi}(s)|}{\Gamma(\alpha-1)}\,ds+|t|\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}|u_{\mu,\xi}(s)|}{\Gamma(\alpha)}\,ds$$
$$+|t|\sum_{j=1}^{k}|1-t_j|\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}|u_{\mu,\xi}(s)|}{\Gamma(\alpha-1)}\,ds. \tag{17}$$

By (H_6) we have

$$\begin{aligned}\big|u_{\mu,\xi}(t)\big| &\le p_1(t)+q_1(t)\big|(\xi,\mu)\big|+r_1(t)\big|u_{\mu,\xi}(t)\big|\\ &\le p_1^*+q_1^*\varrho+r_1^*|\omega|\\ &=\frac{p_1^*+q_1^*\varrho}{1-r_1^*}=:\chi.\end{aligned} \tag{18}$$

Thus by (H_4), (H_5), and (H_6) from (17) we obtain the following result:

$$\begin{aligned}\big\|T_\alpha(\xi,\mu)\big\|_{\mathbf{B}_1} \le{}& \Bbbk_5+\Bbbk_6+m(\Bbbk_2\varrho_1+\mathcal{N}_2)+m(\Bbbk_2\varrho_1+\mathcal{N}_2)\\ &+m(\Bbbk_1\varrho_1+\mathcal{N}_1)+m(\Bbbk_1\varrho_1+\mathcal{N}_1)+\frac{\chi}{\Gamma(\alpha+1)}+\frac{m\chi}{\Gamma(\alpha+1)}\\ &+\frac{m\chi}{\Gamma(\alpha)}+\frac{(m+1)\chi}{\Gamma(\alpha+1)}+\frac{m\chi}{\Gamma(\alpha)}=:\varsigma_1.\end{aligned} \tag{19}$$

Similarly, we can show that

$$\big\|T_\beta(\mu,\xi)\big\|_{\mathbf{B}_2}\le\varsigma_2. \tag{20}$$

Now if $\max(\varsigma_1,\varsigma_2)=\varsigma$, then we have

$$\big\|T(\xi,\mu)\big\|_{\mathbf{B}}\le\varsigma.$$

This shows that bounded sets are mapped into bounded sets under T.

Step 3: W will show that T is equicontinuous. Let $\mathbb{D}\subseteq\mathbf{B}$. Then for $(\xi,\mu)\in\mathbb{D}$ and $t_1,t_2\in \mathrm{J}$ such that $t_1<t_2$, we consider

$$\begin{aligned}&\big|T_\alpha(\xi,\mu)(t_2)-T_\alpha(\xi,\mu)(t_1)\big|\\ &\quad\le|(t_2-t_1)\big(g(\xi)-g(\xi)\big)-(t_2-t_1))\big(h(\xi)-h(\xi)\big)\\ &\qquad+\sum_{j=1}^{k}(t_2-t_1)\bar{I}_j\big(\xi(t_j)-\xi(t_j)\big)-\sum_{j=1}^{k}(t_2-t_1)\bar{I}_j\big(\xi(t_j)-\xi(t_j)\big)-(t_2-t_1)\\ &\qquad\times\sum_{j=1}^{k}I_j\big(\xi(t_j)-\xi(t_j)\big)\\ &\qquad+\left(\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_2}(t_2-s)^{\alpha-1}u_{\mu,\xi}(s)\,ds-\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_1}(t_1-s)^{\alpha-1}u_{\mu,\xi}(s)\,ds\right)\\ &\qquad+\frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(t_2-t_1)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}u_{\mu,\xi}(s)\,ds\end{aligned}$$

$$
\begin{aligned}
&- \frac{(t_2 - t_1)}{\Gamma(\alpha)} \sum_{j=1}^{k+1} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} u_{\mu,\xi}(s)\, ds \\
&- \frac{(t_2 - t_1)}{\Gamma(\alpha - 1)} \sum_{j=1}^{k} (1 - t_j) \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} u_{\mu,\xi}(s)\, ds| \\
\leq{}& \left| \frac{\chi}{\Gamma(\alpha)} \int_{t_j}^{t_2} (t_2 - s)^{\alpha-1}\, ds - \frac{\chi}{\Gamma(\alpha)} \int_{t_j}^{t_1} (t_1 - s)^{\alpha-1}\, ds \right| \\
&+ \frac{k\chi}{\Gamma(\alpha)} (t_2 - t_1) + \frac{\chi(k+1)(t_2 - t_1)}{\Gamma(\alpha + 1)} + \frac{k\chi(t_2 - t_1)}{\Gamma(\alpha)}.
\end{aligned}
\tag{21}
$$

We can see that the right-hand side of inequality (21) approaches to zero as $t_1 \to t_2$. Hence

$$
\left| T_\alpha(\xi, \mu)(t_2) - T_\alpha(\xi, \mu)(t_1) \right| \to 0 \quad \text{as } t_1 \to t_2.
$$

Similarly, we can show that

$$
\left| T_\beta(\mu, \xi)(t_2) - T_\beta(\mu, \xi)(t_1) \right| \to 0 \quad \text{as } t_1 \to t_2.
$$

Therefore by the Ascoli–Arzelà theorem the operators T_α, T_β are completely continuous, and consequently T is completely continuous.

Step 4: Define the set $\mathcal{Z} = \{(\xi, \mu) \in \mathbf{B} : (\xi, \mu) = \delta T(\xi, \mu), 0 < \delta < 1\}$. We will show that $\mathcal{Z}$ is bounded. If $(\xi, \mu) \in \mathcal{Z}$, then by definition $(\xi, \mu) = \delta T(\xi, \mu)$. Hence for any $t \in \mathrm{J}$, we can write

$$
\begin{aligned}
T_\alpha(\xi, \mu) = \delta \Bigg(& tg(\xi) + (1 - t)h(\xi) + \sum_{j=1}^{k} (t - t_j) \bar{I}_j\big(\xi(t_j)\big) - \sum_{j=1}^{k} t(1 - t_j) \bar{I}_j \xi(t_j) \\
&+ \sum_{j=1}^{k} I_j\big(\xi(t_j)\big) - \sum_{j=1}^{k} t I_j \xi(t_j) + \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t} (t - s)^{\alpha-1} u_{\mu,\xi}(s)\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} u_{\mu,\xi}(s)\, ds \\
&+ \frac{1}{\Gamma(\alpha - 1)} \sum_{j=1}^{k} (t - t_j) \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} u_{\mu,\xi}(s)\, ds \\
&- \frac{t}{\Gamma(\alpha)} \sum_{j=1}^{k+1} \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-1} u_{\mu,\xi}(s)\, ds \\
&- \frac{t}{\Gamma(\alpha - 1)} \sum_{j=1}^{k} (1 - t_j) \int_{t_{j-1}}^{t_j} (t_j - s)^{\alpha-2} u_{\mu,\xi}(s)\, ds \Bigg).
\end{aligned}
\tag{22}
$$

Taking the absolute values of both sides of (22) and using $0 < \delta < 1$, we have

$$\begin{aligned}\left|T_\alpha(\xi,\mu)(t)\right| &\le |t|\left|g(\xi)\right| + |1-t|\left|h(\xi)\right| + \sum_{j=1}^{k}|t-t_j|\left|\bar{I}_j\left(\xi(t_j)\right)\right| \\ &+ \sum_{j=1}^{k}\left|t(1-t_j)\right|\left|\bar{I}_j\xi(t_j)\right| + \sum_{j=1}^{k}\left|I_j\left(\xi(t_j)\right)\right| + \sum_{j=1}^{k}|t|\left|I_j\xi(t_j)\right| \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t}(t-s)^{\alpha-1}\left|u_{\mu,\xi}(s)\right|\,ds \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\left|u_{\mu,\xi}(s)\right|\,ds \\ &+ \frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(t-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}\left|u_{\mu,\xi}(s)\right|\,ds \\ &+ \frac{|t|}{\Gamma(\alpha)}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\left|u_{\mu,\xi}(s)\right|\,ds \\ &+ \frac{|t|}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}\left|u_{\mu,\xi}(s)\right|\,ds. \end{aligned} \tag{23}$$

From inequalities (18) and (19) we have

$$\begin{aligned}\left\|T_\alpha(\xi,\mu)\right\|_{\mathrm{B}_1} &\le \mathbb{k}_5 + \mathbb{k}_6 + m(\mathbb{k}_2\varrho_1 + \mathcal{N}_2) + m(\mathbb{k}_2\varrho_1 + \mathcal{N}_2) \\ &+ m(\mathbb{k}_1\varrho_1 + \mathcal{N}_1) + m(\mathbb{k}_1\varrho_1 + \mathcal{N}_1) + \frac{\chi}{\Gamma(\alpha+1)} \\ &+ \frac{m\chi}{\Gamma(\alpha+1)} + \frac{m\chi}{\Gamma(\alpha)} + \frac{(m+1)\chi}{\Gamma(\alpha+1)} + \frac{m\chi}{\Gamma(\alpha)} =: \varsigma_1. \end{aligned} \tag{24}$$

Similarly, we can obtain

$$\left\|T_\beta(\mu,\xi)\right\|_{\mathrm{B}_2} \le \varsigma_2. \tag{25}$$

From (24) and (25) we have

$$\left\|T_\alpha(\xi,\mu)\right\|_{\mathrm{B}} \le \varsigma,$$

where $\varsigma = \max(\varsigma_1,\varsigma_2)$. Thus the set $\mathcal{S}$ is bounded, and hence, by the Schaefer fixed point Theorem, T has at least one fixed point. Consequently, the considered coupled system (1) has at least one solution. □

4 Stability analysis

Theorem 5 *If assumptions* (H_1)–(H_3) *and inequalities* (11) *are satisfied and if* $\varpi = 1 - \frac{\aleph_1\aleph_2}{(1-\aleph_1)(1-\aleph_2)} > 0$, *then the unique solution of the coupled system* (1) *is HU stable and consequently GHU stable.*

Proof Let $(\xi,\mu)\in\Lambda$ be an approximate solution of inequality (2), and let $(\vartheta,\sigma)\in\Lambda$ be the unique solution of the coupled system given by

$$\begin{cases} {}_0^C\mathrm{D}_{t_j}^{\alpha}\vartheta(t)=\Phi(t,\sigma(t),{}_0^C\mathrm{D}_{t_j}^{\alpha}\vartheta(t)), \quad t\in[0,1], t\neq t_j, j=1,2,\dots,m,\\ {}_0^C\mathrm{D}_{t_i}^{\beta}\sigma(t)=\Psi(t,\vartheta(t),{}_0^C\mathrm{D}_{t_i}^{\beta}\sigma(t)), \quad t\in[0,1], t\neq t_i, i=1,2,\dots,n,\\ \vartheta(0)=h(\vartheta), \qquad \vartheta(1)=g(\vartheta) \quad \text{and} \quad \sigma(0)=\kappa(\sigma), \qquad \sigma(1)=f(\sigma),\\ \Delta\vartheta(t_j)=I_j(\vartheta(t_j)), \qquad \Delta\vartheta'(t_j)=\bar{I}_j(\vartheta(t_j)), \quad j=1,2,\dots,m,\\ \Delta\sigma(t_i)=I_i(\sigma(t_i)), \qquad \Delta\sigma'(t_i)=\bar{I}_i(\sigma(t_i)), \quad i=1,2,\dots,n. \end{cases} \tag{26}$$

By Remark 1 we have

$$\begin{cases} {}_0^C\mathrm{D}_{t_j}^{\alpha}\xi(t)=\Phi(t,\mu(t),{}_0^C\mathrm{D}_{t_j}^{\alpha}\xi(t))+\Theta(t), \quad t\in[0,1], t\neq t_j, j=1,2,\dots,m,\\ \Delta\xi(t_j)=I_j(\xi(t_j))+\Theta_j, \qquad \Delta\xi'(t_j)=\bar{I}_j(\xi(t_j))+\Theta_j, \quad j=1,2,\dots,m,\\ {}_0^C\mathrm{D}_{t_i}^{\beta}\mu(t)=\Psi(t,\xi(t),{}_0^C\mathrm{D}_{t_i}^{\beta}\mu(t))+\theta(t), \quad t\in[0,1], t\neq t_i, i=1,2,\dots,n,\\ \Delta\mu(t_i)=I_i(\mu(t_i))+\theta_i, \qquad \Delta\mu'(t_i)=\bar{I}_i(\mu(t_i))+\theta_i, \quad i=1,2,\dots,n. \end{cases} \tag{27}$$

By Corollary 1 the solution of problem (27) is

$$\begin{cases} \xi(t)=tg(\xi)+(1-t)h(\xi)+\sum_{j=1}^{k}(t-t_j)\bar{I}_j(\xi(t_j))+\sum_{j=1}^{k}(t-t_j)\Theta_j\\ \quad -\sum_{j=1}^{k}t(1-t_j)\bar{I}_j\xi(t_j)-\sum_{j=1}^{k}t(1-t_j)\Theta_j+\sum_{j=1}^{k}I_j(\xi(t_j))\\ \quad +\sum_{j=1}^{k}\Theta_j-\sum_{j=1}^{k}tI_j\xi(t_j)-\sum_{j=1}^{k}t\Theta_j\\ \quad +\int_{t_j}^{t}\frac{(t-s)^{\alpha-1}u_{\mu,\xi}(s)}{\Gamma(\alpha)}\,ds+\int_{t_j}^{t}\frac{(t-s)^{\alpha-1}\Theta(s)}{\Gamma(\alpha)}\,ds+\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}u_{\mu,\xi}(s)}{\Gamma(\alpha)}\,ds\\ \quad +\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\Theta(s)\,ds+\frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(t-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}u_{\mu,\xi}(s)\,ds\\ \quad +\frac{1}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(t-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}\Theta(s)\,ds-\frac{t}{\Gamma(\alpha)}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}u_{\mu,\xi}(s)\,ds\\ \quad -\frac{t}{\Gamma(\alpha)}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\Theta(s)\,ds-\frac{t}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}u_{\mu,\xi}(s)\,ds\\ \quad -\frac{t}{\Gamma(\alpha-1)}\sum_{j=1}^{k}(1-t_j)\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}\Theta(s)\,ds,\\ \mu(t)=tf(\mu)+(1-t)\kappa(\mu)+\sum_{i=1}^{k}(t-t_i)\bar{I}_i(\mu(t_i))+\sum_{i=1}^{k}(t-t_i)\theta_i\\ \quad -\sum_{i=1}^{k}t(1-t_i)\bar{I}_i\mu(t_i)-\sum_{i=1}^{k}t(1-t_i)\theta_i+\sum_{i=1}^{k}I_i(\mu(t_i))\\ \quad -\sum_{i=1}^{k}tI_i\mu(t_i)+\sum_{i=1}^{k}I_i(\mu(t_i))-\sum_{i=1}^{k}t\theta_i\\ \quad +\int_{t_i}^{t}\frac{(t-s)^{\beta-1}v_{\xi,\mu}(s)}{\Gamma(\beta)}\,ds+\int_{t_i}^{t}\frac{(t-s)^{\beta-1}\theta(s)}{\Gamma(\beta)}\,ds+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{\beta-1}v_{\xi,\mu}(s)}{\Gamma(\beta)}\,ds\\ \quad +\frac{1}{\Gamma(\beta)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\theta(s)\,ds+\frac{1}{\Gamma(\beta-1)}\sum_{i=1}^{k}(t-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-2}v_{\xi,\mu}(s)\,ds\\ \quad +\frac{1}{\Gamma(\beta-1)}\sum_{i=1}^{k}(t-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-2}\theta(s)\,ds-\frac{t}{\Gamma(\beta)}\sum_{i=1}^{k+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}v_{\xi,\mu}(s)\,ds\\ \quad -\frac{t}{\Gamma(\beta)}\sum_{i=1}^{k+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-1}\theta(s)\,ds-\frac{t}{\Gamma(\beta-1)}\sum_{i=1}^{k}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-2}v_{\xi,\mu}(s)\,ds\\ \quad -\frac{t}{\Gamma(\beta-1)}\sum_{i=1}^{k}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{\beta-2}\theta(s)\,ds. \end{cases} \tag{28}$$

We consider

$$\begin{aligned}
\left|\xi(t)-\vartheta(t)\right| \le{}& |t|\left|g(\xi)-g(\vartheta)\right| + |1-t|\left|h(\xi)-h(\vartheta)\right| + \sum_{j=1}^{k}|t-t_j|\bar{I}_j\left|\xi(t_j)-\vartheta(t_j)\right| \\
&+ \sum_{j=1}^{k}|t-t_j||\Theta_j| + \sum_{j=1}^{k}|t||1-t_j|\bar{I}_j\left|\xi(t_j)-\vartheta(t_j)\right| + \sum_{j=1}^{k}|t||1-t_j||\Theta_j| \\
&+ \sum_{j=1}^{k}I_j\left|\xi(t_j)-\vartheta(t_j)\right| + \sum_{j=1}^{k}|\Theta_j| + \sum_{j=1}^{k}|t|I_j\left|\xi(t_j)-\vartheta(t_j)\right| + \sum_{j=1}^{k}|t||\Theta_j| \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t}(t-s)^{\alpha-1}\left|u_{\mu,\xi}(s)-\bar{u}_{\mu,\xi}(s)\right|ds + \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t}(t-s)^{\alpha-1}\left|\Theta(s)\right|ds \\
&+ \sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}|u_{\mu,\xi}(s)-\bar{u}_{\mu,\xi}(s)|}{\Gamma(\alpha)}ds + \sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-1}|\Theta(s)|}{\Gamma(\alpha)}ds \\
&+ \sum_{j=1}^{k}|t-t_j|\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}|u_{\mu,\xi}(s)-\bar{u}_{\mu,\xi}(s)|}{\Gamma(\alpha-1)}ds \\
&+ \sum_{j=1}^{k}|t-t_j|\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}|\Theta(s)|}{\Gamma(\alpha-1)}ds \\
&+ \frac{t}{\Gamma(\alpha)}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}\left|u_{\mu,\xi}(s)-\bar{u}_{\mu,\xi}(s)\right|ds \\
&+ \frac{t}{\Gamma(\alpha)}\sum_{j=1}^{k+1}\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-1}|\Theta|(s)\,ds \\
&+ \frac{t}{\Gamma(\alpha-1)}\sum_{j=1}^{k}|1-t_j|\int_{t_{j-1}}^{t_j}(t_j-s)^{\alpha-2}\left|u_{\mu,\xi}(s)-\bar{u}_{\mu,\xi}(s)\right|ds \\
&+ \sum_{j=1}^{k}|1-t_j|\int_{t_{j-1}}^{t_j}\frac{(t_j-s)^{\alpha-2}|\Theta(s)|}{\Gamma(\alpha-1)}ds.
\end{aligned}$$

As in Theorem 3, we get

$$\|\xi-\vartheta\|_{B_1} \le \aleph_1\left(\|\xi-\vartheta\|_{B_1} + \|\mu-\sigma\|_{B_1}\right) + 2(4m+1)\epsilon_\alpha \tag{29}$$

and

$$\|\mu-\sigma\|_{PC} \le \aleph_2\left(\|\xi-\vartheta\|_{PC} + \|\mu-\sigma\|_{PC}\right) + 2(4n+1)\epsilon_\beta. \tag{30}$$

From (29) and (30) we have

$$\|\xi-\vartheta\|_{B_1} - \frac{\aleph_1}{1-\aleph_1}\|\mu-\sigma\|_{B_1} \le \frac{2(4m+1)}{1-\aleph_1}\epsilon_\alpha$$

and

$$\|\mu - \sigma\|_{\mathrm{B}_2} - \frac{\aleph_2}{1-\aleph_2}\|\xi - \vartheta\|_{\mathrm{B}_2} \le \frac{2(4n+1)}{1-\aleph_2}\epsilon_\beta,$$

respectively. Let $\frac{2(4m+1)}{1-\aleph_1} = \mathbf{C}_\alpha$ and $\frac{2(4n+1)}{1-\aleph_2} = \mathbf{C}_\beta$. Then the last two inequalities can be written in matrix form as

$$\begin{bmatrix} 1 & -\frac{\aleph_1}{1-\aleph_1} \\ -\frac{\aleph_2}{1-\aleph_2} & 1 \end{bmatrix} \begin{bmatrix} \|\xi - \vartheta\|_{\mathrm{B}_1} \\ \|\mu - \sigma\|_{\mathrm{B}_2} \end{bmatrix} \le \begin{bmatrix} \mathbf{C}_\alpha \epsilon_\alpha \\ \mathbf{C}_\beta \epsilon_\beta \end{bmatrix},$$

which yields

$$\begin{bmatrix} \|\xi - \vartheta\|_{\mathrm{B}_1} \\ \|\mu - \sigma\|_{\mathrm{B}_2} \end{bmatrix} \le \begin{bmatrix} \frac{1}{\varpi} & \frac{\aleph_1}{\varpi(1-\aleph_1)} \\ \frac{\aleph_2}{\varpi(1-\aleph_2)} & \frac{1}{\varpi} \end{bmatrix} \begin{bmatrix} \mathbf{C}_\alpha \epsilon_\alpha \\ \mathbf{C}_\beta \epsilon_\beta \end{bmatrix}, \tag{31}$$

where

$$\varpi = 1 - \frac{\aleph_1\aleph_2}{(1-\aleph_1)(1-\aleph_2)} > 0.$$

From system (31) we have

$$\|\xi - \vartheta\|_{\mathrm{B}_1} \le \frac{\mathbf{C}_\alpha \epsilon_\alpha}{\varpi} + \frac{\aleph_1 \mathbf{C}_\beta \epsilon_\beta}{\varpi(1-\aleph_1)},$$
$$\|\mu - \sigma\|_{\mathrm{B}_2} \le \frac{\mathbf{C}_\beta \epsilon_\beta}{\varpi} + \frac{\aleph_2 \mathbf{C}_\alpha \epsilon_\alpha}{\varpi(1-\aleph_2)},$$

which imply that

$$\|\xi - \vartheta\|_{\mathrm{B}_1} + \|\mu - \sigma\|_{\mathrm{B}_2} \le \frac{\mathbf{C}_\alpha \epsilon_\alpha}{\varpi} + \frac{\mathbf{C}_\beta \epsilon_\beta}{\varpi} + \frac{\aleph_1 \mathbf{C}_\beta \epsilon_\beta}{\varpi(1-\aleph_1)} + \frac{\aleph_2 \mathbf{C}_\alpha \epsilon_\alpha}{\varpi(1-\aleph_2)}.$$

If $\max\{\epsilon_\alpha, \epsilon_\beta\} = \epsilon$ and $\frac{\mathbf{C}_\alpha}{\varpi} + \frac{\mathbf{C}_\beta}{\varpi} + \frac{\aleph_1 \mathbf{C}_\beta}{\varpi(1-\aleph_1)} + \frac{\aleph_2 \mathbf{C}_\alpha}{\varpi(1-\aleph_2)} = \mathbf{C}_{\alpha,\beta}$, then

$$\big\|(\xi,\mu) - (\vartheta,\sigma)\big\|_{\mathrm{B}} \le \mathbf{C}_{\alpha,\beta}\epsilon.$$

This shows that system (1) is HU stable. Also, if

$$\big\|(\xi,\mu) - (\vartheta,\sigma)\big\|_{\mathrm{B}} \le \mathbf{C}_{\alpha,\beta}\varphi(\epsilon)$$

with $\varphi(0) = 0$, then the solution of system (1) is GHU stable. □

For the next result, we assume that

(H_7) There exist two nondecreasing functions $\gamma_\alpha, \gamma_\beta \in C(\mathrm{J}, \mathrm{R}^+)$ such that

$${}_0\mathrm{I}_t^\alpha \gamma_\alpha(t) \le \mathcal{L}_1 \gamma_\alpha(t) \quad \text{and} \quad {}_0\mathrm{I}_t^\beta \gamma_\beta(t) \le \mathcal{L}_2 \gamma_\beta(t), \quad \text{where } \mathcal{L}_1, \mathcal{L}_2 > 0.$$

Theorem 6 *If assumptions (H_1)–(H_3) and (H_7) and inequalities (11) are satisfied and if $\varpi = 1 - \frac{\aleph_1\aleph_2}{(1-\aleph_1)(1-\aleph_2)} > 0$, then the unique solution of the coupled system (1) is HU-Rassias stable, and consequently it is GHU-Rassias stable.*

Proof We can obtain the result by using Definition 5 and performing the same procedure as in Theorem 5. □

5 Example

To testify our results established in the previous section, we provide an adequate problem.

Example 1

$$\begin{cases} {}^C\mathrm{D}^{\frac{3}{2}}\xi(t) = \frac{|\mu(t)|}{40(t+3)(1+|\mu(t)|)} + \frac{\cos|{}^C\mathrm{D}^{\frac{3}{2}}\xi(t)|}{40+t^2}, \quad t \in \mathrm{J}, t \neq \frac{1}{4}, \\ {}^C\mathrm{D}^{\frac{3}{2}}\mu(t) = \frac{1}{30}(t\cos\xi(t) - \xi(t)\sin(t)) + \frac{|{}^C\mathrm{D}^{\frac{3}{2}}\mu(t)|}{30+|{}^C\mathrm{D}^{\frac{3}{2}}\mu(t)|}, \quad t \in \mathrm{J}, t \neq \frac{1}{5}, \\ \xi(0) = g(\xi) = \sum_{j=1}^{50} \frac{\xi(u_j)}{u_j^2+75}, \qquad \xi(1) = h(\xi) = \sum_{j=1}^{50} \frac{\xi(v_j)}{v_j+25}, \\ \mu(0) = f(\mu) = \sum_{j=1}^{60} \frac{\mu(u_j)}{u_j^4+90}, \qquad \mu(1) = \kappa(\mu) = \sum_{j=1}^{60} \frac{\mu(v_j)}{3v_j+45}, \\ \Delta\xi(\frac{1}{4}) = I\xi(\frac{1}{4}) = \frac{1}{60+|\xi|}, \qquad \Delta\xi'(\frac{1}{4}) = \bar{I}\xi(\frac{1}{4}) = \frac{1}{120+|\xi|}, \\ \Delta\mu(\frac{1}{5}) = I\mu(\frac{1}{4}) = \frac{1}{40+|\mu|}, \qquad \Delta\mu'(\frac{1}{5}) = \bar{I}\mu(\frac{1}{4}) = \frac{1}{80+|\mu|}. \end{cases} \tag{32}$$

In system (32), we see that $\alpha = \beta = \frac{3}{2}$, and $t_j \neq \frac{1}{4}$ for $j = 1, 2, \ldots, 50$. For $t \in [0,1]$ and $\xi, \bar{\xi}, \mu, \bar{\mu} \in \mathrm{R}$, we obtain

$$\left|\Phi(t,\xi,\mu) - \Phi(t,\bar{\xi},\bar{\mu})\right| \leq \frac{1}{40}\left[|\xi - \bar{\xi}| + |\mu - \bar{\mu}|\right]$$

and

$$\left|\Psi(t,\xi,\mu) - \Psi(t,\bar{\xi},\bar{\mu})\right| \leq \frac{1}{30}\left[|\xi - \bar{\xi}| + |\mu - \bar{\mu}|\right].$$

From this we get $L_{\Phi 1} = L_{\Phi 2} = \frac{1}{40}$ and $L_{\Psi 1} = L_{\Psi 2} = \frac{1}{30}$. Also,

$$\begin{aligned} &\left|g(\xi) - g(\bar{\xi})\right| \leq \frac{1}{75}|\xi - \bar{\xi}|, \qquad \left|h(\xi) - h(\bar{\xi})\right| \leq \frac{1}{25}|\xi - \bar{\xi}|, \\ &\left|f(\mu) - f(\bar{\mu})\right| \leq \frac{1}{90}|\mu - \bar{\mu}|, \qquad \|\kappa(\mu) - \kappa(\bar{\mu})| \leq \frac{1}{45}|\mu - \bar{\mu}|, \\ &\left|I\xi(t_j) - I\bar{\xi}(t_j)\right| \leq \frac{1}{60}|\xi - \bar{\xi}|, \qquad \left|\bar{I}\xi(t_j) - \bar{I}\bar{\xi}(t_j)\right| \leq \frac{1}{120}|\xi - \bar{\xi}|, \\ &\left|I\mu(t_i) - I\bar{\mu}(t_i)\right| \leq \frac{1}{40}|\mu - \bar{\mu}|, \qquad \left|\bar{I}\mu(t_i) - \bar{I}\mu(t_i)\right| \leq \frac{1}{80}|\mu - \bar{\mu}|. \end{aligned}$$

From this we obtain that $K_g = \frac{1}{75}$, $K_h = \frac{1}{25}$, $K_f = \frac{1}{90}$, $K_\kappa = \frac{1}{45}$, $A_1 = \frac{1}{60}$, $A_2 = \frac{1}{120}$, $A_3 = \frac{1}{40}$, $A_4 = \frac{1}{80}$, and $m = 1$. Calculating

$$\aleph_1 = \left[K_g + K_h + 2m(A_1 + A_2) + \frac{2L_{\Phi_1}}{1 - L_{\Phi_2}}\left(\frac{1+m}{\Gamma(\alpha+1)} + \frac{m}{\Gamma(\alpha)}\right)\right]$$

and

$$\aleph_2 = \left[K_f + K_\kappa + 2n(A_3 + A_4) + \frac{2L_{\Psi_1}}{1 - L_{\Psi_2}}\left(\frac{1+n}{\Gamma(\beta+1)} + \frac{n}{\Gamma(\beta)}\right)\right],$$

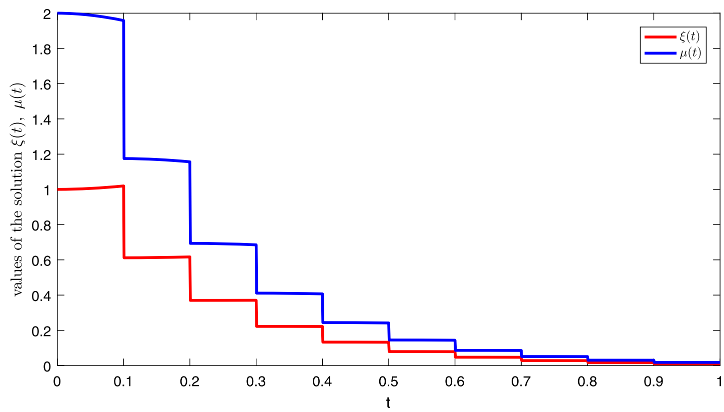

Figure 1 Graphical representation of HU-stability results for Example 1

we have $\aleph_1 = 0.407 < 1$ and $\aleph_2 = 0.467 < 1$, that is, $\max(\aleph_1, \aleph_2) < 1$. Therefore by Theorem 3 the coupled system (32) has a unique solution. Also, $\varpi = 1 - \frac{\aleph_1\aleph_2}{(1-\aleph_1)(1-\aleph_2)} = 0.8096104 > 0$, and hence by Theorem 5 the coupled system (32) is HU stable and thus GHU stable. Similarly, we can verify the conditions of Theorems 6 and 4. Next, we take the initial values for the required solution $\xi = 1$, $\mu = 2$, and at the given fractional order the stability graph is given in Fig. 1 corresponding to the parametric values computed.

6 Conclusion

We successfully applied the Schaefer and Banach fixed point theorems to develop sufficient conditions for the existence of at least one solution and its uniqueness, respectively. Then we obtained some results for different kinds of HU stability. The whole analysis was demonstrated by an example.

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Abbreviations
IBVP, implicit boundary value problem; FODEs, fractional-order differential equations; IFODEs, implicit fractional-order differential equations; HU, Hyers Ulam; GHU, generalized Hyers Ulam; GHUR, generalized Hyers Ulam Rassias.

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The authors declare that they have no competing interests.

Authors' contributions
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