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Common Fixed Point Theorem via Cyclic $(\alpha, \beta) - (\psi, \varphi)_S$-Contraction with Applications

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Abstract: In this paper, we introduce the notion of cyclic $(\alpha, \beta) - (\psi, \varphi)_s$-rational-type contraction in b-metric spaces, and using this contraction, we prove common fixed point theorems. Our work generalizes many existing results in the literature. In order to highlight the usefulness of our results, applications to functional equations are given.

Keywords: functional equations; common fixed points; b-metric spaces; cyclic-(α, β)-admissible mapping; b-(CLR) property

MSC: 47H09; 54H25

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1. Introduction

Throughout this work, $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and the set of real numbers, respectively. Furthermore, opt indicates inf or sup; $\mathbb{B}_1$ and $\mathbb{B}_2$ are Banach spaces; $\mathcal{D} \subseteq \mathbb{B}_1$ is the decision space; $\mathcal{S} \subseteq \mathbb{B}_2$ is the state space; $\mathbb{B}d(\mathcal{S})$ stands for the Banach space of all bounded real-valued functions on $\mathcal{S}$ with sup b-metric defined by:

$$d(x, y) = \sup_{t \in \mathcal{S}} |x(t) - y(t)|^p,$$

for all $x, y \in \mathbb{B}d(\mathcal{S})$ with coefficient $s = 2^{p-1}$ and with norm defined by:

$$\|f\| = \sup\{|f(t)| : t \in \mathcal{S}\},$$

where $f \in \mathbb{B}d(\mathcal{S})$.

The Banach contraction principle [1] is one of the most important results in functional analysis. It is the most widely-applied fixed point result in many branches of mathematics, and it was generalized in different directions.

Bakhtin [2] and Czerwik [3] generalized the metric space with non-Hausdorff topology called the b-metric space to overcome the problem of measurable functions with respect to the measure and their convergence. They proved the Banach contraction principle in b-metric spaces. Afterwards, several papers were published by many authors dealing with the existence of a fixed point in b-metric spaces (see [4–17]).

The contractive conditions on underlying functions play an important role in fixed point theorems. Over the years, different contractive conditions were established by several mathematicians. One of the interesting contractive conditions was given by Samet et al. [18] by introducing the notions of α-admissible and $\alpha - \psi$-contractive-type mappings. They established various fixed point theorems for such mappings in complete metric spaces. Furthermore, several authors considered the generalizations of this new approach (see [19–24]). Isik et al. [19] proved fixed point theorems under the T-cyclic-(α, β)-contractive condition in metric space. Recently, Yamaod and Sintunavarat [7] proposed the notion of $(\alpha, \beta) - (\psi, \varphi)$-contraction in b-metric spaces and proved fixed point theorems for this class of contraction.

On the other hand, the existence of unique solutions to functional equations has been examined using various fixed point results (see [25,26] and the references therein). In particular, Isik et al. [19] and Latif et al. [20] studied the existence of a unique bounded common solution to the following system of functional equations:

$$\begin{Bmatrix} f(x) = \sup\limits_{y \in \mathcal{D}} \{\tau_1(x,y) + H_1(x,y,f(a_1(x,y)))\} \ \forall \ x \in \mathcal{S}, \\ g(x) = \sup\limits_{y \in \mathcal{D}} \{\tau_2(x,y) + H_2(x,y,g(a_2(x,y)))\} \ \forall \ x \in \mathcal{S}, \end{Bmatrix} \tag{1}$$

where x is the state vector, y is the decision vector, and $f(x)$ and $g(x)$ denote the optimal profit functions with the opening state x and transformations of the process a_1, a_2. Moreover, $\tau_1, \tau_2 : \mathcal{S} \times \mathcal{D} \to \mathbb{R}$, $a_i : \mathcal{S} \times \mathcal{D} \to \mathcal{S}$, $H_1, H_2 : \mathcal{S} \times \mathcal{D} \times \mathbb{R} \to \mathbb{R}$.

Motivated by the work in [7,19], we present the notion of cyclic $(\alpha, \beta) - (\psi, \varphi)_s$-rational-type contraction in b-metric space, and using this notion we study common fixed point theorems, which generalize many recent results. As an application of our work, we study the existence of a unique bounded common solution to the system of functional equations that arise in dynamic programming, mathematical optimization, and in computer programming.

2. Preliminaries

In this section, we recall some basic notions and results.

Definition 1. *A function $\theta : [0, \infty) \to [0, \infty)$ is called an altering distance function if it satisfies the following conditions:*

1. *θ is continuous and nondecreasing;*
2. *$\theta(t) = 0$ if and only if $t = 0$.*

Definition 2 ([19]). *Let X be a nonempty set and $\alpha, \beta : X \to [0, \infty)$. If $f, g : X \to X$, then the mapping f is g-cyclic-(α, β)-admissible if:*

1. $\alpha(gt) \geq 1 \Longrightarrow \beta(ft) \geq 1$ *for some $t \in X$;*
2. $\beta(gt) \geq 1 \Longrightarrow \alpha(ft) \geq 1$ *for some $t \in X$.*

Definition 3 ([2,3]). *Let X be a nonempty set and $s \geq 1$ be a fixed real number. Then, the function $d : X \times X \to [0, \infty)$ is a b-metric if for all $x, y, z \in X$:*

1. $d(x,y) = 0 \Leftrightarrow x = y$
2. $d(x,y) = d(y,x)$;
3. $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X, d) is called a b-metric space.

Remark 1. *Every metric space is a b-metric space, but the converse is not true in general (see [4]). Thus, b-metric spaces are superior to ordinary metric spaces.*

Example 1. *Let (M, ρ) be a metric space and $p \in \mathbb{R}$ with $p \geq 1$. Then, $d(u,v) = [\rho(u,v)]^p$ is a b-metric with parameter $s = 2^{p-1}$.*

Definition 4 ([17]). *A sequence $\{\alpha_n\}$ in a b-metric space X is:*

(a) *b-convergent if there exists $\alpha \in X$ such that $d(\alpha_n, \alpha) \to 0$ as $n \to \infty$.*
(b) *b-Cauchy if $d(\alpha_n, \alpha_m) \to 0$ as $n, m \to \infty$.*

It is well known that, in b-metric spaces, every b-convergent sequence is a b-Cauchy sequence. Moreover, a b-metric is not continuous in general. Thus, to establish fixed point theorems, one needs the following Lemma.

Lemma 1 ([27]). *Let $\{u_n\}$ and $\{v_n\}$ be two sequences in a b-metric space (X, d) with coefficient $s \geq 1$ such that $u_n \to u \in X$ and $v_n \to v \in X$. Then:*

$$\frac{1}{s^2} d(u, v) \leq \liminf_{n\to\infty} d(u_n, v_n) \leq \limsup_{n\to\infty} d(u_n, v_n) \leq s^2 d(u, v),$$

and $\lim_{n\to\infty} d(u_n, v_n) = 0$ if $u = v$. Furthermore, for every $w \in X$, we have:

$$\frac{1}{s} d(u, w) \leq \liminf_{n\to\infty} d(u_n, w) \leq \limsup_{n\to\infty} d(u_n, w) \leq s d(u, w).$$

To study common fixed point theorems, Jungck [28] launched the idea of weakly-compatible mappings as: two self-maps are weakly compatible if they commute at their coincidence points.

Proposition 1 ([29]). *Two weakly-compatible self-maps have a unique common fixed point if they have a unique point of coincidence.*

Lemma 2 ([30]). *Let A be a nonempty set and $f, g : A \to \mathbb{R}$ be two mappings such that $\underset{t\in A}{opt}\, f(t)$ and $\underset{t\in A}{opt}\, g(t)$ are bounded, then:*

$$\left| \underset{t\in A}{opt}\, f(t) - \underset{t\in A}{opt}\, g(t) \right| \leq \sup_{t\in A} |f(t) - g(t)|.$$

3. Results

In this section, we present our main results. First, we introduce the concept of cyclic-$(\alpha, \beta) - (\psi, \varphi)_s$-rational contraction in b-metric space as follows.

Definition 5. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $\alpha, \beta : X \to [0, \infty)$ be two mappings. Let f and g be two self-mappings defined on X such that f is a g-cyclic-(α, β)-admissible mapping. Then, f is a g-cyclic-$(\alpha, \beta) - (\psi, \varphi)_s$-rational contraction if for all $u, v \in X$,*

$$\alpha(gu)\beta(gv) \geq 1 \implies \psi(s^3 d(fu, fv)) \leq \psi(\Delta_s(u, v)) - \varphi(\Delta_s(u, v)), \tag{2}$$

where:

$$\Delta_s(u, v) = \max\left(d(gu, gv), \frac{1}{s} d(gv, fu), \frac{d(fu, gv)d(gu, fv)}{2s^3[1 + d(gu, gv)]}, \frac{d(fu, gu)d(fv, gu)}{2s[1 + d(gu, gv)]}, \frac{d(fv, gv)d(fu, gv)}{2s[1 + d(gu, gv)]} \right),$$

and ψ, φ are altering distance functions.

Now, we present our main result.

Theorem 1. *Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$ and $\alpha, \beta : X \to [0, \infty)$ be two mappings. If f and g are two self-mappings defined on X such that f is a g-cyclic-$(\alpha, \beta) - (\psi, \varphi)$-rational contraction satisfying the following conditions:*

- (i) *$fX \subseteq gX$ with gX are closed subspaces of X;*
- (ii) *there exists $u_0 \in X$ with $\alpha(gu_0) \geq 1$ and $\beta(gu_0) \geq 1$;*
- (iii) *if $\{u_n\}$ is a sequence in X with $\beta(u_n) \geq 1$ for all n and $u_n \to u$, then $\beta(u) \geq 1$;*
- (iv) *$\alpha(ga) \geq 1$ and $\beta(gb) \geq 1$ whenever $fa = ga$ and $fb = gb$.*

Then, f and g have a unique point of coincidence in X. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let $u_0 \in X$; then, using Conditions (i) and (ii), we can construct two sequences $\{u_n\}$ and $\{v_n\}$ in X such that:

$$v_n = fu_n = gu_{n+1}, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3}$$

If $v_{\widehat{n}} = v_{\widehat{n}+1}$, then $v_{\widehat{n}+1}$ is a point of coincidence of g and f. Therefore, we assume that $v_n \neq v_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\alpha(gu_0) \geq 1$ and f is a g-cyclic-$(\alpha, \beta) - (\psi, \varphi)$-admissible mapping, we have:

$$\beta(gu_1) = \beta(fu_0) \geq 1 \Longrightarrow \alpha(gu_2) = \alpha(fu_1) \geq 1,$$

and:

$$\beta(gu_3) = \beta(fu_2) \geq 1 \Longrightarrow \alpha(gu_4) = \alpha(fu_3) \geq 1.$$

By continuing this procedure, we obtain that:

$$\alpha(gu_{2j}) \geq 1 \text{ and } \beta(gu_{2j+1}) \geq 1 \text{ for all } j \in \mathbb{N} \cup \{0\}. \tag{4}$$

Similarly, since $\beta(gu_0) \geq 1$ and f is g-cyclic-$(\alpha, \beta) - (\psi, \varphi)$-admissible, we have:

$$\alpha(gu_1) = \alpha(fu_0) \geq 1 \Longrightarrow \beta(gu_2) = \beta(fu_1) \geq 1,$$

and:

$$\alpha(gu_3) = \alpha(fu_2) \geq 1 \Longrightarrow \beta(gu_4) = \beta(fu_3) \geq 1.$$

By continuing this procedure, we get:

$$\beta(gu_{2j}) \geq 1 \text{ and } \alpha(gu_{2j+1}) \geq 1 \text{ for all } j \in \mathbb{N} \cup \{0\}. \tag{5}$$

From (4) and (5), it follows that:

$$\alpha(gu_n) \geq 1 \text{ and } \beta(gu_n) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Consequently,

$$\alpha(gu_n) \geq 1 \text{ and } \beta(gu_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\},$$

which implies that:

$$\alpha(gu_n)\beta(gu_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{6}$$

Using (2) and (3), we have:

$$\psi(s^3 d(fu_n, fu_{n+1})) \leq \psi(\Delta_s(u_n, u_{n+1})) - \varphi(\Delta_s(u_n, u_{n+1})), \tag{7}$$

where:

$$\begin{aligned}\Delta_s(u_n,u_{n+1}) &= \max\left(d(gu_n,gu_{n+1}),\frac{1}{s}d(gu_{n+1},fu_n),\frac{d(fu_n,gu_{n+1})d(gu_n,fu_{n+1})}{2s^3[1+d(gu_n,gu_{n+1})]},\right.\\&\qquad\left.\frac{d(fu_n,gu_n)d(fu_{n+1},gu_n)}{2s^3[1+d(gu_n,gu_{n+1})]},\frac{d(fu_{n+1},gu_{n+1})d(fu_n,gu_{n+1})}{2s[1+d(gu_n,gu_{n+1})]}\right)\\&= \max\left(d(v_{n-1},v_n),\frac{1}{s}d(v_n,v_n),\frac{d(v_n,v_n)d(v_{n-1},v_{n+1})}{2s^3[1+d(v_{n-1},v_n)]},\right.\\&\qquad\left.\frac{d(v_n,v_{n-1})d(v_{n+1},v_{n-1})}{2s[1+d(v_{n-1},v_n)]},\frac{d(v_{n+1},v_n)d(v_n,v_n)}{2s[1+d(v_{n-1},v_n)]}\right).\\&= \max\left(d(v_{n-1},v_n),\frac{s[d(v_{n+1},v_n)+d(v_n,v_{n-1})]}{2s}\right)\\&= \max\left(d(v_{n-1},v_n),d(v_n,v_{n+1})\right).\end{aligned}$$

If $\Delta_s(u_i,u_{i+1})=d(v_i,v_{i+1})$, for some $i\in\mathbb{N}\cup\{0\}$, then from inequality (7), we have:

$$\begin{aligned}\psi(d(v_i,v_{i+1})) &= \psi(d(fu_i,fu_{i+1}))\\&\le \psi(s^3d(fu_i,fu_{i+1}))\\&\le \psi(d(v_i,v_{i+1}))-\varphi(d(v_i,v_{i+1}))\\&< \psi(d(v_i,v_{i+1})),\end{aligned}$$

which is a contradiction. Thus,

$$\Delta_s(u_n,u_{n+1})=d(v_{n-1},v_n),\ \text{for all}\ \ n\in\mathbb{N}\cup\{0\},$$

and hence, from (7), we can write:

$$\begin{aligned}\psi(d(v_n,v_{n+1})) &= \psi(d(fu_n,fu_{n+1}))\\&\le \psi(s^3d(fu_n,fu_{n+1}))\\&\le \psi(d(v_{n-1},v_n))-\varphi(d(v_{n-1},v_n))\\&< \psi(d(v_{n-1},v_n)),\end{aligned}\tag{8}$$

for all $n\in\mathbb{N}\cup\{0\}$. That is:

$$\psi(d(v_n,v_{n+1}))<\psi(d(v_{n-1},v_n)),\tag{9}$$

but ψ is non-decreasing, so that:

$$d(v_n,v_{n+1})<d(v_{n-1},v_n),\ \text{for all}\ \ n\in\mathbb{N}\cup\{0\}.$$

Thus, the sequence $\{d(v_{n+1},v_n)\}$ is decreasing bounded below in X, and hence, there exists some $r\ge 0$ such that:

$$\lim_{n\to\infty}d(v_n,v_{n+1})=r.$$

Taking the limit as $n\to\infty$ in (8), we get:

$$\psi(r)\le\psi(r)-\varphi(r)\le\psi(r).$$

This implies that $\psi(r) = 0$, and thus $r = 0$. Hence, we have:

$$\lim_{n\to\infty} d(v_n, v_{n+1}) = 0. \tag{10}$$

Next, we will show that $\{v_n\}$ is a b-Cauchy sequence. Let on contrary $\{v_n\}$ not be a b-Cauchy sequence, then for some $\epsilon > 0$, there exists two subsequences $\{v_{m(\jmath)}\}$ and $\{v_{n(\jmath)}\}$ of $\{v_n\}$ such that:

$$d(v_{m(\jmath)}, v_{n(\jmath)}) \geq \epsilon, \tag{11}$$

where $n(\jmath) > m(\jmath) \geq \jmath$ with $n(\jmath)$ is odd and $m(\jmath)$ is even. Corresponding to $m(\jmath)$, one can choose the smallest number $n(\jmath)$ with $n(\jmath) > m(\jmath) \geq \jmath$ such that:

$$d(v_{m(\jmath)}, v_{n(\jmath)-1}) < \epsilon. \tag{12}$$

Using Inequalities (11) and (12) and the triangle inequity, we have:

$$\begin{aligned} \epsilon \leq & d(v_{m(\jmath)}, v_{n(\jmath)}) \\ \leq & s\left[d(v_{m(\jmath)}, v_{n(\jmath)-1}) + d(v_{n(\jmath)-1}, v_{n(\jmath)})\right] \\ < & s\left[\epsilon + d(v_{n(\jmath)-1}, v_{n(\jmath)})\right]. \end{aligned} \tag{13}$$

From above and (10), we get:

$$\epsilon \leq \limsup_{\jmath\to\infty} d(v_{m(\jmath)}, v_{n(\jmath)}) < s\epsilon. \tag{14}$$

It follows from the triangle inequity that:

$$d(v_{m(\jmath)}, v_{n(\jmath)}) \leq s\left[d(v_{m(\jmath)}, v_{n(\jmath)+1}) + d(v_{n(\jmath)+1}, v_{n(\jmath)})\right], \tag{15}$$

and:

$$d(v_{m(\jmath)}, v_{n(\jmath)+1}) \leq s\left[d(v_{m(\jmath)}, v_{n(\jmath)}) + d(v_{n(\jmath)}, v_{n(\jmath)+1})\right]. \tag{16}$$

Taking the limit supremum as $\jmath \to \infty$ in (15), (16) and using (10), (14), we get:

$$\epsilon \leq s \limsup_{\jmath\to\infty} d(v_{m(\jmath)}, v_{n(\jmath)+1}),$$

and:

$$\limsup_{\jmath\to\infty} d(v_{m(\jmath)}, v_{n(\jmath)+1}) \leq s^2\epsilon.$$

From here, we can write:

$$\frac{\epsilon}{s} \leq \limsup_{\jmath\to\infty} d(v_{m(\jmath)}, v_{n(\jmath)+1}) \leq s^2\epsilon. \tag{17}$$

Similarly, we can show that:

$$\frac{\epsilon}{s} \leq \limsup_{\jmath\to\infty} d(v_{n(\jmath)}, v_{m(\jmath)+1}) \leq s^2\epsilon. \tag{18}$$

Again using the triangular inequality, we get:

$$d(v_{m(\jmath)}, v_{n(\jmath)+1}) \leq s\left[d(v_{m(\jmath)}, v_{m(\jmath)+1}) + d(v_{m(\jmath)+1}, v_{n(\jmath)+1})\right], \tag{19}$$

and:

$$d(v_{m(\jmath)+1}, v_{n(\jmath)+1}) \leq s\left[d(v_{m(\jmath)+1}, v_{m(\jmath)}) + d(v_{m(\jmath)}, v_{n(\jmath)+1})\right]. \tag{20}$$

Applying the limit supremum as $\jmath \to \infty$ in (19), (20) and using (10), (17), we get:

$$\frac{\epsilon}{s} \leq s \limsup_{\jmath\to\infty} d(v_{m(\jmath)+1}, v_{n(\jmath)+1}),$$

and:

$$\limsup_{\jmath\to\infty} d(v_{m(\jmath)+1}, v_{n(\jmath)+1}) \leq s^3\epsilon.$$

This implies that:

$$\frac{\epsilon}{s^2} \leq \limsup_{\jmath\to\infty} d(v_{m(\jmath)+1}, v_{n(\jmath)+1}) \leq s^3\epsilon. \tag{21}$$

From (6), we obtain $\alpha(gu_{m(\jmath)})\beta(gu_{n(\jmath)}) \geq 1$, and from (2), we can write:

$$\begin{aligned}\psi(s^3 d(v_{m(\jmath)+1}, v_{n(\jmath)+1})) =& \psi(s^3 d(fu_{m(\jmath)+1}, fu_{n(\jmath)+1}))\\ \leq& \psi(\Delta_s(u_{m(\jmath)+1}, u_{n(\jmath)+1})) - \varphi(\Delta_s(u_{m(\jmath)+1}, u_{n(\jmath)+1})),\end{aligned} \tag{22}$$

where:

$$\begin{aligned}\Delta_s(u_{m(\jmath)+1}, u_{n(\jmath)+1}) =& \max\Bigg(d(gu_{m(\jmath)+1}, gu_{n(\jmath)+1}), \frac{1}{s}d(gu_{n(\jmath)+1}, fu_{m(\jmath)+1}),\\ &\frac{d(fu_{m(\jmath)+1}, gu_{n(\jmath)+1})d(gu_{m(\jmath)+1}, fu_{n(\jmath)+1})}{2s^3[1 + d(gu_{m(\jmath)+1}, gu_{n(\jmath)+1})]},\\ &\frac{d(fu_{m(\jmath)+1}, gu_{m(\jmath)+1})d(fu_{n(\jmath)+1}, gu_{m(\jmath)+1})}{2s[1 + d(gu_{m(\jmath)+1}, gu_{n(\jmath)+1})]},\\ &\frac{d(fu_{n(\jmath)+1}, gu_{n(\jmath)+1})d(fu_{m(\jmath)+1}, gu_{n(\jmath)+1})}{2s[1 + d(gu_{m(\jmath)+1}, gu_{n(\jmath)+1})]}\Bigg)\\ =& \max\Bigg(d(v_{m(\jmath)}, v_{n(\jmath)}), \frac{1}{s}d(v_{n(\jmath)}, v_{m(\jmath)+1}),\\ &\frac{d(v_{m(\jmath)+1}, v_{n(\jmath)})d(v_{m(\jmath)}, v_{n(\jmath)+1})}{2s^3[1 + d(v_{m(\jmath)}, v_{n(\jmath)})]},\\ &\frac{d(v_{m(\jmath)+1}, v_{m(\jmath)})d(v_{n(\jmath)+1}, v_{m(\jmath)})}{2s[1 + d(v_{m(\jmath)}, v_{n(\jmath)})]},\\ &\frac{d(v_{n(\jmath)+1}, v_{n(\jmath)})d(v_{m(\jmath)+1}, v_{n(\jmath)})}{2s[1 + d(v_{m(\jmath)}, v_{n(\jmath)})]}\Bigg).\end{aligned}$$

Taking the limit supremum as $\jmath \to \infty$ in the above and using (10), (14), (17), and (18), we have:

$$\epsilon = \max\left\{\epsilon, \frac{\epsilon}{s^2}, \frac{\epsilon^2}{2s^5[1 + s\epsilon]}\right\} \leq \limsup_{\jmath\to\infty} \Delta_s(u_{m(\jmath)+1}, u_{n(\jmath)+1}) \leq \max\left\{s\epsilon, s\epsilon, \frac{s\epsilon^2}{2[1+\epsilon]}\right\} = s\epsilon, \tag{23}$$

and:

$$\epsilon = \max\left\{\epsilon, \frac{\epsilon}{s^2}, \frac{\epsilon^2}{2s^5[1 + s\epsilon]}\right\} \leq \liminf_{\jmath\to\infty} \Delta_s(u_{m(\jmath)+1}, u_{n(\jmath)+1}) \leq \max\left\{s\epsilon, s\epsilon, \frac{s\epsilon^2}{2[1+\epsilon]}\right\} = s\epsilon. \tag{24}$$

By applying the limit supremum as $\jmath \to \infty$ in (22) and using (21), (23), and (24), we have:

$$\begin{aligned}\psi(s\epsilon) = \psi\left(s^3\left(\frac{\epsilon}{s^2}\right)\right) &\le \psi\left(s^3 \limsup_{\jmath\to\infty} d(v_{m(\jmath)+1}, v_{n(\jmath)+1})\right) \\ &\le \psi\left(\limsup_{\jmath\to\infty} \Delta_s(u_{m(\jmath)+1}, u_{n(\jmath)+1})\right) - \varphi\left(\liminf_{\jmath\to\infty} \Delta_s(u_{m(\jmath)+1}, u_{n(\jmath)+1})\right) \\ &\le \psi(s\epsilon) - \varphi(\epsilon),\end{aligned} \tag{25}$$

which is possible only if $\varphi(\epsilon) = 0$. This implies that $\epsilon = 0$, which contradicts that $\epsilon > 0$. Thus, $\{v_n\}$ is a b-Cauchy sequence in X. However, X is $b-$complete, so there exists $a_0 \in X$ such that $\lim_{n\to\infty} v_n = a_0$, and hence, from (3), we get:

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} gu_{n+1} = a_0. \tag{26}$$

Since gX is closed, so in view of (26), $a_0 \in gX$, and therefore, one can find $a \in X$ such that $ga = a_0$.

Now, we will show that $fa = a_0$. For this, since $v_n \to a_0$, so from (3), it follows that:

$$\beta(v_n) = \beta(gu_{n+1}) \ge 1,$$

for all $n \in \mathbb{N}$. From Condition (iii), we have $\beta(a_0) = \beta(ga) \ge 1$, and thus, by (3), $\alpha(gu_n)\beta(ga) \ge 1$ for all $n \in \mathbb{N}$. In view of (2) with $u = u_n$ and $v = a$, we have:

$$\psi(s^3 d(fu_n, fa)) \le \psi(\Delta_s(u_n, a)) - \varphi(\Delta_s(u_n, a)), \tag{27}$$

where:

$$\begin{aligned}\Delta_s(u_n, a) = \max\Bigg(& d(gu_n, ga), \frac{1}{s} d(ga, fu_n), \frac{d(fu_n, ga)d(gu_n, fa)}{2s^3[1 + d(gu_n, ga)]}, \\ & \frac{d(fu_n, gu_n)d(fa, gu_n)}{2s[1 + d(gu_n, ga)]}, \frac{d(fa, ga)d(fu_n, ga)}{2s[1 + d(gu_n, ga)]}\Bigg).\end{aligned}$$

Taking the limit supremum as $n \to \infty$ in the above and using (26) and Lemma 1, we get:

$$\begin{aligned}\limsup_{n\to\infty} \Delta_s(u_n, a) \le \max\Bigg(& sd(a_0, a_0), d(a_0, a_0), \frac{d(a_0, a_0)d(a_0, fa)}{2s[1 + d(a_0, a_0)]}, \\ & \frac{s^2 d(a_0, a_0)d(fa, a_0)}{2[1 + d(a_0, a_0)]}, \frac{d(fa, a_0)d(a_0, a_0)}{2[1 + d(a_0, a_0)]}\Bigg) \\ =& 0.\end{aligned}$$

Now, taking the limit supremum as $\jmath \to \infty$ in (27) and using the above inequality, we get:

$$\psi(s^3 d(a_0, fa)) \le \psi(0) - \varphi(0) = 0.$$

Therefore,

$$\psi(d(a_0, fa)) \le \psi(s^3 d(a_0, fa)) \le 0,$$

which is possible only if $\psi(d(a_0, fa)) = 0$. Thus, $d(a_0, fa) = 0 \Rightarrow fa = a_0$, and hence:

$$fa = ga = a_0. \tag{28}$$

Next, to show that f and g have a unique point of coincidence a_0, let f and g have another point of coincidence $a_0^* \neq a_0$. Then, there exists $b \in X$ so that:

$$fb = gb = a_0^*. \tag{29}$$

Using Condition (iv), we get $\alpha(ga)\beta(gb) \geq 1$. Thus, from (2) with $u = a, v = b$ and using (28), (29), we have:

$$\psi(s^3 d(fa, fb)) \leq \psi(\Delta_s(a,b)) - \varphi(\Delta_s(a,b)), \tag{30}$$

where:

$$\begin{aligned}
\Delta_s(a,b) =& \max\Bigg(d(ga,gb), \frac{1}{s}d(gb,fa), \frac{d(fa,gb)d(ga,fb)}{2s^3[1+d(ga,gb)]},\\
&\qquad \frac{d(fa,ga)d(fb,ga)}{2s[1+d(ga,gb)]}, \frac{d(fb,gb)d(fa,gb)}{2s[1+d(ga,gb)]}\Bigg)\\
=& \max\Bigg(d(a_0,a_0^*), \frac{1}{s}d(a_0^*,a_0), \frac{d(a_0,a_0^*)d(a_0,a_0^*)}{2s^3[1+d(a_0,a_0^*)]},\\
&\qquad \frac{d(a_0,a_0)d(a_0^*,a_0)}{2s[1+d(a_0,a_0^*)]}, \frac{d(a_0^*,a_0^*)d(a_0,a_0^*)}{2s[1+d(a_0,a_0^*)]}\Bigg)\\
=& d(a_0,a_0^*).
\end{aligned}$$

From (30), we have:

$$\begin{aligned}
\psi(d(a_0,a_0^*)) \leq& \psi(s^3 d(a_0,a_0^*))\\
\leq& \psi(d(a_0,a_0^*)) - \varphi(d(a_0,a_0^*))\\
<& \psi(d(a_0,a_0^*)),
\end{aligned}$$

which is a contradiction, unless $a_0 = a_0^*$. Finally, since the pair (f,g) is weakly compatible, so by Proposition (1), a_0 is a unique common fixed point of f and g. □

From Theorem 1, we deduce the following corollaries.

Corollary 1. *Let (X,d) be a b-complete b-metric space with coefficient $s \geq 1$ and $\alpha, \beta : X \to [0,\infty)$ be two mappings. Let f and g be two self-mappings defined on X such that f is a g-cyclic-$(\alpha,\beta)-(\psi,\varphi)$-admissible mapping satisfying the following contractive condition:*

$$\alpha(gu)\beta(gv)\psi(s^3 d(fu,fv)) \leq \psi(\Delta_s(u,v)) - \varphi(\Delta_s(u,v)), \ \forall \ u,v \in X, \tag{31}$$

where:

$$\begin{aligned}
\Delta_s(u,v) = \max\Bigg(&d(gu,gv), \frac{1}{s}d(gv,fu), \frac{d(fu,gv)d(gu,fv)}{2s^3[1+d(gu,gv)]},\\
&\frac{d(fu,gu)d(fv,gu)}{2s[1+d(gu,gv)]}, \frac{d(fv,gv)d(fu,gv)}{2s[1+d(gv,gu)]}\Bigg),
\end{aligned}$$

and ψ, φ are altering distance functions. If the following assumptions hold:

- (i) *$fX \subseteq gX$ with gX are closed subspaces of X;*
- (ii) *there exists $u_0 \in X$ with $\alpha(gu_0) \geq 1$ and $\beta(gu_0) \geq 1$;*
- (iii) *if $\{v_n\}$ is a sequence in X with $\beta(v_n) \geq 1$ for all n and $v_n \to v$, then $\beta(v) \geq 1$;*
- (iv) *$\alpha(ga) \geq 1$ and $\beta(gb) \geq 1$ whenever $fa = ga$ and $fb = gb$.*

Then, f and g have a unique point of coincidence in X. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Proof. Let $\alpha(gu)\beta(gv) \geq 1$, then from (31), we get:

$$\psi(s^3 d(fu, fv)) \leq \psi(\Delta_s(u, v)) - \varphi(\Delta_s(u, v)).$$

Thus, the inequality (2) is satisfied, and thus, the proof easily follows from Theorem (1). □

If we choose $g = I$(Identity map) in Theorem (1), we have the following corollary.

Corollary 2. *Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$ and $\alpha, \beta : X \to [0, \infty)$ be two mappings. Let $f : X \to X$ be a cyclic $(\alpha, \beta) - (\psi, \varphi)$-admissible mapping such that:*

$$\alpha(u)\beta(v) \geq 1 \;\;\Rightarrow\;\; \psi(s^3 d(fu, fv)) \leq \psi(\Delta_s(u, v)) - \varphi(\Delta_s(u, v)), \;\; \forall \;\; u, v \in X, \tag{32}$$

where:

$$\Delta_s(u, v) \quad = \quad \max\left(d(u, v), \frac{1}{s} d(v, fu), \frac{d(fu, v)d(u, fv)}{2s^3[1 + d(u, v)]}, \frac{d(fu, u)d(fv, u)}{2s[1 + d(u, v)]}, \frac{d(fv, v)d(fu, v)}{2s[1 + d(u, v)]}\right).$$

and ψ, φ are altering distance functions. If the following assumptions hold:

- (i) *there exists $u_0 \in X$ with $\alpha(u_0) \geq 1$ and $\beta(u_0) \geq 1$;*
- (ii) *if $\{v_n\}$ is a sequence in X with $\beta(v_n) \geq 1$ for all n and $v_n \to v$, then $\beta(v) \geq 1$;*
- (iii) *$\alpha(a) \geq 1$ and $\beta(b) \geq 1$ whenever $fa = a$ and $fb = b$;*

then f has a unique fixed point in X.

Remark 2. *In Theorem 3.2 of [7], the continuity of mapping is necessary; however, we relaxed this condition in Corollary 2.*

Corollary 3. *Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$ and $\alpha, \beta : X \to [0, \infty)$ be two mappings. Let f and g be two self-mappings defined on X such that f is a g-cyclic-(α, β)-admissible mapping satisfying the following contractive condition:*

$$\alpha(gu)\beta(gv) \geq 1 \;\;\Rightarrow\;\; d(fu, fv) \leq \frac{k}{s^3}\Delta_s(u, v), \;\; \forall \;\; u, v \in X, \tag{33}$$

where $k \in [0, 1)$. If the following assumptions hold:

- (i) *$fX \subseteq gX$ with gX are closed subspaces of X;*
- (ii) *there exists $u_0 \in X$ with $\alpha(gu_0) \geq 1$ and $\beta(gu_0) \geq 1$;*
- (iii) *if $\{v_n\}$ is a sequence in X with $\beta(v_n) \geq 1$ for all n and $v_n \to v$, then $\beta(v) \geq 1$;*
- (iv) *$\alpha(ga) \geq 1$ and $\beta(gb) \geq 1$ whenever $fa = ga$ and $fb = gb$;*

then f and g have a unique point of coincidence in X. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Corollary 4. *Let (X, d) be a b-complete b–metric space with coefficient $s \geq 1$ and $f, g : X \to X$ be mappings such that:*

$$\psi(s^3 d(fu, fv)) \leq \psi(\Delta_s(u, v)) - \varphi(\Delta_s(u, v)), \;\; \forall \;\; u, v \in X, \tag{34}$$

where ψ, φ are altering distance functions. If $fX \subseteq gX$ and gX are closed subspaces of X, then, f and g have a unique point of coincidence in X. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

Corollary 5. *Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$ and $f, g : X \to X$ be mappings such that:*

$$d(fu, fv) \leq \frac{k}{s^3}\Delta_s(u, v), \;\; \forall \;\; u, v \in X, \tag{35}$$

where $k \in [0,1)$. If $fX \subseteq gX$ and gX are closed subspaces of X, then f and g have a unique point of coincidence in X. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

4. Applications in Dynamic Programming

In this section, we present the existence result as an application of Theorem (1) to the following system of functional equations arising in computer programming, mathematical optimization, and in dynamic programming.

$$\left\{\begin{aligned} f(x) &= \underset{y\in\mathcal{D}}{opt}\{\tau_1(x,y)+H_1(x,y,f(a_1(x,y)))\} \ \ \forall \ x\in\mathcal{S}, \\ g(x) &= \underset{y\in\mathcal{D}}{opt}\{\tau_2(x,y)+H_2(x,y,g(a_2(x,y)))\} \ \ \forall \ x\in\mathcal{S}, \end{aligned}\right\} \tag{36}$$

where x and y signify the state and decision vectors, respectively, a_1, a_2 represent the transformations of the process, and $f(x), g(x)$ denote the optimal return functions with the initial state x.

Let $K, L : \mathbb{B}d(\mathcal{S}) \to \mathbb{B}d(\mathcal{S})$ be the mappings defined by:

$$\begin{aligned} Kh(x) &= \underset{y\in\mathcal{D}}{opt}\{\tau_1(x,y)+H_1(x,y,h(a_1(x,y)))\}, \\ Lh(x) &= \underset{y\in\mathcal{D}}{opt}\{\tau_2(x,y)+H_2(x,y,h(a_2(x,y)))\}, \end{aligned} \tag{37}$$

where $(x,h) \in \mathcal{S} \times \mathbb{B}d(\mathcal{S})$.

For the forthcoming analysis, let $\xi_1, \xi_2 : \mathbb{B}d(\mathcal{S}) \to \mathbb{R}$ and assume that

C_0 : $K(\mathbb{B}d(\mathcal{S})) \subseteq L(\mathbb{B}d(\mathcal{S}))$ such that $L(\mathbb{B}d(\mathcal{S}))$ are closed subspaces of $\mathbb{B}d(\mathcal{S})$;
C_1 : there exists $h_0 \in \mathbb{B}d(\mathcal{S})$ such that $\xi_1(Lh_0) \geq 0$ and $\xi_2(Lh_0) \geq 0$;
C_2 : $\{h_n\}$ is a sequence in $\mathbb{B}d(\mathcal{S})$ such that $h_n \to h$ and $\xi_2(h_n) \geq 0$ for all n, then $\xi_2(h) \geq 0$.
C_3 : if $\xi_1(Kh) \geq 0$ and $\xi_2(Lw) \geq 0$, for all $h, w \in \mathbb{B}d(\mathcal{S})$, then for all $(x,y,h,w) \in \mathcal{S} \times \mathcal{D} \times \mathbb{B}d(\mathcal{S}) \times \mathbb{B}d(\mathcal{S})$, we have:

$$|H_1(x,y,h(a_1(x,y))) - H_2(x,y,w(a_2(x,y)))| + |\tau_1(x,y) - \tau_2(x,y)| \leq \left(2^{3-3p}\Upsilon\left(|Lh - Lw|\right)\right)^{\frac{1}{p}},$$

where $\Upsilon : [0,\infty) \to [0,\infty)$ are altering distance functions defined by $\frac{d}{dt}(\Upsilon(t)) < 1$ and $\Upsilon(t) < t$ for all $t > 0$;

C_4 :

$$\xi_1(Lh) \geq 0 \text{ for some } \ h \in \mathbb{B}d(\mathcal{S}) \ \Rightarrow \ \xi_2(Kh) \geq 0$$

and:

$$\xi_2(Lh) \geq 0 \text{ for some } \ h \in \mathbb{B}d(\mathcal{S}) \ \Rightarrow \ \xi_1(Kh) \geq 0;$$

C_5 : $\xi_1(Lu) \geq 0$ and $\xi_2(Lv) \geq 0$ whenever $Ku = Lu$ and $Kv = Lv$;
C_6 : for some $h \in \mathbb{B}d(\mathcal{S})$, $KLh = LKh$, whenever $Kh = Lh$;
C_7 : for $i = 1, 2$, τ_i and H_i are bounded.

Now, we are in a position to present the existence result.

Theorem 2. *Let $K, L : \mathbb{B}d(\mathcal{S}) \to \mathbb{B}d(\mathcal{S})$ given by (37) be mappings for which Conditions (C_0)–(C_7) holds. Then, the system of functional equations (36) has a unique bounded common solution in $\mathbb{B}d(\mathcal{S})$.*

Proof. Let $\epsilon > 0$ be any number and $x \in \mathcal{S}$, $h, w \in \mathbb{B}d(\mathcal{S})$ such that $\xi_1(Lh) \geq 0$ and $\xi_2(Lw) \geq 0$. Then, since for $i = 1, 2$, τ_i and H_i are bounded, we can find $M > 0$ such that:

$$\sup\{\|\tau_1(x,y)\|, \|\tau_2(x,y)\|, \|H_i(x,y,t)\| : (x,y,t) \in \mathcal{S} \times \mathcal{D} \times R\} \leq M. \tag{38}$$

Thus, with the help of Lemma 2, Equation (37), and Inequality (38), K and L are self-mappings in $\mathbb{B}d(\mathcal{S})$.

Now, we show that K is an L-cyclic-$(\alpha, \beta) - (\psi, \varphi)$-rational contraction. For this, define $\alpha, \beta : \mathbb{B}d(\mathcal{S}) \to [0, \infty)$ by:

$$\alpha(h) = \begin{cases} 1, & \text{if } \xi_1(h) > 0 \text{ where } h \in \mathbb{B}d(\mathcal{S}), \\ 0, & \text{otherwise,} \end{cases}$$

and:

$$\beta(h) = \begin{cases} 1, & \text{if } \xi_2(h) > 0 \text{ where } h \in \mathbb{B}d(\mathcal{S}), \\ 0, & \text{otherwise.} \end{cases}$$

From Condition (C_3), if $\xi_1(Kh) \geq 0$ and $\xi_2(Lw) \geq 0$, for all $h, w \in \mathbb{B}d(\mathcal{S})$, then clearly, $\alpha(Lh)\beta(Lw) \geq 1$.

Next, consider the altering distance functions $\psi, \varphi : [0, \infty) \to [0, \infty)$ defined by:

$$\psi(t) = t^p \text{ and } \varphi(t) = t^p - (\Upsilon(t))^p,$$

for all $t \in [0, \infty)$.

Suppose that $\underset{y \in \mathcal{D}}{opt} = \inf_{y \in \mathcal{D}}$. Then, using (37), we can find $y \in \mathcal{D}$ and $(x, h, w) \in \mathcal{S} \times \mathbb{B}d(\mathcal{S}) \times \mathbb{B}d(\mathcal{S})$ such that:

$$Kh(x) > \tau_1(x,y) + H_1(x,y,h(a_1(x,y))) - \epsilon; \tag{39}$$

$$Kw(x) > \tau_2(x,y) + H_2(x,y,w(a_2(x,y))) - \epsilon; \tag{40}$$

$$Kh(x) \leq \tau_1(x,y) + H_1(x,y,h(a_1(x,y))); \tag{41}$$

$$Kw(x) \leq \tau_2(x,y) + H_2(x,y,w(a_2(x,y))); \tag{42}$$

where $(x, h) \in \mathcal{S} \times \mathbb{B}d(\mathcal{S})$.

Next, with the help of Inequalities (39) and (42), we have:

$$\begin{aligned} Kh(x) - Kw(x) &> H_1(x,y,h(a_1(x,y))) - H_2(x,y,w(a_2(x,y))) + \tau_1(x,y) - \tau_2(x,y) - \epsilon \\ &\geq -\{|H_1(x,y,h(a_1(x,y))) - H_2(x,y,w(a_2(x,y)))| + |\tau_1(x,y) - \tau_2(x,y)| + \epsilon\}. \end{aligned}$$

Analogously, with help of Inequalities (40) and (41), we have:

$$\begin{aligned} Kh(x) - Kw(x) &< H_1(x,y,h(a_1(x,y))) - H_2(x,y,w(a_2(x,y))) + \tau_1(x,y) - \tau_2(x,y) + \epsilon \\ &\leq |H_1(x,y,h(a_1(x,y))) - H_2(x,y,w(a_2(x,y)))| + |\tau_1(x,y) - \tau_2(x,y)| + \epsilon. \end{aligned}$$

Therefore, we can write:

$$|Kh(x) - Kw(x)| < |H_1(x,y,h(a_1(x,y))) - H_2(x,y,w(a_2(x,y)))| + |\tau_1(x,y) - \tau_2(x,y)| + \epsilon. \tag{43}$$

Similarly, if we take $\underset{y \in \mathcal{D}}{opt} = \sup_{y \in \mathcal{D}}$, then one can easily obtain the above inequality. Taking the limit as $\epsilon \to 0^+$ in Inequality (43), we get:

$$|Kh(x) - Kw(x)| \leq |H_1(x,y,h(a_1(x,y))) - H_2(x,y,w(a_2(x,y)))| + |\tau_1(x,y) - \tau_2(x,y)|,$$

using Condition (4) of Theorem 2, we have:

$$\begin{aligned}|Kh(x) - Kw(x)| &< \left(2^{3-3p}\Upsilon\left(|Lh(x) - Lw(x)|^p\right)\right)^{\frac{1}{p}} \\ &\leq \left(2^{3-3p}\Upsilon\left(\sup_{x\in\mathcal{S}}|Lh - Lw|^p\right)\right)^{\frac{1}{p}} \\ &= \left(2^{3-3p}\Upsilon\left(d\left(Lh, Lw\right)\right)\right)^{\frac{1}{p}} \\ &\leq \left(2^{3-3p}\Upsilon\left(\Delta_s(h,w)\right)\right)^{\frac{1}{p}}.\end{aligned}$$

Equivalently,

$$|Kh(x) - Kw(x)|^p < 2^{3-3p}\Upsilon\left(\Delta_s(h,w)\right). \tag{44}$$

Now, for all $h, w \in \mathbb{B}d(\mathcal{S})$, we have:

$$\begin{aligned}\psi\left(s^3 d(Kh(x), Kw(x))\right) &= \left(s^3 d(Kh(x), Kw(x))\right)^p \\ &\leq \left(2^{3p-3}\sup_{x\in\mathcal{S}}|Kh(x) - Kw(x)|^p\right)^p \\ &\leq \left(\Upsilon\left(\Delta_s(h,w)\right)\right)^p \\ &= \left(\Delta_s(h,w)\right)^p - \left[\left(\Delta_s(h,w)\right)^p - \left(\Upsilon\left(\Delta_s(h,w)\right)\right)^p\right] \\ &= \psi\left(\Delta_s(h,w)\right) - \varphi\left(\Delta_s(h,w)\right).\end{aligned}$$

That is:

$$\alpha(Lh)\beta(Lw) \geq 1 \;\Rightarrow\; \psi\left(s^3 d(Kh(x), Kw(x))\right) \leq \psi\left(\Delta_s(h,w)\right) - \varphi\left(\Delta_s(h,w)\right).$$

Moreover, from Conditions (C_0), (C_1), (C_2), and (C_5), one can easily obtain Conditions (i)–(iv) of Theorem 1, respectively. Finally, Condition (C_6) implies that the pair (K, L) is weakly compatible. Therefore, by Theorem 1, there exists a unique common fixed point of K and L in $\mathbb{B}d(\mathcal{S})$; consequently, the System (36) of functional equations has a unique bounded common solution. □

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