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On the optimality of the trigonometric system ☆

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ABSTRACT

We study a new phenomenon of the behaviour of widths with respect to the optimality of trigonometric system. It is shown that the trigonometric system is optimal in the sense of Kolmogorov widths in the case of "super-high" and "super-small" smoothness but is not optimal in the intermediate cases. Bernstein's widths behave differently when compared with Kolmogorov in the case of "super-small" smoothness. However, in the case of "super-high" smoothness Kolmogorov and Bernstein widths behave similarly, i.e. are realized by trigonometric polynomials.

1. Introduction

n-Widths were introduced to compare and classify the power of approximation of a wide range of algorithms. Optimality of the trigonometric system is a frequently discussed topic in the theory of n-widths [18,22]. We present a new phenomenon in behaviour of the trigonometric system in the "usual" order, i.e. $1, \cos kx, \sin kx, k \in \mathbb{N}$. Namely, the sequence of subspaces $\mathcal{T}_n$, $n \in \mathbb{N}$ of trigonometric polynomials of degree at most n is optimal in the sense of order of Kolmogorov n-widths $d_n\left(K * U_p, L_q\right)$ of convolution classes $K*U_p$ in L_q (where U_p is the unit ball in L_p) for all $1 < p, q < \infty$ in the case of "super-high" smoothness (analytic and entire), that is in the case

$$K \sim \sum_{k=1}^{\infty} \lambda(k) \cos kx,$$

where

$$\lambda(k) = \exp\left(-\mu k^{\varrho}\right) \quad \mu > 0, \quad \varrho \geq 1$$

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and in the case

$$\lambda(k) = k^{-\left(\frac{1}{p}-\frac{1}{q}\right)} (\ln (k+1))^{-\nu}, \nu > 0, \tag{1}$$

where $\{2 \leq p < q < \infty\} \cup \{1 < p < 2 \leq q < \infty\}$. Observe that the rate of decay of $\lambda(k)$ as $k \to \infty$ determines the smoothness of convolution classes $K * U_p$. Hence, it is natural to call the classes $K * U_p$ generated by the kernels K of the type (1) as sets of "super-small" smoothness since for any $r > \frac{1}{p} - \frac{1}{q}$, $p < q$ we have

$$k^{-r} \ll k^{-\left(\frac{1}{p}-\frac{1}{q}\right)} (\ln (k+1))^{-\nu}.$$

Let $\lambda(k) = k^{-r}$, where $r > 0$. In the known intermediate cases of "small" and "finite" smoothness, which will be discussed later, the sequence of subspaces $\mathcal{T}_n$, $n \in \mathbb{N}$ is not optimal for Kolmogorov n-widths $d_n\left(K * U_p, L_q\right)$ if $r > \frac{1}{p} - \frac{1}{q}$ and

$$\{2 \leq p < q < \infty\} \cup \{1 < p < 2 \leq q < \infty\}.$$

Similarly, $\mathcal{T}_n$, $n \in \mathbb{N}$ is not optimal for linear n-widths $\delta_n\left(K * U_p, L_q\right)$ if

$$\{1 < p < 2 \leq q < \infty\}, \quad r > \frac{1}{p} - \frac{1}{q}.$$

We show that Bernstein n-widths behave differently when compared with Kolmogorov and linear n-widths in the case of "super-small" smoothness. Namely, in the case of "finite" smoothness the sequence of subspaces $\mathcal{T}_n$ is optimal for Bernstein n-widths $b_n\left(K * U_p, L_q\right)$ in the region $2 \leq q \leq p < \infty$ and is not optimal in the case of "small" and "super-small" smoothness, where we have different weak asymptotics. We conjecture that homological widths, introduced in [10], which are intermediate between Kolmogorov and Bernstein widths, behave like Kolmogorov widths. Although we consider here just Kolmogorov, linear and Bernstein n-widths, the problem can be considered for a wider range of n-widths.

2. Sets of smooth functions

Let $\mathcal{T}_n$ be the sequence of subspaces of trigonometric polynomials with the "usual" order. As a model case, we consider usual spaces L_p of p-integrable functions ϕ on the unit circle $\mathbb{T}$ with the Lebesgue measure dx, i.e. such that $\|\phi\|_p < \infty$, where

$$\|\phi\|_p := \left(\int_{\mathbb{T}} |\phi|^p \, dx\right)^{\frac{1}{p}}, \ 1 \leq p \leq \infty.$$

Let $\phi \in L_p$ with the formal Fourier series

$$\phi \sim \sum_{k=1}^{\infty} a_k(\phi) \cos kx + b_k(\phi) \sin kx,$$

where

$$a_k(\phi) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(t) \cos ktdt, \ \ b_k(\phi) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(t) \sin ktdt, \ \ k \in \mathbb{N}$$

and

$$S_n(\phi, x) = \sum_{k=1}^{n} a_k(\phi) \cos kx + b_k(\phi) \sin kx$$

be its nth Fourier sum. A wide range of sets of smooth functions on the unit circle $\mathbb{T}$ can be introduced using multipliers $\Lambda := \{\lambda(k), k \in \mathbb{N}\}$ [19]. We say that $f \in \Lambda_\beta U_p$, $\beta \in \mathbb{R}$ if

$$f \sim \sum_{k=1}^{\infty} \lambda(k) \left(a_k(\phi) \cos\left(kx - \frac{\beta\pi}{2}\right) + b_k(\phi) \sin\left(kx - \frac{\beta\pi}{2}\right)\right), \ \phi \in U_p,$$

where U_p is the unit ball in L_p. We shall consider multiplier operator Λ_β which acts as $L_p \ni \phi \to \Lambda\phi = f \in L_q$. In the case $\beta = 0$ we will write $\Lambda_\beta = \Lambda$. In particular, if there exists $K \in L_1$ such that

$$K \sim \sum_{k=1}^{\infty} \lambda(k) \cos\left(kx - \frac{\beta\pi}{2}\right) \tag{2}$$

then the set $\Lambda_\beta U_p$ can be represented in the convolution form

$$f(x) = \int_{\mathbb{T}} K(x-y)\,\phi(y)\,dy.$$

In this case $\Lambda_\beta U_p = K * U_p$. In general, if $K \notin L_1$ we should consider generalized convolutions. The smoothness of the function classes $\Lambda_\beta U_p$ is mainly governed by the rate of decay of the sequence Λ. For instance, if $\lambda(k) = k^{-r}$, $\beta = r$, $r > 0$ we get Sobolev classes W_p^r. If $\lambda(k) = \exp(-\mu k^\gamma)$, $\beta \in \mathbb{R}$, $\mu > 0$, $0 < \gamma < 1$, then the class $\Lambda_\beta U_p = K * U_p$ consists of infinitely differentiable functions. In the cases $\gamma = 1$ and $\gamma > 1$, we get classes of analytic and entire functions respectively. To simplify technical notations we present our new results just in the case $1 < p, q < \infty$ and $\beta = 0$. We will need a simple norm estimate for multiplier operators [7].

Lemma 1. *Let $1 < p < \infty$, then*

$$\left\| \Lambda \left| L_p \to L_p \right. \right\|$$

$$\leq \chi_p \left(\sum_{k=1}^{\infty} |\lambda(k) - \lambda(k+1)| + \sup_{m\in\mathbb{N}} |\lambda(m)| \right), \tag{3}$$

where

$$\chi_p = 1 + 2 \begin{Bmatrix} \cot\frac{\pi}{2p}, & 2 < p < \infty, \\ \tan\frac{\pi}{2p}, & 1 < p \leq 2. \end{Bmatrix}.$$

Proof. Let $U : \phi \to \widetilde{\phi}$, where

$$U\phi := \widetilde{\phi} \sim \sum_{k=1}^{\infty} -b_k(\phi)\cos kx + a_k(\phi)\sin kx$$

and

$$S_m^*(\phi, x) = \frac{1}{2}\left(S_m(\phi, x) + S_{m-1}(\phi, x)\right).$$

Then

$$S_m^*(\phi, x) = \sin mx\widetilde{g}_m(x) - \cos mx\widetilde{h}_m(x)$$

$$= \sin mxUg_m(x) - \cos mxUh_m(x),$$

where

$$g_m(x) := f(x)\cos mx,\ \ h_m(x) := f(x)\sin mx.$$

Clearly,

$$\left\| S_m^* \left| L_p \to L_p \right. \right\| \leq 2 \left\| U \left| L_p \to L_p \right. \right\|,$$

and

$$\left\| S_m - S_m^* \left| L_p \to L_p \right. \right\| \leq 1.$$

Hence, for any $m \in \mathbb{N}$ we get

$$\left\| S_m \left| L_p \to L_p \right\| = \left\| S_m - S_m^* + S_m^* \left| L_p \to L_p \right\| \right.\right.$$

$$\leq 1 + \left\| S_m^* \left| L_p \to L_p \right\| \leq 1 + 2 \left\| U \left| L_p \to L_p \right\| . \right.\right.\right. \tag{4}$$

It is known [16] that

$$\left\| U \left| L_p \to L_p \right\| = \begin{cases} \cot \dfrac{\pi}{2p}, & 2 < p < \infty, \\ \tan \dfrac{\pi}{2p}, & 1 < p \leq 2. \end{cases} \right. \tag{5}$$

Comparing (4) and (5) we get

$$\sup_{m \in \mathbb{N}} \left\| S_m \left| L_p \to L_p \right\| \leq 1 + 2 \begin{Bmatrix} \cot \dfrac{\pi}{2p}, & 2 < p < \infty, \\ \tan \dfrac{\pi}{2p}, & 1 < p \leq 2. \end{Bmatrix} := \chi_p. \right. \tag{6}$$

Application of Abel transform to $S_m \Lambda \phi$ yields

$$S_m \Lambda \phi (x) = \sum_{k=1}^{m} \lambda (k) \left(a_k (\phi) \cos kx + b_k (\phi) \sin kx \right)$$

$$= \sum_{k=1}^{m-1} \left(\lambda (k) - \lambda (k+1) \right) S_k (\phi, x) + \lambda (m) S_m \phi (x) . \tag{7}$$

Comparing (6) and (7) we find

$$\sup_{m \in \mathbb{N}} \left\| S_m \Lambda \left| L_p \to L_p \right\| \right.$$

$$\leq \chi_p \sup_{m \in \mathbb{N}} \left(\sum_{k=1}^{m-1} \left| \lambda (k) - \lambda (k+1) \right| + \left| \lambda (m) \right| \right)$$

$$\leq \chi_p \left(\sum_{k=1}^{\infty} \left| \lambda (k) - \lambda (k+1) \right| + \sup_{m \in \mathbb{N}} \left| \lambda (m) \right| \right) .$$

Consequently, by the Banach–Steinhaus Theorem we get the proof. ■

An analogue of this estimate in the case of compact globally symmetric spaces of rank one is given in [1], Theorem 2. Observe that the estimate (3) is sufficient for a wide range of applications in the theory of n -widths instead of commonly used Marcinkiewicz result [14] which is based on the Littlewood–Paley theory.

3. *n*-widths in the case of small, finite and infinite smoothness

Let X be a Banach space and let A be a convex, compact and centrally symmetric subset of X. The Kolmogorov n-width of A in X is defined by

$$d_n (A, X) := \inf_{X_n \subset X} \sup_{f \in A} \inf_{g \in X_n} \| f - g \|_X ,$$

where X_n runs over all subspaces of X of dimension $\leq n$. The linear n-width of A in X is defined by

$$\delta_n (A, X) := \inf_{L_n} \sup_{f \in A} \| f - L_n f \|_X ,$$

where L_n varies over all linear operators of rank at most n that map X into itself.

We use several universal constants which enter into the estimates. These positive constants are mostly denoted by $C_{1,p}$, $C_{1,q}$ etc., to underline their dependence on parameters p and q respectively. We did not carefully distinguish between the different constants, neither did we try to get good estimates for them. For the ease of notation we will write $a_n \ll b_n$ for two sequences, if $a_n \leq Cb_n$ for all $n \in \mathbb{N}$, and $a_n \asymp b_n$, if $C_1 b_n \leq a_n \leq C_2 b_n$ for all $n \in \mathbb{N}$ and some constants C, C_1 and C_2. For $a \in \mathbb{R}$ we put $(a)_+ = \max\{a, 0\}$.

Let $A \subset L_q$,

$$\mathcal{E}_n\left(A, L_q\right) := \sup_{f \in A} \|f - S_n(f)\|_q$$

and

$$\mathrm{E}_n\left(A, L_q\right) := \sup_{f \in A} \inf_{g \in \mathcal{T}_n} \|f - g\|_q .$$

If $\lambda(k)$ is a positive and monotone decreasing to zero sequence such that $\lambda(k) \asymp \lambda(2k)$ and $\lim_{k\to\infty} \lambda(k) k^{\left(\frac{1}{p}-\frac{1}{q}\right)_+} = 0$ then from Theorem 8 [20] we get

$$\mathrm{E}_n\left(K * U_p, L_q\right) \asymp \mathcal{E}_n\left(K * U_p, L_q\right) \asymp \lambda(n)\, n^{\left(\frac{1}{p}-\frac{1}{q}\right)_+}, \tag{8}$$

where $1 < p, q < \infty$. If $\lambda(k)$, $k \in \mathbb{N}$ is a positive and monotone decreasing to zero sequence such that $\sum_{k=1}^{\infty} \lambda(k) < \infty$ then it follows from Theorem 1 in [11] that

$$\mathrm{E}_n\left(K * U_p, L_q\right) \asymp \mathcal{E}_n\left(K * U_p, L_q\right)$$

$$\asymp (\lambda(n+1))^{1-\left(\frac{1}{p}-\frac{1}{q}\right)_+} \left(\sum_{k=n+1}^{\infty} \lambda(k)\right)^{\left(\frac{1}{p}-\frac{1}{q}\right)_+}, \ 1 < p, q < \infty. \tag{9}$$

In particular, let $\lambda(k) = k^{-r}$ in (8) then

$$\mathrm{E}_n\left(K * U_p, L_q\right) \asymp \mathcal{E}_n\left(K * U_p, L_q\right) \asymp n^{-r+\left(\frac{1}{p}-\frac{1}{q}\right)_+}, \tag{10}$$

where $r > \left(\frac{1}{p} - \frac{1}{q}\right)_+$. If we put in (9) $\lambda(k) = \exp(-\mu k^{\gamma})$, $\mu > 0$, $\gamma > 0$, then

$$\mathrm{E}_n\left(K * U_p, L_q\right) \asymp \mathcal{E}_n\left(K * U_p, L_q\right) \asymp \exp(-\mu n^{\gamma})\, n^{(1-\gamma)_+ + \left(\frac{1}{p}-\frac{1}{q}\right)_+}. \tag{11}$$

Let $\lambda(k) = k^{-r}$, $\beta = r$, then it is well-known [4,22,23] (the case of "finite" smoothness) that

$$d_n\left(W_p^r, L_q\right) \asymp \begin{cases} n^{-r}, & r > 0, & 1 < q \leq p < \infty, \\ n^{-r+\frac{1}{p}-\frac{1}{q}}, & r > \dfrac{1}{p} - \dfrac{1}{q}, & 1 < p \leq q \leq 2, \\ n^{-r+\frac{1}{p}-\frac{1}{2}}, & r > \dfrac{1}{p}, & 1 < p < 2 \leq q < \infty, \\ n^{-r}, & r > \dfrac{1}{2}\left(\dfrac{1}{p} - \dfrac{1}{q}\right) \Big/ \left(\dfrac{1}{2} - \dfrac{1}{q}\right), & 2 \leq p \leq q < \infty. \end{cases}$$

Hence, the sequence of subspaces of trigonometric polynomials $\mathcal{T}_n$, $n \in \mathbb{N}$ is optimal in the sense of order of Kolmogorov n-widths in the "Ismagilov triangle"

$$\left\{1 < p \leq q \leq 2, r > \frac{1}{p} - \frac{1}{q}\right\}$$

and in the "Makovoz triangle"

$$\{1 < q \leq p < \infty, r > 0\}.$$

In the remaining "Kashin cases",

$$\left\{1 < p < 2 \le q < \infty, r > \frac{1}{p}\right\} \cup \left\{2 \le p < q < \infty, r > \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) / \left(\frac{1}{2} - \frac{1}{q}\right)\right\}$$

the sequence $\mathcal{T}_n$ is not optimal. In the case of linear n-widths we have (see [22,23] for more details)

$$\delta_n\left(W_p^r, L_q\right) \asymp n^{-r+\left(\frac{1}{p}-\frac{1}{q}\right)_+}$$

if $r > \left(\frac{1}{p} - \frac{1}{q}\right)_+$ and

$$\{1 < p \le q \le 2\} \cup \{2 \le p \le q < \infty\} \cup \{1 \le q \le p < \infty\}.$$

If

$$r > \theta := \max\left\{1 - \frac{1}{q}, \frac{1}{p}\right\} \quad 1 < p < 2 \le q < \infty$$

then

$$\delta_n\left(W_p^r, L_q\right) \asymp n^{-r+\theta-\frac{1}{2}}. \tag{12}$$

Comparing (12) and (10) we get that $\mathcal{T}_n$ is not optimal if $1 < p < 2 \le q < \infty$.

If $\frac{1}{p} - \frac{1}{q} < r < \frac{1}{p}$, $1 < p < 2 \le q < \infty$ then Kolmogorov n-widths change the order of decay. Namely,

$$d_n\left(W_p^r, L_q\right) \asymp n^{\frac{q}{2}\left(-r+\frac{1}{p}-\frac{1}{q}\right)}. \tag{13}$$

Also, if

$$\frac{1}{p} - \frac{1}{q} < r < \frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) / \left(\frac{1}{2} - \frac{1}{q}\right), \quad 2 \le q < p < \infty,$$

then again

$$d_n\left(W_p^r, L_q\right) \asymp n^{\frac{q}{2}\left(-r+\frac{1}{p}-\frac{1}{q}\right)}. \tag{14}$$

Similarly, in the case of linear n-widths we have [23]

$$\delta_n\left(W_p^r, L_q\right) \asymp n^{\frac{1}{2}\left(-r+\frac{1}{p}-\frac{1}{q}\right)\min\{p',q\}}, \tag{15}$$

where $1 < p < 2 \le q < \infty$, $\frac{1}{p} - \frac{1}{q} < r < \theta$, $\frac{1}{p} + \frac{1}{p'} = 1$. This phenomenon was discovered by Kashin [5] (see also [6]) and is known as "small" smoothness. Comparing (11) with (13)–(15) we see that $\mathcal{T}_n$ is not optimal in these cases.

If in (2)

$$\lambda(k) = \exp\left(\mu k^{\gamma}\right), \mu > 0, \beta = 0, 0 < \gamma < 1,$$

i.e. $K * U_p$ is a class of infinitely differentiable functions, then

$$d_{2n}\left(K * U_p, L_q\right)$$

$$\asymp \begin{cases} \exp\left(-\mu n^{\gamma}\right) n^{(1-\gamma)\left(\frac{1}{p}-\frac{1}{q}\right)}, & 1 < p \le q \le 2, \\ \exp\left(-\mu n^{\gamma}\right), & 1 < q \le p \le 2 \text{ or } 2 \le p, q < \infty, \\ \exp\left(-\mu n^{\gamma}\right) n^{(1-\gamma)\left(\frac{1}{p}-\frac{1}{2}\right)}, & 1 < p < 2 \le q < \infty \end{cases} \tag{16}$$

(see [8–11,21], for details). Comparing (16) and (11) we get that the sequence $\mathcal{T}_n$ is not optimal if $1 < p < 2 \le q < \infty$ and $2 \le p < q < \infty$ similarly to the case of "small" and "finite" smoothness.

If in (2) $\lambda(k) = \exp(\mu k^{\gamma})$, $\mu > 0$, $\beta = 0$, $\gamma \geq 1$, then we have the case of "super-high" smoothness. Comparing (11) and [8] we get

$$d_{2n}\left(K * U_p, L_q\right) \asymp \mathrm{E}_n\left(K * U_p, L_q\right) \asymp \mathcal{E}_n\left(K * U_p, L_q\right)$$

$$\asymp \exp\left(-\mu k^{\gamma}\right), \; 1 < p, q < \infty.$$

Hence, in this case the sequence $\mathcal{T}_n$ is optimal for any $1 < p, q < \infty$ like in the case of "super-small" smoothness.

4. *n*-widths in the case of "super-small" smoothness

We consider the case of "super-small" smoothness here.

Theorem 1. *Let $\phi(k)$, $k \in \mathbb{N}$, be a sequence of positive numbers which is decreasing for $k \geq N$ for some N and satisfies the following conditions: $\lim_{k\to\infty} \phi(k) = 0$ and $\phi(k^s) \asymp \phi(k)$ for any fixed $s > 0$. Let $\lambda(k) = \phi(k)\, k^{-\left(\frac{1}{p}-\frac{1}{q}\right)_+}$ and*

$$K(x) \sim \sum_{k=1}^{\infty} \lambda(k) \cos kx$$

be the associated kernel. Then

$$\mathrm{E}_n\left(K * U_p, L_q\right) \asymp \mathcal{E}_n\left(K * U_p, L_q\right) \asymp d_n\left(K * U_p, L_q\right)$$

$$\asymp \delta_n\left(K * U_p, L_q\right) \asymp \phi(n), \; 1 < p, q < \infty$$

and the sequence of subspaces $\mathcal{T}_n$ of trigonometric polynomials in the "usual" order is optimal for any $1 < p, q < \infty$.

Proof. We present the proof only for Kolmogorov widths, the statement for linear widths follows similarly. The following upper bounds follow from (8),

$$d_n\left(K * U_p, L_q\right) \ll \phi(n), \;\; 1 < p, q < \infty. \tag{17}$$

We turn to the lower bounds now. As usual, we reduce the problem to a finite dimensional one. For given $m \in \mathbb{N}$, consider the multiplier operator

$$\Lambda_m^{-1} = \left\{\frac{1}{\lambda(1)}, \frac{1}{\lambda(2)}, \ldots, \frac{1}{\lambda(m)}, 0, 0, \ldots\right\}. \tag{18}$$

Applying (3) we get $\left\| \Lambda_m^{-1} \left| L_p \longrightarrow L_p \right\| \leq C_p \frac{1}{\lambda(m)}\right.$, which implies

$$C_p \lambda(m) \cdot U_p \cap \mathcal{T}_m \subset K * U_p. \tag{19}$$

From the definition of Kolmogorov n-widths and (19) we get

$$d_n\left(K * U_p, L_q\right) \geq C_p \lambda(m)\, d_n\left(U_p \cap \mathcal{T}_m, L_q\right).$$

Next, we need to reduce L_q to $L_q \cap \mathcal{T}_m$. Let $S_m(\phi)$ be the Fourier sum of $\phi \in L_q$ of order m. Since the projection operator $S_m : \phi \longrightarrow S_m(\phi)$ is bounded if $1 < q < \infty$, i.e. $\left\| S_m | L_q \longrightarrow L_q \cap \mathcal{T}_m \right\| < C_q$, then for any $t_m \in \mathcal{T}_m$ and $y \in L_q$ we get $\|S_m(t_m - y)\|_q = \|t_m - S_m y\|_q \leq C_q \|t_m - y\|_q$ or $\|t_m - y\|_q \geq C_q^{-1} \|t_m - S_m y\|_q$ (Ismagilov lemma on projections [3]). Since $S_m y \in L_q \cap \mathcal{T}_m$ and for any $X_n \subset L_q$, $\dim X_n = n$ we have $\dim S_m X_n \leq n$, then by the definition of n-widths

$$d_n\left(U_p \cap \mathcal{T}_m, L_q\right) \geq C_q^{-1} d_n\left(U_p \cap \mathcal{T}_m, L_q \cap \mathcal{T}_m\right). \tag{20}$$

Finally, applying Marcinkiewicz inequality [12,13],

$$C_{1,p} \|t_m\|_p \leq \left(\frac{1}{m} \sum_{k=1}^{2m+1} \left| t_m\left(\frac{2\pi k}{2m+1}\right)\right|^p\right)^{\frac{1}{p}} \leq C_{2,p} \|t_m\|_p\,, \tag{21}$$

which is valid for any $t_m \in \mathcal{T}_m$ and $1 < p < \infty$, we get

$$d_n\left(K * U_p, L_q\right) \gg \lambda(m)\, m^{\frac{1}{p}-\frac{1}{q}} d_n\left(B(p, 2m+1), l(q, 2m+1)\right)$$

$$= \phi(m)\, d_n\left(B(p, 2m+1), l(q, 2m+1)\right),$$

where the norm in $l(q, 2m+1)$ is defined as usual,

$$\|x\|_{l(q,2m+1)} = \left(\sum_{k=1}^{2m+1} |x_k|^q\right)^{\frac{1}{q}},$$

$$x = (x_1, \ldots, x_{2m+1}) \in \mathbb{R}^{2m+1},\ 1 \le q \le \infty$$

and $B(p, 2m+1)$ is the unit ball in $l(p, 2m+1)$. To get the lower bounds for Kolmogorov n-widths we will need the following result [22]

$$C_{1,p,q} \le \frac{d_n\left(B(p, 2m+1), l(q, 2m+1)\right)}{\Phi(m, n, p, q)} \le C_{2,p,q}$$

for any $m > n$, where

$$\Phi(m, n, p, q) := \left(\min\left\{1, m^{\frac{1}{q}} n^{-1/2}\right\}\right)^{\left(\frac{1}{p}-\frac{1}{q}\right)/\left(\frac{1}{2}-\frac{1}{q}\right)} \tag{22}$$

if $2 \le p \le q \le \infty$ and

$$\Phi(m, n, p, q) := \max\left\{m^{\frac{1}{q}-\frac{1}{p}},\ \min\left\{1, m^{\frac{1}{q}} n^{-\frac{1}{2}}\right\}\left(1 - \frac{n}{m}\right)^{\frac{1}{2}}\right\}, \tag{23}$$

if $1 \le p < 2 \le q \le \infty$. Let $2 \le p \le q \le \infty$. Let us put in (22) $m = n^{\frac{q}{2}}$ then $\min\left\{1, m^{\frac{1}{q}} n^{-\frac{1}{2}}\right\} = 1$ and

$$d_n\left(B\left(p, 2n^{\frac{q}{2}}+1\right), l(q, 2m+1)\right) \ge C_{1,p,q}\Phi\left(n^{\frac{q}{2}}, n, p, q\right) = C_{1,p,q}.$$

Hence, in this case

$$d_n\left(K * U_p, L_q\right) \ge C_{p,q}\phi\left(n^{\frac{q}{2}}\right) d_n\left(B\left(p, 2n^{\frac{q}{2}}+1\right), l\left(q, 2n^{\frac{q}{2}}+1\right)\right)$$

$$\ge C_{p,q}\phi\left(n^{\frac{q}{2}}\right) \asymp \phi(n).$$

Similarly, if $1 \le p < 2 \le q \le \infty$ then we put in (23) $m = n^{\frac{q}{2}}$. This gives

$$\max\left\{m^{\frac{1}{q}-\frac{1}{p}},\ \min\left\{1, m^{\frac{1}{q}} n^{-\frac{1}{2}}\right\}\left(1 - \frac{n}{m}\right)^{\frac{1}{2}}\right\} \asymp 1$$

and

$$d_n\left(K * U_p, L_q\right) \ge C_{p,q}\lambda\left(n^{\frac{q}{2}}\right) n^{\frac{q}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} d_n\left(B\left(p, 2n^{\frac{q}{2}}+1\right), l\left(q, 2n^{\frac{q}{2}}+1\right)\right)$$

$$\ge C_{p,q}\phi\left(n^{\frac{q}{2}}\right) \asymp \phi(n).$$

In the case of linear n-widths $\delta_n\left(K * U_p, L_q\right)$ we just need to apply a finite dimensional result [22]

$$C_{1,p,q} \le \frac{\delta_n\left(B(p, 2m+1), l(q, 2m+1)\right)}{\Psi(m, n, p, q)} \le C_{2,p,q},$$

where

$$\Psi(m, n, p, q) := \begin{cases} \Phi(m, n, p, q), & 1 \le p < q \le p', \\ \Phi(m, n, p, q), & \max\left\{p, p'\right\} < q < \infty, \end{cases}$$

and $\Phi(m,n,p,q)$ was defined in (22) and (23) and repeat the line of arguments we used for Kolmogorov n-widths.

The upper bounds in the "Makovoz triangle", i.e. if $1 < q < p < \infty$ follow from (17). The respective lower bounds are the consequence of (19), (20), (21) and a well-known result [22], p. 209,

$$d_n\left(B(p,2m+1), l(q,2m+1)\right) = (2m-n)^{-\left(\frac{1}{p}-\frac{1}{q}\right)}, \quad 1 \le q \le p \le \infty. \quad \blacksquare$$

A typical example of the sequence $\lambda(k)$, $k \in \mathbb{N}$ which satisfies the conditions of Theorem 1 is given by

$$\lambda(k) = k^{-\left(\frac{1}{p}-\frac{1}{q}\right)_+}(\ln(k+1))^{-\nu}(\ln\ln(k+3))^{\varrho}, \quad k \in \mathbb{N},$$

where $\nu > 0$ and $\varrho \in \mathbb{R}$.

The Gel'fand n-width of $A \subset X$ is defined by

$$d^n(A,X) := \inf_{X^n} \sup_{x \in A \cap X^n} \|x\|_X$$

where X^n runs over all subspaces of X of codimension at most n. Let $\widetilde{X}$ be a Banach space and let $i : X \longrightarrow \widetilde{X}$ be a linear isometry. We denote the pairing of these objects by $\left(\widetilde{X}, i\right)$. The absolute linear n-width, $\Lambda_n(A,X)$ is defined as

$$\Lambda_n(A,X) := \inf_{(\widetilde{X},i)} \delta_n(A,X),$$

where inf is taken over all extensions $\left(\widetilde{X}, i\right)$ of X. It is known [3] that

$$\Lambda_n(A,X) = d^n(A,X).$$

Applying duality between Kolmogorov and Gelfand n-widths [22],

$$d^n\left(K * U_p, L_q\right) = d_n\left(K * U_{q'}, L_{p'}\right), \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1,$$

we get

$$\Lambda_n\left(K * U_p, L_q\right) = d_n\left(K * U_{q'}, L_{p'}\right)$$

which implies that an analogue of Theorem 1 remains valid for absolute linear n-widths.

To underline an opposite situation with respect to the "super-small" smoothness we consider Bernstein n-width of A in X defined as

$$b_n(A,X) := \sup_{X_{n+1}} \sup_{\epsilon > 0} \left\{\epsilon B_X \cap X_{n+1} \subset A\right\},$$

where X_{n+1} is any $(n+1)$-dimensional subspace of X and B_X is the unit ball of X.

It is known (see [15] Remark 5.6, Theorem 2.2 and [22] p. 212) that in the case of "finite" smoothness

$$b_n\left(W_p^r, L_q\right) \asymp \begin{cases} n^{-r}, & r > \frac{1}{p} - \frac{1}{q}, \ 1 < p \le q < \infty, \\ n^{-r}, & r > 0, \ 1 < q \le p \le 2, \ 1 < q = p < \infty, \\ n^{-r+\frac{1}{p}-\frac{1}{q}}, & r > \dfrac{\frac{1}{q}-\frac{1}{p}}{\frac{p}{2}-1}, \ 2 \le q < p < \infty, \\ n^{-r+\frac{1}{p}-\frac{1}{2}}, & r > \frac{1}{p}, \ 1 < q \le 2 < p < \infty \end{cases} \tag{24}$$

and in the case of “small” smoothness

$$b_n\left(W^r_p, L_q\right) \asymp n^{-\frac{rp}{2}}, \tag{25}$$

where

$$2 \le q < p < \infty,\ \ 0 < r < \frac{\frac{1}{q}-\frac{1}{p}}{\frac{p}{2}-1},$$

or

$$1 < q \le 2 < p < \infty,\ \ 0 < r < \frac{1}{p}.$$

Comparing (24) and (3) we get that in the case of “finite” smoothness the sequence $\mathcal{T}_n$ is not optimal if either $1 < q < p \le 2$ or $1 < q \le 2 < p < \infty$ and is optimal if either $1 < p \le q < \infty$, $r > \frac{1}{p} - \frac{1}{q}$ or $2 \le q < p < \infty$, $r > 0$. In the case of “small” smoothness (25) $\mathcal{T}_n$ is not optimal in both cases $2 \le q < p < \infty$ and $1 < q \le 2 < p < \infty$.

Next statement shows that in the case of “super-small” smoothness the asymptotic behaviour of Bernstein widths changes and the sequence $\mathcal{T}_n$ remains not optimal if $2 \le q < p < \infty$.

Theorem 2. *Let $\lambda(k)$, $k \in \mathbb{N}$, be a sequence of positive numbers which is decreasing for $k \ge N$ for some N and satisfies the following conditions: $\lim_{k\to\infty}\lambda(k) = 0$ and $\lambda(k^s) \asymp \lambda(k)$ for any fixed $s > 0$. Let*

$$K(x) \sim \sum_{k=1}^{\infty} \lambda(k)\cos kx$$

be the associated kernel. Then

$$\mathcal{E}_n\left(K * U_p, L_q\right) \asymp b_n\left(K * U_p, L_q\right) \asymp \lambda(n),\ \ 2 \le q < p < \infty$$

and the sequence of subspaces $\mathcal{T}_n$ of trigonometric polynomials in the “usual” order is not optimal.

Proof. Let $2 \le q < p < \infty$. Since $b_n\left(K * U_p, L_q\right) \le d_n\left(K * U_p, L_q\right)$, the upper bounds follow from (8),

$$b_n\left(K * U_p, L_q\right) \ll \lambda(n).$$

We turn to the lower bounds now. From the definition of Bernstein n-widths and (19) we get

$$b_n\left(K * U_p, L_q\right) \ge b_n\left(K * U_p \cap \mathcal{T}_m, L_q \cap \mathcal{T}_m\right)$$

$$\ge \lambda(m)\, b_n\left(U_p \cap \mathcal{T}_m, L_q \cap \mathcal{T}_m\right) \tag{26}$$

for a given $m > n$. Let $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{k=1}^n x_k y_k$ be the canonic scalar product of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and $|\mathbf{x}| = \langle \mathbf{x}, \mathbf{x}\rangle^{1/2}$. Let $\mathbb{S}^{n-1} = \{\mathbf{x}\,|\,\mathbf{x} \in \mathbb{R}^n,\ |\mathbf{x}| = 1\}$ be the unit sphere with the normalized invariant surface measure $d\mu$. Denote by $E = (\mathbb{R}^n, \|\cdot\|)$ a Banach space with the norm $\|\cdot\|$. Let $\mathrm{I} : \mathbb{R}^{2n} \longrightarrow \mathcal{T}_{2n}$ be the coordinate isomorphism that assigns to $\mathbf{a} = (a_1, \ldots, a_{2n}) \in \mathbb{R}^{2n}$ the polynomial $\mathrm{I}\mathbf{a} = t^{\mathbf{a}}_n(\cdot) = \sum_{k=1}^n a_k \cos k(\cdot) + a_{n+k}\sin k(\cdot)$. The definition $\|\mathbf{a}\|_{(p)} := \left\|t^{\mathbf{a}}_n(\cdot)\right\|_p$ induces a norm on $\mathbb{R}^{2n}$. We will get an upper bound for the expectation,

$$\mathbf{E}\left[\|\cdot\|\right] = \int_{\mathbb{S}^{2n-1}} \|\mathbf{a}\|\, d\mu(\mathbf{a}),$$

or the Lévy mean, of the function $\|\cdot\| : \mathbb{S}^{2n-1} \longrightarrow \mathbb{R}_+$ in the case $\|\cdot\| = \|\cdot\|_{(p)}$, $p \ge 2$ with respect to the normalized invariant measure $d\mu$. Let $r_k(\theta)$, $\theta \in (0, 1)$, $k \in \mathbb{N}$ be the Rademacher functions. Since $d\mu$ is invariant on $\mathbb{S}^{2n-1}$ then for any $\theta \in (0, 1)$

$$\mathbf{E}\left[\|\cdot\|_{(p)}\right] = \int_{\mathbb{S}^{2n-1}} \|\mathrm{I}\mathbf{a}\|_p\, d\mu(\mathbf{a}) = \int_{\mathbb{S}^{2n-1}} \left\|t^{\mathbf{a}}_n(\cdot)\right\|_p d\mu(\mathbf{a})$$

$$= \int_{\mathbb{S}^{2n-1}} \left(\int_{\mathbb{T}} \left|\sum_{k=1}^n a_k \cos k\tau + a_{n+k}\sin k\tau\right|^p d\tau\right)^{\frac{1}{p}} d\mu(\mathbf{a})$$

$$= \int_{\mathbb{S}^{2n-1}} \left(\int_{\mathbb{T}} \left| \sum_{k=1}^{n} r_k(\theta) a_k \cos k\tau + r_{n+k}(\theta) a_{n+k} \sin k\tau \right|^p d\tau \right)^{\frac{1}{p}} d\mu(\mathbf{a})$$

$$= \int_0^1 \int_{\mathbb{S}^{2n-1}} \left(\int_{\mathbb{T}} \left| \sum_{k=1}^{n} r_k(\theta) a_k \cos k\tau + r_{n+k}(\theta) a_{n+k} \sin k\tau \right|^p d\tau \right)^{\frac{1}{p}} d\mu(\mathbf{a})\, d\theta.$$

Hence, by Jensen inequality and Fubini theorem,

$$\mathbf{E}\left[\|\cdot\|_{(p)}\right]$$

$$\leq \left(\int_{\mathbb{S}^{2n-1}} \int_{\mathbb{T}} \int_0^1 \left| \sum_{k=1}^{n} r_k(\theta) a_k \cos k\tau + r_{n+k}(\theta) a_{n+k} \sin k\tau \right|^p d\theta d\tau d\mu(\mathbf{a}) \right)^{\frac{1}{p}}.$$

Applying Khinchin inequality [17, p.41],

$$\left(\int_0^1 \left| \sum_{k=1}^{n} c_k r_k(\theta) \right|^p d\theta \right)^{\frac{1}{p}} \leq C(p) \left(\sum_{k=1}^{n} c_k^2 \right)^{\frac{1}{2}},$$

where

$$C(p) := 2^{\frac{1}{2}} \left(\frac{\Gamma\left(\frac{1+p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \right)^{\frac{1}{p}}$$

we obtain

$$\mathbf{E}\left[\|\cdot\|_{(p)}\right] \leq C(p) \left(\int_{\mathbb{S}^{2n-1}} \int_{\mathbb{T}} \left(\sum_{k=1}^{n} a_k^2 \cos^2 k\tau + a_{n+k}^2 \sin^2 k\tau \right)^{\frac{p}{2}} d\tau d\mu(\mathbf{a}) \right)^{\frac{1}{p}}$$

$$< C(p) \left(\int_{\mathbb{S}^{2n-1}} \left(\sum_{k=1}^{n} a_k^2 + a_{n+k}^2 \right)^{\frac{p}{2}} d\mu(\mathbf{a}) \right)^{\frac{1}{p}} = C(p) \left(\int_{\mathbb{S}^{2n-1}} |\mathbf{a}|^p d\mu(\mathbf{a}) \right)^{\frac{1}{p}}$$

$$= C(p) \left(\int_{\mathbb{S}^{2n-1}} d\mu(\mathbf{a}) \right)^{\frac{1}{p}} = 2^{\frac{1}{2}} \left(\frac{\Gamma\left(\frac{1+p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \right)^{\frac{1}{p}} \asymp p^{\frac{1}{2}}. \tag{27}$$

Let X_m be an m-dimensional Banach space with the norm $\|\cdot\|$ and $|\mathbf{x}| \leq \|\mathbf{x}\| \leq b\,|\mathbf{x}|$ for any $\mathbf{x} \in X$. Then there is a subspace

$$Y_n \subset X_m,\ \dim Y_n = n \geq \left[C_{X_m} m \left(\mathbf{E}\left[\|\cdot\|\right] \right)^2 b^{-2} \right] \tag{28}$$

such that $\|\mathbf{x}\| \leq C_X |\mathbf{x}|$ for all $\mathbf{x} \in Y_n$ [2]. In particular, let $\|\mathbf{x}\| = \|\mathbf{x}\|_{(p)}$, then $b = Cm^{\frac{1}{2}-\frac{1}{p}}$ and by (27) $\mathbf{E}\left[\|\cdot\|_{(p)}\right] < Cp^{\frac{1}{2}}$. Hence, by (28) there is such subspace $Y_n \subset \left(\mathbb{R}^{2m},\ \|\mathbf{x}\|_{(p)}\right)$, $\dim Y_n = n = \left[C_p m^{\frac{2}{p}} \right]$ that $\|\mathbf{x}\|_{(p)} \leq |\mathbf{x}|$ for all $\mathbf{x} \in Y_k$ or there is $L_n = \mathrm{I} Y_n \subset \mathcal{T}_m$ such that $\|t_m\|_p \leq \|t_m\|_2$ for any $t_m \in L_n$. This implies

$$b_n\left(U_p \cap \mathcal{T}_m, L_2 \cap \mathcal{T}_m\right) \geq C_p.$$

Consequently, from (26) we get $b_n\left(K * U_p, L_2\right) \geq \lambda(m)\, C_p$ and by embedding

$$b_n\left(K * U_p, L_q\right) \geq C_{p,q} \lambda(m),\ 2 \leq q < p < \infty,$$

or

$$b_n\left(K * U_p, L_q\right) \geq C_{p,q} \lambda\left(C_p n^{\frac{p}{2}}\right) \gg \lambda(n),\ 2 \leq q < p < \infty.$$

Finally, applying (3) for the multiplier sequence (18) we get

$$\|\Lambda_n^{-1}t_n\|_p \leq \frac{C_p}{\lambda(n)}\|t_n\|_p \leq \frac{C_{p,q}n^{\frac{1}{q}-\frac{1}{p}}}{\lambda(n)}\|t_n\|_q \tag{29}$$

for any $t_n \in \mathcal{T}_n$ or $C_{p,q}\lambda(n)n^{\frac{1}{p}-\frac{1}{q}}U_q \cap \mathcal{T}_n \subset K * U_p$. Observe that (29) is sharp in the sense of order. Hence $\mathcal{T}_n$ is not optimal if $2 \leq q < p < \infty$. ■

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