Applied Mathematics and Computation 208 (2009) 238–248

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Third order differential equations with fixed critical points

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ARTICLE INFO

Keywords:
Differential equations in complex domain
Painlevé property
Painlevé equations
Singular point analysis
Painlevé test
Fuches indices

ABSTRACT

The singular point analysis of third order ordinary differential equations which are algebraic in y and y' is presented. Some new third order ordinary differential equations that pass the Painlevé test as well as the known ones are found.

1. Introduction

Painlevé and his school addressed a question raised by E. Picard concerning a second order first degree ordinary differential equation of the form

$$y'' = F(z, y, y'), \tag{1.1}$$

where F is rational in y', algebraic in y and locally analytic in z and has the property that singularities other than poles of any of the solutions are fixed [1–5]. This property is known as the Painlevé property. Within the Möbius transformation, there are fifty such equations, and six of them are irreducible and define classical Painlevé transcendents $P_{\mathrm{I}} - P_{\mathrm{VI}}$.

The first order first degree equation, which has the Painlevé property, is the Riccati equation. Before the work of Painlevé and his school, Fuchs (see [4]) considered the equation of the form

$$F(z, y, y') = 0, \tag{1.2}$$

where F is polynomial in y and y' and locally analytic in z, such that the movable branch points are absent, that is, the generalization of the Riccati equation. Briot and Bouquet (see [4]) considered the subcase of (1.2), that is, first order binomial equations of degree $m \in \mathbb{Z}_+$:

$$(y')^m + F(z, y) = 0, \tag{1.3}$$

where $F(z, y)$ is a polynomial of degree at most $2m$ in y. It was found that there are six types of equations of the form (1.3). All of these equations, however, are either reducible to a linear equation or solvable by means of elliptic functions [4]. Second order binomial-type equations of degree $m \geqslant 3$

$$(y'')^m + F(z, y, y') = 0, \tag{1.4}$$

where F is polynomial in y and y' and locally analytic in z, were considered by Cosgrove [6]. He found nine such classes. Only two of these classes have arbitrary degree m and the others have degree three, four and six. All nine classes are solvable in terms of the first, second and fourth Painlevé transcendents, elliptic functions or by means of quadratures.

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doi:10.1016/j.amc.2008.11.044

Second order second-degree Painlevé type equations of the following form

$$(y'')^2 = E(z,y,y')y'' + F(z,y,y'), \tag{1.5}$$

where E and F are assumed to be rational in y, y' and locally analytic in z were the subject of the articles [7–11]. In [7,8], the special form, $E = 0$, and hence F is polynomial in y and y' of (1.5) was considered. In addition, in this case, no new Painlevé type equation was discovered, since all of them can be solved either in terms of the known functions or one of the six Painlevé equations. In [9–11], it was shown that all the second-degree equations obtained in [7,8], and some of the new second-degree equations such that $E \neq 0$ can be obtained from $P_{\mathrm{I}} - P_{\mathrm{VI}}$ by using the Riccati and Fuchsian type transformations, both of which preserve the Painlevé property.

Chazy [12], Garnier [13] and Bureau [14] considered the third order differential equations possessing the Painlevé property of the following form

$$y''' = F(z,y,y',y''), \tag{1.6}$$

where F is assumed to be rational in y, y', y'' and locally analytic in z. In [14], the special form of $F(z,y,y',y'')$

$$F(z,y,y',y'') = f_1(z,y)y'' + f_2(z,y)(y')^2 + f_3(z,y)y' + f_4(z,y), \tag{1.7}$$

where $f_k(z,y)$ are polynomials in y of degree k with analytic coefficients in z, was considered. In this class, no new Painlevé transcendents were discovered, and all of them were solvable either in terms of the known functions or one of the six Painlevé transcendents. The case in which F is a polynomial in y and its derivatives was also investigated in [15,16]. Eq. (1.6) with F analytic in z and rational in its other arguments, was considered in [17–21]. Fourth and higher order equations with the Painlevé property were investigated in many articles [14–16,22–32]. Kudryashov [23], Clarkson et al. [33], and Gordoa et al. [34,35] obtained first, second and fourth Painlevé hierarchy, by using the non-isospectral scattering problems. In [35] the associated linear equations (Lax pairs) for the second and fourth Painlevé hierarchies are given.

In this article, we consider the following equation

$$y'''(\delta y' + \alpha y^2) = \beta y''^2 + a_1 yy'y'' + a_2 y^3 y'' + a_3 y'^3 + a_4 y^2 y'^2 + a_5 y^4 y' + a_6 y^6 + a_7 y^2 y'' + a_8 yy'^2 + a_9 y^4 + a_{10} y^2 y' + a_{11} y^3. \tag{1.8}$$

and determine the coefficients α, β, δ and $a_i, i = 1, 2, \ldots, 11$ by using the Painlevé ODE test, singular point analysis. Singular point analysis is an algorithm introduced by Ablowitz et al. [36,37] to test whether a given ordinary differential equation satisfies the necessary conditions to be of Painlevé type. Some special cases of (1.8) were studied in the literature. Incomplete investigation of the case $\beta = \delta = 0$ was given in [17], where some of the equations were incorrectly stated as being of Painlevé type. The case of $\beta = \delta = 0$ was also considered in [21]. In [18] a special case of (1.8), and only for the leading order $m = -1$ as $z \to z_0$ was considered.

If we let $z \to z_0 + \epsilon z$ and take the limit as $\epsilon \to 0$, (1.8) yields the following *reduced* equation

$$\delta y' y''' = \beta y''^2; \tag{1.9}$$

without loss of generality, one can take $\delta = 1$. If one lets $v = y'/y$, then (1.9) yields

$$v'' = \beta \frac{v'^2}{v} + (2\beta - 3)vv' + (\beta - 1)v^3. \tag{1.10}$$

Eq. (1.10) was considered by Painlevé (see [4]) and Bureau [5], and it was shown that β should be either 1 or $(\eta - 1)/\eta$, $\eta \in \mathbb{Z} - \{-1, 0\}$.

Substituting

$$y = y_0(z - z_0)^m, \quad \text{as } z \to z_0, \ m \in \mathbb{Z}, \tag{1.11}$$

where z_0 is arbitrary into (1.8), for certain values of m, two or more terms may balance (depending on y_0), and the rest can be ignored as $z \to z_0$. For each choice of m, the terms that can balance are called *leading terms*. In the following sections, the *simplified* equations that retain only leading terms as $z \to z_0$ will be considered for $m = -1, -2, -3$ and -4 with distinct Fuchs indices (resonances). For all cases of m, we search for the existence of at least one *principal branch* (a branch that has two positive distinct integer resonances, except $r_0 = -1$). In the case of $m = -1$, it is possible to find at least one principal branch. There is no principal branch when $m = -3, 4$ and for certain cases of $m = -2$. In cases where there is no principal branch, we consider the *maximal branch* [38,39] (a branch that has two distinct integer resonances, except $r_0 = -1$), but for all cases, the compatibility conditions at the positive resonances are identically satisfied.

2. Leading order $m = -1$

For $m = -1$, the simplified equation is

$$y'''(\delta y' + \alpha y^2) = \beta y''^2 + a_1 yy'y'' + a_2 y^3 y'' + a_3 y'^3 + a_4 y^2 y'^2 + a_5 y^4 y' + a_6 y^6. \tag{2.1}$$

Two cases $\beta = 0$ and $\beta \neq 0$ should be considered separately.

I. $\beta = 0$:

For $\beta = 0$, reduced Eq. (1.9) implies that $\delta = 0$. Hence, if $\alpha = 0$, (2.1) reduces to the second order equation of the following form:

$$y'' = F(y, y', z), \tag{2.2}$$

where F is a rational function in y', algebraic in y with analytic coefficients in z. Eq. (2.2) was considered by Painlevé and his school [1,3–5].

If $\alpha \neq 0$, the simplified equation is

$$y^2y''' = a_1yy'y'' + a_2y^3y'' + a_3y'^3 + a_4y^2y'^2 + a_5y^4y' + a_6y^6. \tag{2.3}$$

Eq. (2.3) was investigated by Exton [17], Martynov [19], and Mugan and Jrad [21].

II. $\beta \neq 0$:

II.a. If $\delta = 0$, and $\alpha = 0$, (2.1) can be written as

$$y''^2 = A(y, y', z)y'' + B(y, y', z), \tag{2.4}$$

where A and B are assumed to be rational in y and y' and locally analytic in z. Eq. (2.4) was considered by Bureau [7], Cosgrove and Scoufis [8], and Sakka and Mugan [9–11].

II.b. For $\delta = 0$, and $\alpha \neq 0$, there is no equation that possesses the Painlevé property [12].

II.c. If $\delta \neq 0, \alpha = 0$, without loss of generality, we can take $\delta = 1$. Substituting

$$y \cong y_0(z - z_0)^{-1} + \kappa(z - z_0)^{r-1}, \tag{2.5}$$

where $y_0 \neq 0$, into (2.1), we obtain the following equations for the Fuchs indices r and y_0

$$(r + 1)[r^2 + (a_2y_0^2 - a_1y_0 + 4\beta - 7)r + a_5y_0^3 - 2(a_4 + 2a_2)y_0^2 + 3(2a_1 + a_3)y_0 - 8(2\beta - 3)] = 0, \tag{2.6}$$

$$a_6y_0^4 - a_5y_0^3 + (a_4 + 2a_2)y_0^2 + (6\alpha - 2a_1 - a_3)y_0 + 2(2\beta - 3) = 0, \tag{2.7}$$

respectively. Eq. (2.7) implies that there are four branches if $a_6 \neq 0$. Now, we determine $y_{0j}, j = 1, \ldots, 4$ and $a_i, i = 1, \ldots, 6$, such that at least one branch is the principal branch.

If we let

$$P(y_{0j}) = \prod_{k=1}^{3} r_{jk} = a_5y_{0j}^3 - 2(a_4 + 2a_2)y_{0j}^2 + 3(2a_1 + a_3)y_{0j} - 8(2\beta - 3), \tag{2.8}$$

$j = 1, 2, 3, 4$. Then $P(y_{0j}) = P_j$ satisfy the following Diophantine equation:

$$\sum_{j=1}^{4} \frac{1}{P_j} = -\frac{1}{2(2\beta - 3)} = \frac{\eta}{2(2 + \eta)}. \tag{2.9}$$

Depending on the number of branches, we have the following subcases:

II.c.i. In the case of the single branch, that is $a_5 = a_6 = 2a_2 + a_4 = 0$, Eqs. (2.8) and (2.7) give

$$P_1 = \prod_{k=1}^{3} r_{1k} = 2\left(1 + \frac{2}{\eta}\right), \quad (2a_1 + a_3)y_{01} = -2\left(1 + \frac{2}{\eta}\right), \tag{2.10}$$

respectively. Thus, η must divide 4, and we obtain the following equations:

$$\begin{aligned} &y'y''' = \tfrac{3}{2}y''^2, && (r_{11}, r_{12}) = (0, 1), \\ &y'y''' = \tfrac{1}{2}y''^2 + 4y'^3, && (r_{11}, r_{12}) = (1, 4), \\ &y'y''' = \tfrac{3}{4}y''^2 + 3y'^3, && (r_{11}, r_{12}) = (1, 3), \\ &y'y''' = \tfrac{1}{2}y''^2 + 2y'^3, && (r_{11}, r_{12}) = (1, 2). \end{aligned} \tag{2.11}$$

By letting $y' = w$, (2.11).a-c can be reduced to second order equations, which all possess the Painlevé property [4]. By differentiating once and letting $y' = w$, (2.11).d yields a third order equation, which has the Painlevé property [4].

II.c.ii. If $a_5 = a_6 = 0$ and $2a_2 + a_4 \neq 0$, then there are two branches. In this case $y_{0j}, j = 1, 2$ satisfies the following equation:

$$(a_4 + 2a_2)y_0^2 - (2a_1 + a_3)y_0 + 2(2\beta - 3) = 0, \tag{2.12}$$

and the resonances $r_{jk}, r_{jk}, j, k = 1, 2$ satisfy

$$r^2 + (a_2y_{0j}^2 - a_1y_{0j} + 4\beta - 7)r_1 + P(y_{0j}) = 0, \quad j = 1, 2. \tag{2.13}$$

From (2.12), one has

$$2a_2 + a_4 = \frac{(4\beta - 6)}{y_{01}y_{02}}, \quad (2a_1 + a_3) = \frac{(4\beta - 6)(y_{01} + y_{02})}{y_{01}y_{02}}, \tag{2.14}$$

and thus

$$P_1 = -(4\beta - 6)\left(1 - \frac{y_{01}}{y_{02}}\right), \quad P_2 = -(4\beta - 6)\left(1 - \frac{y_{02}}{y_{01}}\right). \tag{2.15}$$

For $P_1 P_2 \neq 0$, P_j satisfy the following Diophantine equation:

$$\frac{1}{P_1} + \frac{1}{P_2} = \frac{\eta}{2(\eta + 2)}. \tag{2.16}$$

For $\eta = 5, \infty$, there are no equations passing the Painlevé test. For $\eta = 2, 3, 4$, the following equations pass the Painlevé test. For $\eta = 2$:

$$\begin{aligned}
&y'y''' = \frac{1}{2}y''^2 + 3yy'y'' + 6y'^3 + 2y^3y'' - 12y^2y'^2, \\
&(r_{11}, r_{12}) = (1, 2), \quad (r_{21}, r_{22}) = (-2, 2). \\
&y'y''' = \frac{1}{2}y''^2 + \frac{5}{6}yy'y'' + \frac{10}{3}y'^3 + \frac{1}{6}y^3y'' - \frac{4}{3}y^2y'^2, \\
&(r_{11}, r_{12}) = (1, 3), \quad (r_{21}, r_{22}) = (-4, 3). \\
&y'y''' = \frac{1}{2}y''^2 + yy'y'' + 3y'^3 - y^2y'^2, \\
&(r_{11}, r_{12}) = (1, 3), \quad (r_{21}, r_{22}) = (-3, 4). \\
&y'y''' = \frac{1}{2}y''^2 + \frac{1}{3}yy'y'' - \frac{8}{3}y'^3 - \frac{1}{3}y^3y'' + \frac{8}{3}y^2y'^2, \\
&(r_{11}, r_{12}) = (2, 3), \quad (r_{21}, r_{22}) = (3, 4).
\end{aligned} \tag{2.17}$$

For $\eta = 3$:

$$\begin{aligned}
&y'y''' = \frac{2}{3}y''^2 + \frac{1}{3}yy'y'' + 3y'^3 - \frac{1}{3}y^2y'^2, \\
&(r_{11}, r_{12}) = (1, 3), \quad (r_{21}, r_{22}) = (-5, 6).
\end{aligned} \tag{2.18}$$

For $\eta = 4$:

$$\begin{aligned}
&y'y''' = \frac{3}{4}y''^2 + 5yy'y'' - 10y'^3 + 4y^2y'' - 5y^2y'^2, \\
&(r_{11}, r_{12}) = (2, 3), \quad (r_{21}, r_{22}) = (-2, -3). \\
&y'y''' = \frac{3}{4}y''^2 + \frac{1}{2}yy'y'' - 3y'^3 - \frac{1}{2}y^2y'' + 2y^2y'^2, \\
&(r_{11}, r_{12}) = (1, 4), \quad (r_{21}, r_{22}) = (3, 4). \\
&y'y''' = \frac{3}{4}y''^2 - y^3y'' + 5y^2y'', \\
&(r_{11}, r_{12}) = (r_{21}, r_{22}) = (2, 4).
\end{aligned} \tag{2.19}$$

II.c.iii. If $a_6 = 0$ and $a_5 \neq 0$, there are three branches corresponding to the roots $y_{0j}, j = 1, 2, 3$ of (2.7). Similar to the previous case, P_j satisfy the following Diophantine equation:

$$\sum_{j=1}^{3} \frac{1}{P_j} = \frac{\eta}{2(\eta + 2)}, \tag{2.20}$$

and if $\prod P_j \neq 0$, then

$$\prod_{j=1}^{3} P_j = \left[\frac{2(\eta + 2)}{\eta}\right]^3 \frac{[(y_{01} - y_{02})(y_{01} - y_{03})(y_{02} - y_{03})]^2}{(y_{01}y_{02}y_{03})^2}. \tag{2.21}$$

As an example, we obtain the following equations, which have at least one principal branch and pass the Painlevé test for $\eta = 2, 3, 4, 5$ and all the equations for $\eta = \infty$. For $\eta = 2$:

$$\begin{aligned}
&y'y''' = \frac{1}{2}y''^2 - yy'y'' - y^3y'' + y'^3 + 6y^2y'^2 + y^4y', \\
&(r_{11}, r_{12}) = (2, 3), \quad (r_{21}, r_{22}) = (2, 5), \quad (r_{31}, r_{32}) = (-3, 20).
\end{aligned} \tag{2.22}$$

For $\eta = 3$:

$$y'y''' = \frac{2}{3}y''^2 + \frac{10}{3}yy'y'' + \frac{2}{3}y^3y'' - \frac{23}{3}y'^3 + \frac{2}{3}y^2y'^2 - \frac{1}{3}y^4y',$$
$$(r_{11}, r_{12}) = (1, 6), \quad (r_{21}, r_{22}) = (1, -30), \quad (r_{31}, r_{32}) = (-2, -3). \tag{2.23}$$

For $\eta = 4$:

$$y'y''' = \frac{3}{4}y''^2 + (6 - 2\sqrt{6})yy'y'' + (5 - 2\sqrt{6})y^3y'' - \left(\frac{63 - 17\sqrt{6}}{5}\right)y'^3 - \left(\frac{31 - 14\sqrt{6}}{5}\right)y^2y'^2 + \left(\frac{7 - 3\sqrt{6}}{5}\right)y^4y',$$
$$(r_{11}, r_{12}) = (1, 4), \quad (r_{21}, r_{22}) = (1, -12), \quad (r_{31}, r_{32}) = (-2, -3). \tag{2.24}$$

For $\eta = 5$:

$$y'y''' = \frac{4}{5}y''^2 + \frac{28}{5}yy'y'' + \frac{16}{5}y^3y' + \left(\frac{234 + 80\sqrt{5}}{5}\right)y'^3 + \left(\frac{336 + 160\sqrt{5}}{5}\right)y^2y'^2 + \left(\frac{176 + 80\sqrt{5}}{5}\right)y^4y',$$
$$(r_{11}, r_{12}) = (-2, -3), \quad (r_{21}, r_{22}) = (1, -7), \quad (r_{31}, r_{32}) = (1, 3). \tag{2.25}$$

For $\eta = \infty$:

$$y'y''' = y''^2 + 2yy'y'' - y^3y'' - 4y'^3 + 2y^2y'^2 + \frac{1}{4}y^4y',$$
$$(r_{11}, r_{12}) = (1, 6), \quad (r_{21}, r_{22}) = (-3, -2), \quad (r_{31}, r_{32}) = (1, 6).$$
$$y'y''' = y''^2 - 2iyy'y'' - 2(1 + i)y^3y'' + \frac{2 + 24i}{5}y'^3 + \frac{24 + 28i}{5}y^2y'^2 - \frac{4 + 8i}{5}y^4y',$$
$$(r_{11}, r_{12}) = (2, 3), \quad (r_{21}, r_{22}) = (-3, -4), \quad (r_{31}, r_{32}) = (1, 4). \tag{2.26}$$

II.c.iv. If $a_6 \neq 0$, then there are four branches corresponding to the roots $y_{0j}, j = 1, \ldots, 4$ of (2.7), and product P_j of the resonances for each branch satisfy the following Diophantine equation:

$$\sum_{j=1}^{4} \frac{1}{P_j} = \frac{\eta}{2(\eta + 2)}. \tag{2.27}$$

For $\eta = 3, 4, \infty$, there are no equations that pass the Painlevé test; for $\eta = 2$ we obtain the following equations:

$$y'y''' = \frac{1}{2}y''^2 - \frac{5}{3}y'^3 + \frac{5}{2}y^2y' - \frac{1}{6}y^6,$$
$$y'y''' = \frac{1}{2}y''^2 + 5y^2y' - y^6, \tag{2.28}$$

with the resonances $(r_{11}, r_{12}) = (2, 3), (r_{21}, r_{22}) = (-2, 7), (r_{31}, r_{32}) = (12, -7), \quad (r_{41}, r_{42}) = (2, 3)$, and $(r_{11}, r_{12}) = (2, 3)$, $(r_{21}, r_{22}) = (2, 3), (r_{31}, r_{32}) = (-3, 8), (r_{41}, r_{42}) = (-3, 8)$ respectively. Differentiating the Eq. (2.28) once gives the following equations

$$y^{(4)} = -5y'y'' + 5y^2y'' + 5yy'^2 - y^5,$$
$$y^{(4)} = 10y^2y'' + 10yy'^2 - 6y^5, \tag{2.29}$$

respectively. Eq. (2.29) were considered in [14,16,27,32]. For $\eta = 5$:

$$y'y''' = \frac{4}{5}y''^2 + \frac{3}{10}yy'y'' + \frac{1}{10}y^3y'' - \frac{113}{36}y'^3 + \frac{7}{60}y^2y'^2 + \frac{1}{20}y^4y' - \frac{1}{180}y^6,$$
$$(r_{11}, r_{12}) = (1, 3), \quad (r_{21}, r_{22}) = (-7, 8),$$
$$(r_{31}, r_{32}) = (-2, -3), \quad (r_{41}, r_{42}) = (1, -8). \tag{2.30}$$

II.d. $\delta \neq 0$ and $\alpha \neq 0$. Without loss of generality, we may choose $\delta = 1$, then the simplified equation becomes

$$y'''(y' + \alpha y^2) = \beta y''^2 + a_1yy'y'' + a_2y^3y'' + a_3y'^3 + a_4y^2y'^2 + a_5y^4y' + a_6y^6. \tag{2.31}$$

A special case of (2.31) was investigated in [18]. In this case, Fuchs indices satisfy the following equation:

$$(r + 1)[(1 - \alpha y_0)r^2 + H(y_0)r + G(y_0)] = 0, \tag{2.32}$$

where

$$H(y_0) = a_2y_0^2 + (7\alpha - a_1)y_0 + (4\beta - 7),$$
$$G(y_0) = a_5y_0^3 - 2(a_4 + 2a_2)y_0^2 - 3(6\alpha - 2a_1 - a_3)y_0 - 8(2\beta - 3), \tag{2.33}$$

and y_0 satisfies

$$a_6y_0^4 - a_5y_0^3 + (a_4 + 2a_2)y_0^2 + (6\alpha - 2a_1 - a_3)y_0 + 2(2\beta - 3) = 0. \quad (2.34)$$

Without loss of generality, one may assume that one of the roots y_{01} of (2.34) is -1. Then, the equation for the resonances corresponding to the branch $y_{01} = -1$ reads

$$(r_1 + 1)[(\alpha + 1)r_1^2 + Mr_1 - N] = 0, \quad (2.35)$$

where $M = H(-1)$, and $N = G(-1)$. In the following subsections, we consider the case of $\alpha = -1$.

II.d.i $\alpha = -1$, and $M = N = 0$: The Eq. (2.31) takes the form of

$$y''' = \beta\frac{(y'' - 2yy')^2}{y' - y^2} + c_1yy'' + c_2y'^2 + c_3y'y^2 + c_4y^4. \quad (2.36)$$

This case was considered in detail by Martynov [18]. The following equation was given in [18] (see eq. 41)):

$$y''' = \frac{(y'' - 2yy')^2}{y' - y^2} + 4yy'' - 2y'^2, \quad (2.37)$$

but, it was incorrectly stated that the equation has a moving singular point. Let y be,

$$2y = \frac{d}{dz}\left[Log\left(\frac{v'}{v(v-1)}\right)\right] = \frac{v''}{v'} - \left(\frac{1}{v} + \frac{1}{v-1}\right)v', \quad (2.38)$$

where $v(z)$ is the general solution of the Schwartzian ordinary differential equation [4]

$$v'v''' = \frac{3}{2}v''^2 - \frac{1}{2}\left[\frac{1}{v^2} + \frac{1}{(v-1)^2} - \frac{1}{v(v-1)}\right]v'^4, \quad (2.39)$$

or

$$-\left[\frac{v'''}{v'} - \frac{3}{2}\left(\frac{v''}{v'}\right)^2\right]\frac{1}{v'^2} = \frac{1}{2}\left[\frac{1}{v^2} + \frac{1}{(v-1)^2} - \frac{1}{v(v-1)}\right] \equiv \frac{1}{2}I(v). \quad (2.40)$$

By letting $' = d/dz$, and $^* = d/dv$, the Schwartzian ordinary differential Eq. (2.39) can be reduced to the hypergeometric equation

$$\left[\frac{z^{***}}{z^*} - \frac{3}{2}\left(\frac{z^{**}}{z^*}\right)\right] = \frac{1}{2}I(v) = \{z, v\}. \quad (2.41)$$

By setting $W(v) = z^{**}/z^*$ one gets the following Riccati equation

$$\frac{dW}{dv} = \frac{1}{2}W^2 + \frac{1}{2}I(v). \quad (2.42)$$

If we let $W(v) = -2w^*/w$, then (2.42) yields the following linear equation for w:

$$v(v-1)w^{**} + (2v-1)w^* + \frac{1}{4}w = 0. \quad (2.43)$$

Hence, (2.43), and consequently, (2.42), (2.39) and (2.37) have the Painlevé property.

II.d.ii. $\alpha = -1$, and $(M,N) \neq (0,0)$: In this case, we consider the Eq. (2.31), which has at least one principal branch, the other branches may be *non-maximal* branches [39] (a branch that has less than two distinct integer resonances, except $r_0 = -1$).

If $a_5 = a_6 = 0$ and $2a_2 + a_4 = 0$, with the choice of $y_0 = -1$, then there is only one branch that is non-maximal, and the corresponding equation is

$$y'''(y' - y^2) = \frac{1}{2}y''^2 - 2y'^3, \quad r_1 = -2. \quad (2.44)$$

Note that $r_0 = -1$ is a root of (2.32).

If $a_5 = a_6 = 0$ and $2a_2 + a_4 \neq 0$, then there are two branches, one of which is a non-maximal branch, and we have the following equations:

$$\begin{aligned} &y'''(y' - y^2) = \frac{1}{2}y''^2 - yy'y'' + y^3y' - 4y'^3 + 2y^2y'^2, \\ &r_{11} = -4, \quad (r_{21}, r_{22}) = (1, 4), \\ &y'''(y' - y^2) = y''^2 - 3yy'y'' + y''y^3, \\ &r_{11} = -2, \quad (r_{21}, r_{22}) = (1, 4). \end{aligned} \quad (2.45)$$

If $a_6 = 0, a_5 \neq 0$, then there is one non-maximal branch corresponding to $y_{01} = -1$ and two maximal branches, one of which is a principal branch corresponding to $y_{0j}, j = 2, 3$. For $\eta = 2, 4, \infty$, we obtain the following equations:

For $\eta = 2$:

$$
\begin{aligned}
&y'''(y'-y^2) = \frac{1}{2}y''^2 - \frac{2}{3}yy'y'' - \frac{8}{3}y^3y'' + \frac{10}{3}y'^3 + \frac{28}{3}y^2y'^2 - 8y^4y', \\
&r_{11} = -6, \quad (r_{21}, r_{22}) = (3,4), \quad (r_{31}, r_{32}) = (2,3), \\
&y'''(y'-y^2) = \frac{1}{2}y''^2 - 2yy'y'' - 4\frac{8}{3}y^3y'' + 6y'^3 + 12y^2y'^2 - 8y^4y', \\
&r_{11} = -2, \quad (r_{21}, r_{22}) = (3,4), \quad (r_{31}, r_{32}) = (1,6), \\
&y'''(y'-y^2) = \frac{1}{2}y''^2 - \frac{14}{3}yy'y'' + \frac{28}{3}y^3y'' - \frac{38}{3}y'^3 - \frac{44}{3}y^2y'^2 + 16y^4y', \\
&r_{11} = -6, \quad (r_{21}, r_{22}) = (3,-4), \quad (r_{31}, r_{32}) = (1,3), \\
&y'''(y'-y^2) = \frac{1}{2}y''^2 - \frac{2n+3}{2n}yy'y'' + \frac{6n+1}{n^2}y^3y'' - \frac{3}{n}y'^3 - \frac{3}{n^2}y^2y'^2 + \frac{2}{n^2}y^4y', \quad n \in \mathbb{Z}, \\
&r_{11} = -4, \quad (r_{21}, r_{22}) = (-2,2), \quad (r_{31}, r_{32}) = (1,2).
\end{aligned} \tag{2.46}
$$

For $\eta = 4$:

$$
\begin{aligned}
&y'''(y'-y^2) = \frac{3}{4}y''^2 - 6\frac{n-1}{n}yy'y'' - 3\frac{2n-1}{n^2}y^3y'' + 3\frac{3n-4}{n}y'^3 + 3\frac{4n-1}{n^2}y^2y'^2 - \frac{3}{n^2}y^4y', \\
&r_{11} = 1, \quad (r_{21}, r_{22}) = (-3,-2), \quad (r_{31}, r_{32}) = (1,6), \\
&y'''(y'-y^2) = \frac{3}{4}y''^2 - 3\frac{n+1}{n}yy'y'' - 3\frac{n+3}{2n^2}y^3y'' + \frac{9}{n}y'^3 + 3\frac{n-6}{n^2}y^2y'^2 - \frac{9}{n^2}y^4y', \\
&r_{11} = -2, \quad (r_{21}, r_{22}) = (3,4), \quad (r_{31}, r_{32}) = (1,4), \\
&y'''(y'-y^2) = \frac{3}{4}y''^2 - 2\frac{n+1}{n}yy'y'' + 3\frac{2n+1}{n^2}y^3y'' + \frac{n-8}{n}y'^3 - 3\frac{15}{n^2}y^2y'^2 + \frac{9}{n^2}y^4y', \\
&r_{11} = -3, \quad (r_{21}, r_{22}) = (-3,2), \quad (r_{31}, r_{32}) = (1,2), \\
&y'''(y'-y^2) = \frac{3}{4}y''^2 - 2yy'y'' - \frac{1}{n^2}y^3y'' + y'^3 + \frac{5}{n^2}y^2y'^2 - \frac{3}{n^2}y^4y', \\
&r_{11} = -3, \quad (r_{21}, r_{22}) = (r_{31}, r_{32}) = (2,3),
\end{aligned} \tag{2.47}
$$

where $n \in \mathbb{Z}$.

For $\eta = \infty$:

$$
\begin{aligned}
&y'''(y'-y^2) = y''^2 - 7y'^3 + 12y^2y'^2 - 9y^4y', \\
&r_{11} = 1, \quad (r_{21}, r_{22}) = (-3,-2), \quad (r_{31}, r_{32}) = (1,3).
\end{aligned} \tag{2.48}
$$

In each of the above cases, the compatibility conditions at the positive resonances are identically satisfied. For $\eta = 3$ and 5, there are no equations that pass the Painlevé test.

If $a_6 \neq 0$, there are four branches. Similar to the previous case, one branch is non-maximal, and the others are maximal. We consider the cases in which one of the maximal branches is a principal branch. In this case, $P_j, j = 2,3,4$ satisfy the following equation:

$$
\frac{1}{P_2} + \frac{1}{P_3} + \frac{1}{P_4} = \frac{2(\eta+2)}{\eta}, \tag{2.49}
$$

where $P_j = r_{j1}r_{j2}$ and is given as

$$
\begin{aligned}
&P_2 = -a_6y_{02}(y_{02}-y_{03})(y_{02}-y_{04}), \quad P_3 = -a_6y_{03}(y_{03}-y_{02})(y_{03}-y_{04}), \\
&P_4 = -a_6y_{04}(y_{04}-y_{02})(y_{04}-y_{03}).
\end{aligned} \tag{2.50}
$$

As an example, we obtain the following equations with a principal branch for $\eta = 2,3,4$ and ∞.

For $\eta = 2$:

$$
\begin{aligned}
&y'''(y'-y^2) = \frac{1}{2}y''^2 + 30yy'y'' + 40y^3y'' - 51(y'^3 + y^2y'^2) - 15y^4y' - 25y_6, \\
&r_{11} = 1, \quad (r_{21}, r_{22}) = (-2,3), \quad (r_{31}, r_{32}) = (1,4), \quad (r_{41}, r_{42}) = (-2,-3).
\end{aligned} \tag{2.51}
$$

For $\eta = 3$:

$$
\begin{aligned}
&y'''(y'-y^2) = \frac{2}{3}y''^2 + \frac{433}{60}yy'y'' - \frac{13}{2}y^3y'' + \frac{1863}{20}y'^3 + \frac{1649}{20}y^2y'^2 + 540y^4y' - 225y_6, \\
&r_{11} = 5, \quad (r_{21}, r_{22}) = (1,3), \quad (r_{31}, r_{32}) = (-6,3), \quad (r_{41}, r_{42}) = (-3,-15).
\end{aligned} \tag{2.52}
$$

For $\eta = 4$:

$$y'''(y' - y^2) = \frac{3}{4}y''^2 + 9yy'y'' + 18y^3y'' - \frac{39}{2}(y'^3 + y^2y'^2) + \frac{9}{2}y^4y' - \frac{27}{2}y_6, \quad (2.53)$$
$$r_{11} = 1, \quad (r_{21}, r_{22}) = (-2, 3), \quad (r_{31}, r_{32}) = (1, 3), \quad (r_{41}, r_{42}) = (-2, -3).$$

For $\eta = \infty$:

$$y'''(y' - y^2) = y''^2 + 8y^3y'' - 6y'^3 - 6y^2y'^2 - 8y^6, \quad (2.54)$$
$$r_{11} = 1, \quad (r_{21}, r_{22}) = (1, 2), \quad (r_{31}, r_{32}) = (2, 3), \quad (r_{41}, r_{42}) = (-2, 3).$$

If we let $y = w'/2w$ in (2.54), and integrate once, then we obtain

$$kw''' = 2ww' - 3w'^2, \quad (2.55)$$

where k is an integration constant, and (2.55) was considered by Chazy [12].

3. Leading order $m = -2$

For leading order $m = -2$, there are two simplified equations corresponding to $\delta = 0$, and $\alpha = 0$:

$$\alpha y''' = a_1yy'y'' + a_3y'^3, \quad (3.1)$$
$$\delta y'y''' = \beta y''^2 + a_7y^2y'' + a_8yy'^2 + a_9y^4,$$

respectively. (3.1).a was studied in [15]. In this section, we consider (3.1).b. Without loss of generality, we may choose $\delta = 1$. Substituting

$$y \cong y_0(z - z_0)^{-2} + \kappa(z - z_0)^{r-2} \quad (3.2)$$

in (3.1).b gives the following equations of the resonances r and y_0

$$(r + 1)[2r^2 + (a_7y_0 + 12\beta - 20)r - (6a_7 + 4a_8)y_0 + 96 - 72\beta] = 0, \quad (3.3)$$
$$a_9y_0^2 + (6a_7 + 4a_8)y_0 - 48 + 36\beta = 0, \quad (3.4)$$

respectively. In general, there are two branches if $a_9 \neq 0$. Now, we determine $y_{0j}, j = 1, 2,$. According to the number of branches, the following cases should be considered separately.

I. $a_9 = 0$:

In this case, there is one branch, and if the resonances are r_1, r_2, then $r_1r_2 = 24 - 18\beta = 6 + \frac{18}{\eta}$. Thus, $\eta = \infty, \pm2, \pm3, \pm6, \pm9, \pm18$. The equations with a principal branch that pass the Painlevé test are as follows:

For $\eta = -2$, there is no principal branch, and we have the following equation with a maximal branch:

$$y'y''' = \frac{3}{2}y''^2 + 6y^2y'' - \frac{21}{2}yy'^2, \quad (r_1, r_2) = (-3, 1). \quad (3.5)$$

For $\eta = 2$:

$$y'y''' = \frac{1}{2}y''^2 - 18y^2y'' + \frac{69}{2}yy'^2, \quad (r_1, r_2) = (1, 15), \quad (3.6)$$
$$y'y''' = \frac{1}{2}y''^2 - 2y^2y'' + \frac{21}{2}yy'^2, \quad (r_1, r_2) = (3, 5).$$

For $\eta = -3$:

$$y'y''' = \frac{4}{3}y''^2, \quad (r_1, r_2) = (0, 2). \quad (3.7)$$

For $\eta = 3$:

$$y'y''' = \frac{2}{3}y''^2 - 14y^2y'' + 27yy'^2, \quad (r_1, r_2) = (1, 12), \quad (3.8)$$
$$y'y''' = \frac{3}{2}y''^2 - 4y^2y'' + 12yy'^2, \quad (r_1, r_2) = (2, 6),$$
$$y'y''' = \frac{2}{3}y''^2 - 2y^2y''^2 + 9yy'^2, \quad (r_1, r_2) = (3, 4).$$

For $\eta = -6$:

$$y'y''' = \frac{7}{6}y''^2 - 2y^2y'' + \frac{9}{2}yy'^2, \quad (r_1, r_2) = (1, 3). \quad (3.9)$$

For $\eta = 6$:

$$y'y''' = \frac{5}{6}y''^2 - 10y^2y'' + \frac{39}{2}yy'^2, \quad (r_1, r_2) = (1, 9). \quad (3.10)$$

For $\eta = -9$:

$$y'y''' = \frac{10}{9}y''^2 - \frac{10}{3}y^2y'' + 7yy'^2, \quad (r_1, r_2) = (1, 4). \tag{3.11}$$

For $\eta = 9$:

$$\begin{aligned} y'y''' &= \frac{8}{9}y''^2 - \frac{26}{3}y^2y'' + 17yy'^2, \quad (r_1, r_2) = (1, 8), \\ y'y''' &= \frac{8}{9}y''^2 - \frac{8}{3}y^2y'' + 8yy'^2, \quad (r_1, r_2) = (2, 4). \end{aligned} \tag{3.12}$$

For $\eta = -18$:

$$y'y''' = \frac{19}{18}y''^2 - \frac{8}{3}y^2y'' + \frac{13}{2}yy'^2, \quad (r_1, r_2) = (1, 5). \tag{3.13}$$

For $\eta = 18$:

$$y'y''' = \frac{17}{18}y''^2 - \frac{22}{3}y^2y'' + \frac{29}{2}yy'^2, \quad (r_1, r_2) = (1, 7). \tag{3.14}$$

II. $a_9 \neq 0$:

There are two branches corresponding to the roots $y_{0j}, j = 1, 2$ of (3.4). In this case, we examine the equations when $\eta = \pm 2, 3, 4, 5$.

For $\eta = -2$:

$$\begin{aligned} & y'y''' = \frac{3}{2}y''^2 + 2y^2y'' - 4yy'^2 - 2y^4, \\ & (r_{11}, r_{12}) = (-2, 2), (r_{21}, r_{22}) = (-2, 6), \\ & y'y''' = \frac{3}{2}y''^2 + 12y^2y'' - 24yy'^2 + 18y^4, \\ & (r_{11}, r_{12}) = (-2, -3), \quad (r_{21}, r_{22}) = (-2, 1), \\ & y'y''' = \frac{3}{2}y''^2 - \frac{1}{6}y^4, \\ & (r_{11}, r_{12}) = (-2, 3), \quad (r_{21}, r_{22}) = (-2, 3). \end{aligned} \tag{3.15}$$

For $\eta = 2$:

$$\begin{aligned} & y'y''' = \frac{1}{2}y''^2 - 4y^2y'' + 14yy'^2 - 2y^4, \\ & (r_{11}, r_{12}) = (2, 7), \quad (r_{21}, r_{22}) = (-5, 42), \\ & y'y''' = \frac{1}{2}y''^2 + 9yy'^2 - 6y^4, \\ & (r_{11}, r_{12}) = (3, 4), \quad (r_{21}, r_{22}) = (-5, 12), \\ & y'y''' = \frac{1}{2}y''^2 + 10yy'^2 - 10y^4, \\ & (r_{11}, r_{12}) = (2, 5), \quad (r_{21}, r_{22}) = (-3, 10). \end{aligned} \tag{3.16}$$

The canonical forms (equations that also contain the non-dominant terms as $z \to z_0$) of (3.16).b and (3.16).c are given in [20],[40].

For $\eta = 3$,

$$\begin{aligned} & y'y''' = \frac{2}{3}y''^2 + 2y^2y'' + 6yy'^2 - 12y^4, \\ & (r_{11}, r_{12}) = (2, 3), \quad (r_{21}, r_{22}) = (-2, 6). \end{aligned} \tag{3.17}$$

For $\eta = 4$,

$$\begin{aligned} & y'y''' = \frac{3}{4}y''^2 - 3y^2y'' + 10yy'^2 - y^4, \\ & (r_{11}, r_{12}) = (2, 5), \quad (r_{21}, r_{22}) = (-5, 42). \end{aligned} \tag{3.18}$$

For $\eta = 5$,

$$\begin{aligned} & y'y''' = \frac{4}{5}y'^2 - \frac{18}{5}y^2y'' + 10yy'^2 + \frac{4}{5}y^4, \\ & (r_{11}, r_{12}) = (2, 5), \quad (r_{21}, r_{22}) = (-30, -8). \end{aligned} \tag{3.19}$$

For $\eta = \infty$, no equation passes the Painlevé test.

4. Leading order $m = -3, -4$

When $m = -3$, there are two simplified equations corresponding to $\delta = 0$, and $\alpha = 0$

$$\begin{aligned} &\alpha y^2 y''' = a_1 y y' y'' + a_3 y'^3 + a_9 y^4, \\ &\delta y y' y''' = \beta y''^2 + a_{10} y^2 y', \end{aligned} \tag{4.1}$$

respectively. (4.1).a was studied in [15], and hence we consider (4.1).b. Without loss of generality, we may choose $\delta = 1$. Substituting

$$y \cong y_0 (z - z_0)^{-3} + \kappa (z - z_0)^{r-3} \tag{4.2}$$

in (4.1.b) gives the following equations of the resonances r and y_0:

$$(r+1)[r^2 + (8\beta - 13)r - 48\beta + 60] = 0, \tag{4.3}$$

$$a_{10} y_0 - 48\beta + 60 = 0. \tag{4.4}$$

There is only one branch, and $5 + (8/\eta)$ should be integer in order to have integer resonance. That is, $\eta = \pm 2, \pm 4, \pm 8, \infty$.

For these values of η, there is no principal branch, and only for $\eta = -2$ is there a maximal branch. The equation for $\eta = -2$ is as follows:

$$y' y''' = \frac{3}{2} y''^2 + 12 y^2 y', \quad (r_1, r_2) = (-3, 4). \tag{4.5}$$

Similar to the previous case, for the leading order $m = -4$, the simplified equation with $\alpha = 0$ is

$$y' y''' = \beta y''^2 + a_{11} y^3. \tag{4.6}$$

Substituting

$$y \cong y_0 (z - z_0)^{-4} + \kappa (z - z_0)^{r-4} \tag{4.7}$$

into (4.6), we obtain the following equations for the Fuchs indices r and y_0

$$(r+1)[r^2 + (10\beta - 16)r - 100\beta + 120] = 0, \tag{4.8}$$

and

$$a_{11} y_0 + 400\beta - 480 = 0, \tag{4.9}$$

respectively. Therefore, for $\eta = \pm 2, \pm 5, \pm 10, \infty$, there are integer resonances. None of these values of η gives rise to an equation with a principal branch. We have only the following equation with a maximal branch, and it passes the Painlevé test:

$$y' y''' = \frac{3}{2} y''^2 - 120 y^3, \quad (r_1, r_2) = (-5, 6). \tag{4.10}$$

5. Conclusion

In conclusion, we investigated the equation of form (1.8), which is more general than equations considered previously in the literature, so that it passes the Painlevé test. In the second, third and fourth sections, we investigated the simplified equations with leading orders of $m = -1, m = -2$ and $m = -3, -4$, respectively, subject to the condition of the existence of the at least one principal branch. In the case of more than one branch however, the compatibility conditions at the positive resonances for the secondary branches are identically satisfied for each case. For $m = -1$, there exists a principal branch, but for the case of $m = -2$ (see Eq. (3.5)), and for $m = -3, -4$, there is no principal branch. In those cases, we considered the maximal branches. The canonical form of all of the given simplified equations can be obtained by adding appropriate non-dominant terms with the coefficients analytic in z. The coefficients of the non-dominant terms can be determined from the compatibility conditions at the positive resonances. Instead of having positive, distinct integer resonances (principal branch), one can consider the case of distinct, negative integer resonances (maximal branch). In this case, it is possible to obtain equations that belong to Chazy classes.

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