Research Article

A New Neumann Series Method for Solving a Family of Local Fractional Fredholm and Volterra Integral Equations

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We propose a new Neumann series method to solve a family of local fractional Fredholm and Volterra integral equations. The integral operator, which is used in our investigation, is of the local fractional integral operator type. Two illustrative examples show the accuracy and the reliability of the obtained results.

1. Introduction

Many initial- and boundary-value problems associated with ordinary differential equations (ODEs) and partial differential equations (PDEs) can be transformed into problems of solving the corresponding approximate integral equations. However, some initial- and boundary-value domains are fractal curves, which are everywhere continuous, but nowhere differentiable. As a result, we cannot employ the classical calculus, which requires that the defined functions should be differentiable, in order to process various classes of ordinary differential equations (ODEs) and partial differential equations (PDEs). Applications of fractional calculus, in general, and fractional differential equations [1–10], in particular, as well as various transport phenomena in complex and disordered media and fractional systems, have attracted considerable attention during the past two decades or so [11–22].

Recently, local fractional calculus [23–40], processing local fractional continuous non-differential functions, was successfully applied to model the stress-strain relation in fractal elasticity [26, 27], fractal release equation [32], wave equations on Cantor sets [34], fractal heat equation [34], diffusion equation arising in discontinuous heat transfer in fractal media [35], Laplace equation within local fractional operators [36], Schrödinger equation in fractal time-space [37], damped wave equation and dissipative wave equation in fractal strings [38], heat-conduction equation on Cantor sets without heat generation in fractal media [39], and so on. There are some analytical and numerical methods for solving local fractional ODEs and PDEs, such as fractional complex transform method with local fractional operator [35], local fractional variational iteration method [37], Cantor-type cylindrical-coordinate method [38], local fractional Fourier series method [39], local fractional series expansion method [40], Fourier and Laplace transforms with local fractional operator [39], and reference therein.

The Neumann series method was applied to solve the integral equations [41, 42]. Recently, the fractional Neumann series method was considered in [43, 44]. This paper focuses on a new Neumann series method for solving the local fractional Fredholm and Volterra integral equation being

here facts in mind. This paper is structured as follows. Section 2 introduces the notations and the basic concepts. Section 3 is devoted to a new Neumann series method via local fractional integral operator. Two illustrative examples are explained in Section 4. Finally, conclusions are reported in Section 5.

2. Preliminaries

In order to investigate the local fractional continuity of non-differential functions, we suggest the result derived from fractal geometry [34, 39].

Let $f(x)$ be local fractional continuous on interval (a, b); then we write [34, 35]

$$f(x) \in C_\alpha(a, b). \tag{1}$$

If $f : (F, d) \to (\Omega', d')$ is a bi-Lipschitz mapping, then

$$\rho^s H^s(F) \le H^s(f(F)) \le \tau^s H^s(F), \tag{2}$$

which leads to

$$\rho^\alpha |x_1 - x_2|^\alpha \le |f(x_1) - f(x_2)| \le \tau^\alpha |x_1 - x_2|^\alpha, \tag{3}$$

so that

$$|f(x_1) - f(x_2)| < \varepsilon^\alpha, \tag{4}$$

where $\rho, \tau > 0$ and $x_1, x_2 \in F$.

The result deduced from fractal geometry is related to fractal coarse-grained mass function $\gamma^\alpha[F, a, b]$, which reads [34] as

$$\gamma^\alpha[F, a, b] = \frac{H^\alpha(F \cap (a, b))}{\Gamma(1 + \alpha)}, \tag{5}$$

with

$$H^\alpha(F \cap (a, b)) = (b - a)^\alpha, \tag{6}$$

where H^α is an α-dimensional Hausdorff measure.

Notice that we consider that the dimensions of any fractal spaces (e.g., Cantor spaces or the Cantor-like spaces) are a positive numbers. It looks like the Euclidean space because its dimension is also positive number. The detailed results were considered in [34].

For $f(x) \in C_\alpha(a, b)$, local fractional integral of $f(x)$ of order α in the interval $[a, b]$ is given by [34, 37, 39]

$$\begin{aligned} {}_aI_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha, \end{aligned} \tag{7}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots\}$ and $[t_j, t_{j+1}]$, $j = 0, \ldots, N - 1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a, b]$.

For any $x \in (a, b)$, we have [34]

$$\ _aI_x^{(\alpha)} f(x), \tag{8}$$

denoted by

$$f(x) \in I_x^{(\alpha)}(a, b). \tag{9}$$

If $f(x) \in I_x^{(\alpha)}(a, b)$, then we have [34]

$$f(x) \in C_\alpha(a, b). \tag{10}$$

For detailed content of fractal geometrical explanation of local fractional integral, we can see [34, 35]. Some properties of local fractional integral operator were suggested in (A.1)–(A.5).

3. A New Neumann Series Method to Deal with the Local Fractional Fredholm and Volterra Integral Equations

In this section, we consider a new Neumann series method to process the local fractional Fredholm and Volterra integral equations.

A new Neumann series method to deal with the local fractional Fredholm integral equation is written in the following form:

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1 + \alpha)} \int_a^b K(x, t) u(t)(dt)^\alpha. \tag{11}$$

It is obtained if we set

$$u_0(x) = f(x), \tag{12}$$

such that

$$\begin{aligned} u_1(x) &= u_0(x) + \frac{\lambda^\alpha}{\Gamma(1 + \alpha)} \int_a^b K(x, t) u_0(t)(dt)^\alpha \\ &= f(x) + \lambda^\alpha \psi_1(x), \end{aligned} \tag{13}$$

where $\psi_1(x) = (1/\Gamma(1 + \alpha)) \int_a^b K(x, t) f(t)(dt)^\alpha$.

The zeroth approximation can be written as

$$\begin{aligned} u_2(x) &= f(x) + \frac{\lambda^\alpha}{\Gamma(1 + \alpha)} \int_a^b K(x, t) u_1(t)(dt)^\alpha \\ &= f(x) + \frac{\lambda^\alpha}{\Gamma(1 + \alpha)} \\ &\quad \times \int_a^b K(x, t) \{f(x) + \lambda^\alpha \psi_1(x)\}(dt)^\alpha \\ &= f(x) + \lambda^\alpha \psi_1(x) + \lambda^{2\alpha} \psi_2(x), \end{aligned} \tag{14}$$

where $\psi_2(x) = (1/\Gamma(1 + \alpha)) \int_a^b K(x, t) \psi_1(x)(dt)^\alpha$.

Proceeding in this manner, the final solution $u(x)$ can be obtained as

$$\begin{aligned} u(x) &= f(x) + \lambda^\alpha \psi_1(x) + \lambda^{2\alpha} \psi_2(x) + \cdots + \lambda^{n\alpha} \psi_n(x) + \cdots \\ &= f(x) + \sum_{n=1}^\infty \lambda^{n\alpha} \psi_n(x), \end{aligned} \tag{15}$$

where $\psi_n(x) = (1/\Gamma(1 + \alpha)) \int_a^b K(x, t) \psi_{n-1}(x)(dt)^\alpha$, $n \ge 1$.

Now we structure a new Neumann series method to handle the local fractional Volterra integral equation, which reads as

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^x K(x,t) u(t) (dt)^\alpha. \quad (16)$$

The method is applicable provided that $u(x)$ is a local fractional analysis function; that is, $u(x)$ have a local fractional Taylor's expansion around $x = 0$.

$u(x)$ can be expressed by a local fractional series expansion; which reads as

$$u(x) = \sum_{n=1}^\infty a_n x^{n\alpha}, \quad (17)$$

where the coefficients a_n and x are constants that are required to be determined.

We have

$$\sum_{n=1}^\infty a_n x^{n\alpha} = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^x K(x,t) \sum_{n=1}^\infty a_n x^{n\alpha} (dt)^\alpha. \quad (18)$$

Thus, using a few terms of the expansion in both sides, we find that

$$a_0 + a_1 x^\alpha + a_2 x^{2\alpha} + \cdots + a_n x^{n\alpha} + \cdots$$

$$= f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^x K(x,t) a_0 (dt)^\alpha$$

$$+ \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^x K(x,t) a_1 x^\alpha (dt)^\alpha \quad (19)$$

$$+ \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^x K(x,t) a_2 x^{2\alpha} (dt)^\alpha + \cdots$$

$$+ \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^x K(x,t) a_n x^{n\alpha} (dt)^\alpha + \cdots.$$

We then write the local fractional Taylor's expansions for $f(x)$ and count the first few integrals in (19). After the integration is performed, we equate the coefficients of the same powers of x^α in both sides of (19). By this way, we can determine completely the unknown coefficients and produce solution in a local fractional series form.

4. Examples

Example 1. Solve the following local fractional Fredholm integral equation:

$$u(x) = \Gamma(1+\alpha) + \frac{1}{\Gamma(1+\alpha)} \int_0^1 x^\alpha u(t) (dt)^\alpha. \quad (20)$$

Let us consider the zeroth approximation given by

$$u_0(t) = \Gamma(1+\alpha). \quad (21)$$

The first approximation can be computed as follows:

$$u_1(x) = \Gamma(1+\alpha) + \frac{1}{\Gamma(1+\alpha)} \int_0^1 x^\alpha \Gamma(1+\alpha) (dt)^\alpha$$

$$= \Gamma(1+\alpha) + x^\alpha. \quad (22)$$

Proceeding in this manner, we find the following local fractional series approximation:

$$u_2(x) = \Gamma(1+\alpha) + \frac{1}{\Gamma(1+\alpha)} \int_0^1 x^\alpha (\Gamma(1+\alpha) + t^\alpha) (dt)^\alpha$$

$$= \Gamma(1+\alpha) + x^\alpha \left(1 + \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right). \quad (23)$$

Similarly, the third approximation reads as follows:

$$u_3(x) = \Gamma(1+\alpha) + \frac{1}{\Gamma(1+\alpha)}$$

$$\times \int_0^1 x^\alpha \left(\Gamma(1+\alpha) + t^\alpha \left(1 + \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\right) (dt)^\alpha$$

$$= \Gamma(1+\alpha) + x^\alpha \left(1 + \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} + \frac{\Gamma^2(1+\alpha)}{\Gamma^2(1+2\alpha)}\right). \quad (24)$$

The fourth approximation yields

$$u_4(x) = \Gamma(1+\alpha) + \frac{1}{\Gamma(1+\alpha)}$$

$$\times \int_0^1 x^\alpha \left(\Gamma(1+\alpha) + t^\alpha\right.$$

$$\times \left.\left(1 + \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} + \frac{\Gamma^2(1+\alpha)}{\Gamma^2(1+2\alpha)}\right)\right) (dt)^\alpha$$

$$= \Gamma(1+\alpha) + x^\alpha \left(1 + \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right.$$

$$\left.+ \frac{\Gamma^2(1+\alpha)}{\Gamma^2(1+2\alpha)} + \frac{\Gamma^3(1+\alpha)}{\Gamma^3(1+2\alpha)}\right). \quad (25)$$

In conclusion, we get

$$u_n(x) = \Gamma(1+\alpha) + x^\alpha \sum_{i=0}^n \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^n. \quad (26)$$

Hence,

$$u(x) = \lim_{n\to\infty} u_n(x)$$

$$= \Gamma(1+\alpha) + x^\alpha \lim_{n\to\infty} \sum_{i=0}^n \left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^n \quad (27)$$

$$= \Gamma(1+\alpha) + \frac{x^\alpha \Gamma(1+\alpha)}{\Gamma(1+2\alpha) - \Gamma(1+\alpha)}.$$

Example 2. Obtain the solution of the following local fractional Volterra equation:

$$u(x) = 1 + \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \int_0^x \frac{(t-x)^\alpha}{\Gamma(1+\alpha)} u(t)(dt)^\alpha. \tag{28}$$

Suppose that there exists the solution in the following local fractional series form:

$$u(x) = \sum_{n=1}^{\infty} a_n x^{n\alpha}. \tag{29}$$

Then, upon substituting the local fractional series into the equation, we find that

$$
\begin{aligned}
\sum_{n=1}^{\infty} & a_n x^{n\alpha} \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \int_0^x \frac{(t-x)^\alpha}{\Gamma(1+\alpha)} \sum_{n=1}^{\infty} a_n x^{n\alpha} (dt)^\alpha \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)} - \sum_{n=1}^{\infty} \frac{\Gamma(n\alpha+1) a_n x^{(n+2)\alpha}}{\Gamma((n+2)\alpha+1)}.
\end{aligned}
\tag{30}
$$

Comparing the coefficients of the same powers of x^α, we get

$$
\begin{gathered}
a_0 = 1, \qquad a_1 = \frac{1}{\Gamma(1+\alpha)}, \\
a_2 = -\frac{a_0}{\Gamma(2\alpha+1)}, \qquad a_3 = -\frac{\Gamma(\alpha+1) a_1}{\Gamma(3\alpha+1)}, \\
\vdots \\
a_n = -\frac{\Gamma((n-2)\alpha+1) a_{n-2}}{\Gamma(n\alpha+1)},
\end{gathered}
\tag{31}
$$

and so on. Thus, the values of the coefficients can be calculated as follows:

$$
\begin{gathered}
a_0 = 1, \qquad a_1 = \frac{1}{\Gamma(1+\alpha)}, \\
a_2 = -\frac{1}{\Gamma(2\alpha+1)}, \qquad a_3 = -\frac{1}{\Gamma(3\alpha+1)}, \\
a_4 = \frac{1}{\Gamma(4\alpha+1)}, \qquad a_5 = \frac{1}{\Gamma(5\alpha+1)}, \\
a_6 = -\frac{1}{\Gamma(6\alpha+1)}, \qquad a_7 = -\frac{1}{\Gamma(7\alpha+1)}, \\
\vdots
\end{gathered}
\tag{32}
$$

Hence, the local fractional series solution is given by

$$
\begin{aligned}
u(x) &= \sum_{n=1}^{\infty} a_n x^{n\alpha} \\
&= \left(1 - \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{x^{5\alpha}}{\Gamma(5\alpha+1)} - \frac{x^{7\alpha}}{\Gamma(7\alpha+1)} + \cdots \right) \\
&\quad + \left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{4\alpha}}{\Gamma(4\alpha+1)} \right. \\
&\qquad \left. - \frac{x^{6\alpha}}{\Gamma(6\alpha+1)} + \cdots \right) \\
&= \cos_\alpha x^\alpha + \sin_\alpha x^\alpha,
\end{aligned}
\tag{33}
$$

which are satisfied with the condition given by [34, 39]

$$E_\alpha(i^\alpha x^\alpha) = \cos_\alpha x^\alpha + i^\alpha \sin_\alpha x^\alpha, \tag{34}$$

where the Mittag-Leffler function defined on fractal set of fractal dimension α is suggested by [34, 39]

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}. \tag{35}$$

5. Conclusions

Local fractional differential and integral operators have proven to be useful tools to deal with everywhere continuous (but nowhere differentiable) functions in fractal areas ranging from fundamental science to engineering. In this paper, it is proven that a new Neumann series method can be used for solving the local fractional Fredholm and Volterra integral equations, and their solutions are fractal functions. The proposed method is efficient and leads to accurate, approximately convergent solutions to local fractional Fredholm and Volterra integral equations. It is demonstrated that the solutions of local fractional Fredholm and Volterra integral equations are fractal functions, which are equipped with local fractional continuities. However, the classical and fractional Neumann series methods [41–44] were only applied to continuous functions.

Appendix

The following properties of local fractional integral operator are valid [34].

(a) For any $f(x) \in C_\alpha(a,b)$, $0 < \alpha \leq 1$, we have local fractional multiple integrals, which are written as [34]

$$_{x_0}I_x^{(k\alpha)} f(x) = \overbrace{_{x_0}I_x^{(\alpha)} \cdots {_{x_0}}I_x^{(\alpha)}}^{k \text{ times}} f(x). \tag{A.1}$$

(b) If $\psi(x,y) \in C_\alpha(a,b) \times C_\alpha(c,d)$, then [34]

$$_aI_b^{(\alpha)}{}_cI_b^{(\alpha)} \psi(x,y) = {_cI_d^{(\alpha)}}{_aI_b^{(\alpha)}} \psi(x,y). \tag{A.2}$$

(c) The sine and cosine subfunctions can, respectively, be written as follows [34, 39]:

$$\sin_\alpha x^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{\alpha(2k+1)}}{\Gamma\left[1 + \alpha\left(2k + 1\right)\right]},$$

$$\cos_\alpha x^\alpha = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2\alpha k}}{\Gamma\left(1 + 2\alpha k\right)}, \quad 0 < \alpha \le 1. \tag{A.3}$$

(d) Suppose that $f(t)$ is local fractional continuous on the interval $[a, b]$. Then

$${}_aI_x^{(\alpha)}{}_aI_\tau^{(\alpha)} f(t) = {}_aI_x^{(\alpha)} \frac{(x-t)^\alpha f(t)}{\Gamma(1+\alpha)} \quad (x \in [a, b]). \tag{A.4}$$
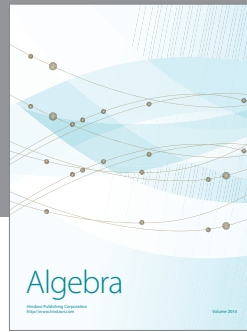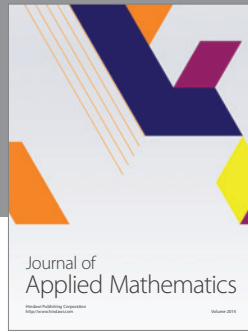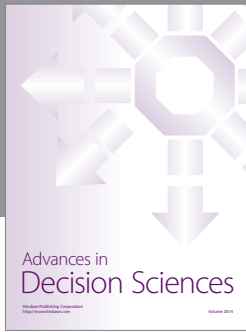
(e) We have

$${}_0I_x^{(\alpha)}{}_0I_\tau^{(\alpha)} \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} = \frac{t^{(k+2)\alpha}}{\Gamma((k+2)\alpha + 1)}. \tag{A.5}$$

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