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Damped wave equation and dissipative wave equation in fractal strings within the local fractional variational iteration method

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Abstract

In this paper, the local fractional variational iteration method is given to handle the damped wave equation and dissipative wave equation in fractal strings. The approximation solutions show that the methodology of local fractional variational iteration method is an efficient and simple tool for solving mathematical problems arising in fractal wave motions.

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1 Introduction

The variational iteration method was effectively applied in various fields of science and engineering [1–15] and the references therein. It is in some cases, more powerful than the existing techniques, *e.g.*, the fractional variational iteration method [6, 16, 17], the homotopy perturbation method [18, 19], the exp-function method [20, 21], the decomposition method [22–24], the homotopy analysis method [25, 26] and others [27]. The wave equation was investigated within some differential methods [7–15, 18–26] and the references therein.

As it is known, the quantum behavior of microphysics in terms of a non-differentiable space-time continuum possesses and has fractal property. Also, it was shown by many authors that a time-space structure of microphysics is non-differentiable. The relativistic quantum mechanics in fractal time space was suggested in [28]. It was pointed out that, while the zero set represents the Cantor point-like quantum particle, the empty set was the basic mathematical representation of the quantum wave [29]. The exact solutions for a class of fractal time random walks were researched in [30]. The questions of a philosophical nature about fractal spacetime and its implications for phenomenology and ontology were shown in [31]. The fractal time-space structure for dealing with the non-differentiability and infinities of fractals derived from local fractional operators was presented in [32–34] and the references therein. A solution of the wave equation in fractal vibrating string by using the local fractional Fourier series was discussed in [35]. The diffusion equation on Cantor time-space was reported in [36] while the diffusion problems on fractal space were suggested in [37]. The heat conduction problem by local fractional variational iteration method was investigated in [38]. The heat conduction equation in fractal time space was structured in [32]. A relaxation equation in fractal space was set up in [39]. The anomalous diffusion equation in the fractal time-space fabric was pointed out in [40]. The Fokker-Planck equation in fractal time was considered in [41].

Recently, fractional calculus analysis and fractional dynamics are hot topics [42–48]. In this paper, we consider a general wave equation of a fractal string within the local fractional operators, namely

$$L_{\zeta\zeta}^{(2\alpha)}u(\zeta,\xi)+R_{\xi}^{(\alpha)}u(\zeta,\xi)-g(\zeta,\xi)=0, \tag{1}$$

where [32–35]

$$\frac{\partial^{2\alpha}}{\partial\xi^{2\alpha}}u(\zeta,\xi)=\overbrace{\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}}^{2\text{ times}}u(\zeta,\xi), \tag{2}$$

and where $R_{\xi}^{(\alpha)}$ is a local fractional linear operator, which has low order local fractional partial derivatives with respect to ξ, subject to fractal initial conditions

$$\frac{\partial^{i\alpha}}{\partial\zeta^{i\alpha}}u(\zeta,\xi)=\varphi_i(\zeta),\quad i\in N_0. \tag{3}$$

Thus, we obtain

$$\begin{aligned}&L_{\xi\xi}^{(2\alpha)}u(\zeta,\xi)=L_{tt}^{(2\alpha)}u(x,t),\qquad R_{\xi}^{(\alpha)}u(\zeta,\xi)=-L_{t}^{(\alpha)}u(x,t),\\&g(\zeta,\xi)=L_{xx}^{(2\alpha)}u(x,t)+L_{x}^{(\alpha)}u(x,t)+m(x,t),\end{aligned} \tag{4}$$

and we have the following dissipative wave equation in fractal time space:

$$L_{tt}^{(2\alpha)}u(x,t)-L_{t}^{(\alpha)}u(x,t)-L_{xx}^{(2\alpha)}u(x,t)-L_{x}^{(\alpha)}u(x,t)-m(x,t)=0,\quad 0\le x\le l, t>0, \tag{5}$$

subject to initial conditions

$$u(x,0)=\mu_1(x),\qquad L_{t}^{(\alpha)}u(x,0)=\mu_2(x),\quad 0\le x\le l. \tag{6}$$

If we start with

$$\begin{aligned}&L_{\xi\xi}^{(2\alpha)}u(\zeta,\xi)=L_{tt}^{(2\alpha)}u(x,t),\qquad R_{\xi}^{(\alpha)}u(\zeta,\xi)=-L_{t}^{(\alpha)}u(x,t),\\&g(\zeta,\xi)=L_{xx}^{(2\alpha)}u(x,t)+n(x,t),\end{aligned} \tag{7}$$

then we obtain the following damped wave equation given by

$$L_{tt}^{(2\alpha)}u(x,t)-L_{t}^{(\alpha)}u(x,t)-L_{xx}^{(2\alpha)}u(x,t)-n(x,t)=0,\quad 0\le x\le l, t>0, \tag{8}$$

where the damping force is proportional to the velocity, a and b are constants, subject to initial conditions, which are suggested by the following expression:

$$u(x,0)=\psi_1(x),\qquad L_{t}^{(\alpha)}u(x,0)=\psi_2(x),\quad 0\le x\le l. \tag{9}$$

More recently, the local fractional variational iteration method, which was structured in [49], was applied to solve heat conduction equation on Cantor sets [38] and the local fractional Laplace equation [50]. The purpose of this paper is to present the solutions of the damped wave equation and the dissipative wave equation in fractal strings equipped with fractal initial conditions.

2 Mathematical tools

In this section, we recall briefly some basic theory of local fractional calculus, and for more details, see [32–36, 49–52].

Local fractional derivative of $f(x)$ at the point $x = x_0$, which is satisfied the condition [32, 35]

$$\left|f(x) - f(x_0)\right| < \varepsilon^{\alpha} \tag{10}$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$, is given by [32–36, 49–52]

$$D_x^{(\alpha)} f(x_0) = f^{(\alpha)}(x_0) = \left.\frac{d^{\alpha} f(x)}{dx^{\alpha}}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^{\alpha}}, \tag{11}$$

where

$$\Delta^{\alpha}\big(f(x) - f(x_0)\big) \cong \Gamma(1 + \alpha)\Delta\big(f(x) - f(x_0)\big). \tag{12}$$

Now, Eq. (11) is written in the form [32]

$$\Delta^{\alpha} f(x) = f^{(\alpha)}(x)(\Delta x)^{\alpha} + \lambda(\Delta x)^{\alpha}, \tag{13}$$

with $\lambda \to 0$ as $\Delta x \to 0$, or

$$d^{\alpha} f = f^{(\alpha)}(x)(dx)^{\alpha}. \tag{14}$$

Suppose that $f(x)$ is satisfied the condition (10) for $x \in [a, b]$, we can denote [32]

$$f(x) \in C_{\alpha}(a, b). \tag{15}$$

The right-hand local fractional derivative is defined as [32–36, 49–52]

$${}_{x_0^-}D_x^{\alpha} f(x) = f^{(\alpha)}\big(x_0^-\big) = \left.\frac{d^{\alpha} f(x)}{dx^{\alpha}}\right|_{x=x_0^-} = \lim_{x \to x_0^-} \frac{\Gamma(1 + \alpha)[f(x) - f(x_0^-)]}{(x - x_0^-)^{\alpha}} \tag{16}$$

if $f(x)$ is satisfied the conditions $x \in (x_0 - \delta, x_0)$ and $f(x) \in C_{\alpha}[a, b]$.

The left-hand local fractional derivative is written as [32–36]

$${}_{x_0^+}D_x^{\alpha} f(x) = f^{(\alpha)}\big(x_0^+\big) = \left.\frac{d^{\alpha} f(x)}{dx^{\alpha}}\right|_{x=x_0^+} = \lim_{x \to x_0^+} \frac{\Gamma(1 + \alpha)[f(x) - f(x_0^+)]}{(x - x_0^+)^{\alpha}} \tag{17}$$

if $f(x)$ is satisfied the conditions $f(x) \in C_{\alpha}[a, b]$ and $x \in (x_0, x_0 + \delta)$.

We can obtain that [32–36]

$$\left.\frac{d^\alpha}{dx^\alpha}f(x)\right|_{x=x_0^+} = \left.\frac{d^\alpha}{dx^\alpha}f(x)\right|_{x=x_0^-} = \left.\frac{d^\alpha}{dx^\alpha}f(x)\right|_{x=x_0}. \tag{18}$$

As an inverse local fractional derivative, local fractional integral of $f(x)$ at the point $x = x_0$ for $f(x) \in C_\alpha[a,b]$, is expressed by [32–36, 49–52]

$${}_aI_b^{(\alpha)}f(x) = \frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha, \tag{19}$$

if there are conditions for a partition of the interval $[a,b]$ given by

$$\Delta t_j = t_{j+1} - t_j \quad \text{and} \quad \Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots\} \quad \text{for } j = 0, \ldots, N-1, t_0 = a, t_N = b.$$

We always give the relation [32–36]

$$f(x) = {}_aI_x^{(\alpha)}f^{(\alpha)}(x) \tag{20}$$

with given conditions $f(x) \in C_\alpha[a,b]$ for $x \in (a,b)$.

Local fractional multiple integrals of $f(x)$ is given by [32–36]

$${}_{x_0}I_x^{(k\alpha)}f(x) = \overbrace{{}_{x_0}I_x^{(\alpha)} \cdots {}_{x_0}I_x^{(\alpha)}}^{k \text{ times}} f(x) \tag{21}$$

for given condition $f(x) \in C_\alpha[a,b]$.

Local fractional Taylor expansion of the following functions is written as [32–36]

$$E_\alpha\left(x^\alpha\right) = \sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+k\alpha)}. \tag{22}$$

3 The method

In this section, we present the local fractional variational iteration method [38, 49, 50] for handling differential equations with the help of the local fractional calculus theory [32–36].

Let us consider a general wave equation (1) subject to initial conditions as

$$u(\zeta,0) = \mu_1(\zeta), \qquad L_\xi^{(\alpha)}u(\zeta,0) = \mu_2(\zeta), \quad 0 \le x \le \zeta. \tag{23}$$

We can construct a correction local fractional iteration algorithm given below

$$u_{n+1}(\zeta,\xi) = u_n(\zeta,\xi) + {}_0I_\rho^{(\alpha)}\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left\{L_{\xi\xi}^{(2\alpha)}u_n(\zeta,\xi) + R_\xi^{(\alpha)}u_n(\zeta,\xi) - g(\zeta,\xi)\right\}, \tag{24}$$

where $\lambda^\alpha/\Gamma(1+\alpha)$ is a general fractal Lagrange's multiplier.

By using the local fractional integration by parts [32], we obtain

$${}_0I_\rho^{(\alpha)}\left\{\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^\alpha u_n(\zeta,\xi)}{\partial\xi^\alpha}\right]\right\} = \left.\frac{\lambda(\tau)^\alpha}{\Gamma(1+\alpha)}u_n(\zeta,\xi)\right|_{\xi=\rho} - {}_0I_\rho^{(\alpha)}\left\{u_n(\zeta,\xi)\frac{\partial^\alpha\xi(\xi)}{\partial\xi^\alpha}\right\},$$

$$\begin{aligned}
&{}_0I_\rho^{(\alpha)}\left\{\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left[\frac{\partial^{2\alpha}u_n(\zeta,\xi)}{\partial\xi^{2\alpha}}\right]\right\} \\
&\quad=\frac{\lambda(\tau)^\alpha}{\Gamma(1+\alpha)}\frac{\partial^\alpha u_n(\zeta,\xi)}{\partial\xi^\alpha}\bigg|_{\xi=\rho}-u_n(\zeta,\xi)\frac{\partial^\alpha}{\partial\xi^\alpha}\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\bigg|_{\xi=\rho} \\
&\qquad+{}_0I_t^{(\alpha)}\left\{u_n(\zeta,\xi)\frac{\partial^{2\alpha}\xi(\xi)}{\partial\xi^{2\alpha}}\right\}.
\end{aligned} \tag{25}$$

For the determination of the fractal Lagrange multiplier, the extremum condition of u_{n+1} lead us to $\delta^\alpha u_{n+1}=0$. By making use of Eq. (25), we have

$$\begin{aligned}
&\delta^\alpha u_{n+1}(\zeta,\xi) \\
&\quad=\delta^\alpha u_n(\zeta,\xi)+\delta^\alpha{}_0I_\rho^{(\alpha)}\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left\{L_{\xi\xi}^{(2\alpha)}u_n(\zeta,\xi)+R_\xi^{(\alpha)}u_n(\zeta,\xi)-g(\zeta,\xi)\right\} \\
&\quad=\delta^\alpha u_n(\zeta,\xi)+\delta^\alpha{}_0I_\rho^{(\alpha)}\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left\{L_{\xi\xi}^{(2\alpha)}u_n(\zeta,\xi)-L_\xi^{(\alpha)}u_n(\zeta,\xi)-g(\zeta,\xi)\right\} \\
&\quad=\left(1-\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\bigg|_{\xi=\rho}-\frac{\partial^\alpha}{\partial\xi^\alpha}\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\bigg|_{\xi=\rho}\right)\delta^\alpha u_n(x,t)+\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\bigg|_{\xi=\rho}\delta^\alpha\left\{\frac{\partial^\alpha u_n(\zeta,\xi)}{\partial\xi^\alpha}\right\} \\
&\qquad+{}_0I_\rho^{(\alpha)}\left\{\delta^\alpha u_n(\zeta,\xi)\frac{\partial^{2\alpha}}{\partial\xi^{2\alpha}}\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\right\}.
\end{aligned} \tag{26}$$

This yields to the stationary conditions listed below:

$$\begin{aligned}
&1-\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\bigg|_{\xi=\rho}-\frac{\partial^\alpha}{\partial\xi^\alpha}\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\bigg|_{\xi=\rho}=0, \\
&\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\bigg|_{\xi=\rho}=0, \\
&\frac{\partial^{2\alpha}}{\partial\xi^{2\alpha}}\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\bigg|_{\xi=\rho}=0.
\end{aligned} \tag{27}$$

Thus, we conclude that

$$\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}=\frac{(\xi-\rho)^\alpha}{\Gamma(1+\alpha)}. \tag{28}$$

From Eq. (28), the recurrence relation becomes

$$u_{n+1}(\zeta,\xi)=u_n(\zeta,\xi)+{}_0I_\rho^{(\alpha)}\frac{(\xi-\rho)^\alpha}{\Gamma(1+\alpha)}\left\{L_{\xi\xi}^{(2\alpha)}u_n(\zeta,\xi)+R_\xi^{(\alpha)}u_n(\zeta,\xi)-g(\zeta,\xi)\right\}. \tag{29}$$

The function $u_0(\zeta,\xi)$ is selected by using the fractal initial conditions given as below:

$$u_0(\zeta,\xi)=\mu_1(\zeta)+\frac{\xi^\alpha}{\Gamma(1+\alpha)}\mu_2(\zeta). \tag{30}$$

Thus, the approximation expression becomes

$$u(x,t)=\lim_{n\to\infty}\phi_n(x,t),\qquad \lim_{n\to\infty}\phi_n(x,t)=\lim_{n\to\infty}\sum_{i=1}^{\infty}u_i(x,t). \tag{31}$$

4 Solution of dissipative wave equation with a fractal string

The dissipative wave equation with local fractional differential operator has the form

$$\begin{aligned} &L_{tt}^{(2\alpha)}u(x,t) - L_t^{(\alpha)}u(x,t) - L_{xx}^{(2\alpha)}u(x,t) - L_x^{(\alpha)}u(x,t) - \frac{t^\alpha}{\Gamma(1+\alpha)} \\ &\quad = 0, \quad 0 \le x \le l, t > 0 \end{aligned} \tag{32}$$

subjected to the fractal initial conditions

$$u(x,0) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \qquad L_t^{(\alpha)}u(x,0) = 0, \quad 0 \le x \le l. \tag{33}$$

Making use of Eq. (29), the recurrence relation reads as

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{tt}^{(2\alpha)}u_n(x,t) - L_t^{(\alpha)}u_n(x,t) \right\} \\ &\quad - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{xx}^{(2\alpha)}u_n(x,t) + L_x^{(\alpha)}u_n(x,t) + \frac{t^\alpha}{\Gamma(1+\alpha)} \right\}. \end{aligned} \tag{34}$$

If the expression from Eq. (30) is given, we can determine the fractal initial conditions, which are expressed through

$$u_0(x,t) = u(x,0) = \frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{35}$$

The first iteration yields

$$\begin{aligned} u_1(x,t) &= u_0(x,t) + {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{tt}^{(2\alpha)}u_0(x,t) - L_t^{(\alpha)}u_0(x,t) \right\} \\ &\quad - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{xx}^{(2\alpha)}u_0(x,t) + L_x^{(\alpha)}u_0(x,t) \right\} - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} \\ &= \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}. \end{aligned} \tag{36}$$

Thus, the second iteration reads

$$\begin{aligned} u_2(x,t) &= u_1(x,t) + {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{tt}^{(2\alpha)}u_1(x,t) - L_t^{(\alpha)}u_1(x,t) \right\} \\ &\quad - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{xx}^{(2\alpha)}u_1(x,t) + L_x^{(\alpha)}u_1(x,t) \right\} - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} \\ &= \frac{x^\alpha}{\Gamma(1+\alpha)} + \left(\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} \right) + \left(\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} \right). \end{aligned} \tag{37}$$

In similar manner, the third iteration is described by

$$\begin{aligned} u_3(x,t) &= u_2(x,t) + {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{tt}^{(2\alpha)}u_2(x,t) - L_t^{(\alpha)}u_2(x,t) \right\} \\ &\quad - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{xx}^{(2\alpha)}u_2(x,t) + L_x^{(\alpha)}u_2(x,t) \right\} - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{x^\alpha}{\Gamma(1+\alpha)} + \left(\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right) \\
&\quad + \left(\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{t^{5\alpha}}{\Gamma(1+5\alpha)}\right) \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)} + \left(\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{2t^{3\alpha}}{\Gamma(1+3\alpha)}\right) + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{t^{5\alpha}}{\Gamma(1+5\alpha)}.
\end{aligned} \tag{38}
$$

The fourth iteration is suggested by

$$
\begin{aligned}
u_4(x,t) &= u_3(x,t) + {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{L_{tt}^{(2\alpha)} u_3(x,t) - L_t^{(\alpha)} u_3(x,t)\right\} \\
&\quad - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{L_{xx}^{(2\alpha)} u_3(x,t) + L_x^{(\alpha)} u_3(x,t)\right\} - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)} + \left(\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right) \\
&\quad + \left(\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{t^{5\alpha}}{\Gamma(1+5\alpha)} + \frac{t^{6\alpha}}{\Gamma(1+6\alpha)}\right) \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)} - \sum_{i=0}^{1} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} - \sum_{i=0}^{2} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \\
&\quad + \sum_{i=0}^{4} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} + \sum_{i=0}^{6} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)}.
\end{aligned} \tag{39}
$$

The fifth approximation is written as follows:

$$
\begin{aligned}
u_5(x,t) &= u_4(x,t) + {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{L_{tt}^{(2\alpha)} u_4(x,t) - L_t^{(\alpha)} u_4(x,t)\right\} \\
&\quad - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{L_{xx}^{(2\alpha)} u_4(x,t) + L_x^{(\alpha)} u_4(x,t)\right\} - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)} + \left(\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{t^{5\alpha}}{\Gamma(1+5\alpha)}\right) \\
&\quad + \left(\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{t^{5\alpha}}{\Gamma(1+5\alpha)} + \frac{t^{6\alpha}}{\Gamma(1+6\alpha)} + \frac{t^{7\alpha}}{\Gamma(1+7\alpha)}\right) \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)} - \sum_{i=0}^{1} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} - \sum_{i=0}^{2} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \\
&\quad + \sum_{i=0}^{5} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} + \sum_{i=0}^{7} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)}.
\end{aligned} \tag{40}
$$

Proceeding in this manner, we can derive the following formula:

$$
\begin{aligned}
u_n(x,t) &= u_{n-1}(x,t) + {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{L_{tt}^{(2\alpha)} u_{n-1}(x,t) - L_t^{(\alpha)} u_{n-1}(x,t)\right\} \\
&\quad - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{L_{xx}^{(2\alpha)} u_{n-1}(x,t) + L_x^{(\alpha)} u_{n-1}(x,t)\right\} - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)}
\end{aligned}
$$

$$= \frac{x^\alpha}{\Gamma(1+\alpha)} - \sum_{i=0}^{1} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} - \sum_{i=0}^{2} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} + \sum_{i=0}^{n} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} + \sum_{i=0}^{n+2} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)}. \tag{41}$$

Finally, the compact solution becomes

$$u(x,t) = \frac{x^\alpha}{\Gamma(1+\alpha)} - \sum_{i=0}^{1} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} - \sum_{i=0}^{2} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} + 2E_\alpha\left(t^\alpha\right). \tag{42}$$

5 Solution of damped wave equation with a fractal string

The damped wave equation with local fractional differential operator can be written in the form

$$L_{tt}^{(2\alpha)} u(x,t) - L_t^{(\alpha)} u(x,t) - L_{xx}^{(2\alpha)} u(x,t) - \frac{x^\alpha}{\Gamma(1+\alpha)} = 0, \quad 0 \le x \le l, t > 0, \tag{43}$$

and it is subjected to the initial conditions described by

$$u(x,0) = 0, \qquad L_t^{(\alpha)} u(x,0) = -\frac{x^\alpha}{\Gamma(1+\alpha)}, \quad 0 \le x \le l. \tag{44}$$

Applying Eq. (29), we arrive at the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + {}_0I_t^{(\alpha)} \frac{(\tau - t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{tt}^{(2\alpha)} u_n(x,t) - L_t^{(\alpha)} u_n(x,t) \right\} - {}_0I_t^{(\alpha)} \frac{(\tau - t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{xx}^{(2\alpha)} u(x,t) + \frac{x^\alpha}{\Gamma(1+\alpha)} \right\}. \tag{45}$$

By using Eq. (35), we obtain

$$u_0(x,t) = -\frac{t^\alpha}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{46}$$

Therefore, we deduce the first approximation as

$$\begin{aligned} u_1(x,t) &= u_0(x,t) + {}_0I_t^{(\alpha)} \frac{(\tau - t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{tt}^{(2\alpha)} u_0(x,t) - L_t^{(\alpha)} u_0(x,t) \right\} \\ &\quad - {}_0I_t^{(\alpha)} \frac{(\tau - t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{xx}^{(2\alpha)} u_0(x,t) + \frac{x^\alpha}{\Gamma(1+\alpha)} \right\} \\ &= -\frac{t^\alpha}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{47}$$

The second approximation has the form

$$\begin{aligned} u_2(x,t) &= u_1(x,t) + {}_0I_t^{(\alpha)} \frac{(\tau - t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{tt}^{(2\alpha)} u_1(x,t) - L_t^{(\alpha)} u_1(x,t) \right\} \\ &\quad - {}_0I_t^{(\alpha)} \frac{(\tau - t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{xx}^{(2\alpha)} u_1(x,t) + \frac{x^\alpha}{\Gamma(1+\alpha)} \right\} \\ &= -\frac{t^\alpha}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{48}$$

By using the same procedure, the third approximation becomes

$$\begin{aligned} u_3(x,t) &= u_2(x,t) + {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \bigl\{ L_{tt}^{(2\alpha)} u_2(x,t) - L_t^{(\alpha)} u_2(x,t) \bigr\} \\ &\quad - {}_0I_t^{(\alpha)} \frac{(\tau-t)^\alpha}{\Gamma(1+\alpha)} \left\{ L_{xx}^{(2\alpha)} u_2(x,t) + \frac{x^\alpha}{\Gamma(1+\alpha)} \right\} \\ &= -\frac{t^\alpha}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{49}$$

Thus, we have

$$\begin{aligned} u_0(x,t) &= -\frac{t^\alpha}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)}, \\ u_1(x,t) &= -\frac{t^\alpha}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)}, \\ u_0(x,t) &= -\frac{t^\alpha}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)}, \\ &\vdots \\ u_n(x,t) &= -\frac{t^\alpha}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)} \end{aligned} \tag{50}$$

and so on.

Finally, the solution is given by

$$u(x,t) = -\frac{t^\alpha}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{51}$$

6 Conclusions

In this manuscript, utilizing the local fractional differential operators, we investigated the damped and the dissipative wave equations in fractal strings. Based on the local fractional variational iteration method, the solutions of the damped and dissipative wave equations were presented. The iteration functions, which is local fractional continuous, is obtained easily within the fractal Lagrange multipliers, which can be optimally determined by the local fractional variational theory [32]. It is shown that the local fractional variational iteration method is an efficient and simple tool for handling partial differential equations with local fractional differential operator.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Authors contributed equally and in writing this article. Authors read and approved the final manuscript.

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