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Mathematical aspects of the Heisenberg uncertainty principle within local fractional Fourier analysis

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Abstract
In this paper, we discuss the mathematical aspects of the Heisenberg uncertainty principle within local fractional Fourier analysis. The Schrödinger equation and Heisenberg uncertainty principles are structured within local fractional operators.

Keywords: Heisenberg uncertainty principle; local fractional Fourier operator; Schrödinger equation; fractal time-space

1 Introduction

As it is known, the fractal curves [1, 2] are everywhere continuous but nowhere differentiable; therefore, we cannot use the classical calculus to describe the motions in Cantor time-space [3–10]. The theory of local fractional calculus [11–20], started to be considered as one of the useful tools to handle the fractal and continuously non-differentiable functions. This formalism was applied in describing physical phenomena such as continuum mechanics [21], elasticity [20–22], quantum mechanics [23, 24], heat-diffusion and wave phenomena [25–30], and other branches of applied mathematics [31–33] and nonlinear dynamics [34, 35].

The fractional Heisenberg uncertainty principle and the fractional Schrödinger equation based on fractional Fourier analysis were proposed [36–48]. Local fractional Fourier analysis [49], which is a generalization of the Fourier analysis in fractal space, has played an important role in handling non-differentiable functions. The theory of local fractional Fourier analysis is structured in a generalized Hilbert space (fractal space), and some results were obtained [26, 49–53]. Also, its applications were investigated in quantum mechanics [23], differentials equations [26, 28] and signals [51].

The main purpose of this paper is to present the mathematical aspects of the Heisenberg uncertainty principle within local fractional Fourier analysis and to structure a local fractional version of the Schrödinger equation.

The manuscript is structured as follows. In Section 2, the preliminary results for the local fractional calculus are investigated. The theory of local fractional Fourier analysis is introduced in Section 3. The Heisenberg uncertainty principle in local fractional Fourier analysis is studied in Section 4. Application of quantum mechanics in fractal space is considered in Section 5. Finally, the conclusions are presented in Section 6.

2 Mathematical tools

2.1 Local fractional continuity of functions

Definition 1 [18–20, 27–30] If there is

$$\left|f(x) - f(x_0)\right| < \varepsilon^\alpha \tag{2.1}$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in \mathbb{R}$. Now $f(x)$ is called a local fractional continuous at $x = x_0$, denoted by $\lim_{x\to x_0} f(x) = f(x_0)$. Then $f(x)$ is called local fractional continuous on the interval (a, b), denoted by

$$f(x) \in C_\alpha(a, b). \tag{2.2}$$

The function $f(x)$ is said to be local fractional continuous at x_0 from the right if $f(x_0)$ is defined, and

$$\lim_{x\to x_0^+} f(x) = f\left(x_0^+\right).$$

The function $f(x)$ is said to be local fractional continuous at x_0 from the left if $f(x_0)$ is defined, and

$$\lim_{x\to x_0^-} f(x) = f\left(x_0^-\right).$$

Suppose that $\lim_{x\to x_0^+} f(x) = f(x_0^+)$, $\lim_{x\to x_0^-} f(x) = f(x_0^-)$ and $f(x_0^+) = f(x_0^-)$, then we have the following relation:

$$\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0} f(x).$$

For other results of theory of local fractional continuity of functions, see [18–20, 27–30].

2.2 Local fractional derivative and integration

Definition 2 [18–20, 27–30] Setting $f(x) \in C_\alpha(a, b)$, a local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha(f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{2.3}$$

where $\Delta^\alpha(f(x) - f(x_0)) \cong \Gamma(1 + \alpha)\Delta(f(x) - f(x_0))$ with a gamma function $\Gamma(1 + \alpha)$.

Definition 3 [18–20, 27–30] Setting$f(x) \in C_\alpha(a, b)$, a local fractional integral of $f(x)$ of order α in the interval $[a, b]$ is defined as

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta t\to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha, \tag{2.4}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots\}$ and $[t_j, t_{j+1}]$, $j = 0, \ldots, N - 1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a, b]$.

Their fractal geometrical explanation of local fractional derivative and integration can be seen in [22, 26, 50–52].

If $f(x) \in C_\alpha[a,b]$, then we have [18, 19]

$$\left| {}_aI_b^{(\alpha)} f(x) \right| \le {}_aI_b^{(\alpha)} \left| f(x) \right| \tag{2.5}$$

with $b - a > 0$.

Lemma 1 [18, 19]

$$\left[{}_{-\infty}I_{\infty}^{(\alpha)} f(x)g(x) \right]^2 \le \left[{}_{-\infty}I_{\infty}^{(\alpha)} \left| g(x) \right|^2 \right] \left[{}_{-\infty}I_{\infty}^{(\alpha)} \left| g(x) \right|^2 g(x) \right]. \tag{2.6}$$

Proof See [18, 19]. □

3 Theory of local fractional Fourier analysis

In this section, we investigate local fractional Fourier analysis [49–53], which is a generalized Fourier analysis in fractal space. Here we discuss the local fractional Fourier series, the Fourier transform and the generalized Fourier transform in fractal space. We start with a local fractional Fourier series.

3.1 Local fractional Fourier series

Definition 4 [18, 19, 49–52] The local fractional trigonometric Fourier series of $f(t)$ is given by

$$f(t) = a_0 + \sum_{i=1}^{\infty} a_k \sin_\alpha \left(k^\alpha \omega_0^\alpha t^\alpha \right) + \sum_{i=1}^{\infty} b_k \cos_\alpha \left(k^\alpha \omega_0^\alpha t^\alpha \right). \tag{3.1}$$

Then the local fractional Fourier coefficients can be computed by

$$\begin{cases} a_0 = \frac{1}{T^\alpha} \int_0^T f(t)(dt)^\alpha, \\ a_k = (\frac{2}{T})^\alpha \int_0^T f(t) \sin_\alpha (k^\alpha \omega_0^\alpha t^\alpha)(dt)^\alpha, \\ b_k = (\frac{2}{T})^\alpha \int_0^T f(t) \cos_\alpha (k^\alpha \omega_0^\alpha t^\alpha)(dt)^\alpha. \end{cases} \tag{3.2}$$

The Mittag-Leffler functions expression of the local fractional Fourier series is described by [18, 19, 49–52]

$$f(x) = \sum_{k=-\infty}^{\infty} C_k E_\alpha \left(\frac{\pi^\alpha i^\alpha (kx)^\alpha}{l^\alpha} \right), \tag{3.3}$$

where the local fractional Fourier coefficients are

$$C_k = \frac{1}{(2l)^\alpha} \int_{-l}^{l} f(x) E_\alpha \left(\frac{-\pi^\alpha i^\alpha (kx)^\alpha}{l^\alpha} \right) (dx)^\alpha \quad \text{with } k \in \mathbb{Z}. \tag{3.4}$$

The above is generalized to calculate the local fractional Fourier series.

3.2 The Fourier transform in fractal space

Definition 5 [18, 19, 49–53] Suppose that $f(x) \in C_\alpha(-\infty,\infty)$, the Fourier transform in fractal space, denoted by $F_\alpha\{f(x)\} \equiv f_\omega^{F,\alpha}(\omega)$, is written in the form

$$F_\alpha\{f(x)\} = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} E_\alpha\left(-i^\alpha\omega^\alpha x^\alpha\right)f(x)(dx)^\alpha, \tag{3.5}$$

where the latter converges.

Definition 6 [18, 19, 49–53] If $F_\alpha\{f(x)\} \equiv f_\omega^{F,\alpha}(\omega)$, its inversion formula is written in the form

$$f(x) = \frac{1}{(2\pi)^\alpha}\int_{-\infty}^{\infty} E_\alpha\left(i^\alpha\omega^\alpha x^\alpha\right)f_\omega^{F,\alpha}(\omega)(d\omega)^\alpha, \quad x > 0. \tag{3.6}$$

3.3 The generalized Fourier transform in fractal space

Definition 7 [18, 19] The generalized Fourier transform in fractal space is written in the form

$$F_\alpha\{f(x)\} = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} f(x)E_\alpha\left(-i^\alpha h_0 x^\alpha\omega^\alpha\right)(dx)^\alpha, \tag{3.7}$$

where $h_0 = \frac{(2\pi)^\alpha}{\Gamma(1+\alpha)}$ with $0 < \alpha \leq 1$.

Definition 8 [18, 19] The inverse formula of the generalized Fourier transform in fractal space is written in the form [18, 19]

$$f(x) = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} f_\omega^{F,\alpha}(\omega)E_\alpha\left(i^\alpha h_0 x^\alpha\omega^\alpha\right)(d\omega)^\alpha, \tag{3.8}$$

where $h_0 = \frac{(2\pi)^\alpha}{\Gamma(1+\alpha)}$ with $0 < \alpha \leq 1$.

3.4 Some useful results

The following formula is valid [18, 19].

Theorem 1 [18, 19]

$$F_\alpha\left\{f^{(\alpha)}(x)\right\} = i^\alpha h_0\omega^\alpha F_\alpha\left\{f(x)\right\}. \tag{3.9}$$

Proof See [18, 19]. □

Theorem 2 [18, 19] *If* $F_\alpha\{f(x)\} = f_\omega^{F,\alpha}(\omega)$, *then we have*

$$\int_{-\infty}^{\infty}\left|f(x)\right|^2(dx)^\alpha = \int_{-\infty}^{\infty}\left|f_\omega^{F,\alpha}(\omega)\right|^2(d\omega)^\alpha. \tag{3.10}$$

Proof See [18, 19]. □

Theorem 3 [18, 19] *If* $F_\alpha\{f(x)\} = f_\omega^{F,\alpha}(\omega)$, *then we have*

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}(dx)^\alpha = \int_{-\infty}^{\infty} f_\omega^{F,\alpha}(\omega)\overline{g_\omega^{F,\alpha}(\omega)}(d\omega)^\alpha. \tag{3.11}$$

Proof See [18, 19]. □

4 Heisenberg uncertainty principles in local fractional Fourier analysis

Theorem 4 *Suppose that* $f \in L_{2,\alpha}[\Re]$, $F_\alpha\{f(x)\} = f_\omega^{F,\alpha}(\omega)$, *then we have*

$$\frac{\Gamma^2(1+\alpha)}{4h_0^2} \le \left[\frac{\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}[f(x)x^\alpha]^2(dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}|f(x)|^2(dx)^\alpha}\right]\cdot\left[\frac{\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}[f_\omega^{F,\alpha}(\omega)\omega^\alpha]^2(d\omega)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}|f(x)|^2(dx)^\alpha}\right], \tag{4.1}$$

with equality only if $f(x)$ *is almost everywhere equal to a constant multiple of*

$$C_0E_\alpha\left(\frac{-x^{2\alpha}}{K}\right), \tag{4.2}$$

with $K > 0$ *and a constant* C_0.

Proof Considering the equality

$$\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f_\omega^{F,\alpha}(\omega)h_0\omega^\alpha\right]^2(d\omega)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f^{(\alpha)}(x)\right]^2(dx)^\alpha, \tag{4.3}$$

we have

$$\begin{aligned}
&\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f(x)x^\alpha\right]^2(dx)^\alpha\right]\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f_\omega^{F,\alpha}(\omega)h_0\omega^\alpha\right]^2(dx)^\alpha\right]\\
&\quad = \left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f(x)x^\alpha\right]^2(dx)^\alpha\right]\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f^{(\alpha)}(x)\right]^2(dx)^\alpha\right]\\
&\quad \ge \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left|\overline{f(x)}f^{(\alpha)}(x)x^\alpha\right|^2(dx)^\alpha\\
&\quad \ge \left|\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\overline{f(x)}f^{(\alpha)}(x)x^\alpha(dx)^\alpha\right|^2.
\end{aligned} \tag{4.4}$$

When $\frac{f(x)x^\alpha}{K} = f^{(\alpha)}(x)$, then we have $f(x) = C_0E_\alpha(-\frac{x^{2\alpha}}{K})$ with a constant C_0.
Since

$$\left(\left|f(x)\right|^2\right)^{(\alpha)} = \left(f(x)\overline{f(x)}\right)^{(\alpha)} = f^{(\alpha)}(x)\overline{f(x)} + f(x)\overline{f^{(\alpha)}(x)} \tag{4.5}$$

and

$$\left|\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}f^{(\alpha)}(x)\overline{f(x)}x^\alpha(dx)^\alpha\right| = \left|\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}f(x)\overline{f^{(\alpha)}(x)}x^\alpha(dx)^\alpha\right|, \tag{4.6}$$

we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left(\left|f(x)\right|^2\right)^{(\alpha)}x^\alpha(dx)^\alpha\\
&\quad = \left[\left(\left|f(x)\right|^2\right)x^\alpha\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left|f(x)\right|^2(dx)^\alpha\\
&\quad = -\int_{-\infty}^{\infty}\left|f(x)\right|^2(dx)^\alpha,
\end{aligned} \tag{4.7}$$

when $(|f(x)|^2)x^\alpha \to 0$, $x \to \infty$.

Hence, there is

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty}\left|f(x)\right|^2(dx)^\alpha\right| \\
&\quad=\left|\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left(\left|f(x)\right|^2\right)^{(\alpha)}x^\alpha(dx)^\alpha\right| \\
&\quad=\left|\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}f^{(\alpha)}(x)\overline{f(x)}x^\alpha(dx)^\alpha+\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}x^\alpha f(x)\overline{f^{(\alpha)}(x)}(dx)^\alpha\right| \\
&\quad\le\frac{2}{\Gamma(1+\alpha)}\left|\int_{-\infty}^{\infty}f^{(\alpha)}(x)\overline{f(x)}x^\alpha(dx)^\alpha\right|
\end{aligned} \tag{4.8}$$

such that

$$\begin{aligned}
&\left|\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left(\left|f(x)\right|^2\right)^{(\alpha)}x^\alpha(dx)^\alpha\right|^2 \\
&\quad=\left|\int_{-\infty}^{\infty}\left|f(x)\right|^2(dx)^\alpha\right|^2 \\
&\quad=\Gamma^2(1+\alpha)\left|\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left|f(x)\right|^2(dx)^\alpha\right|^2 \\
&\quad\le 4\left(\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}f^{(\alpha)}(x)\overline{f(x)}x^\alpha(dx)^\alpha\right)^2 \\
&\quad\le 4\left(\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left|\overline{f(x)}f^{(\alpha)}(x)x^\alpha\right|(dx)^\alpha\right)^2 \\
&\quad\le 4\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f(x)x^\alpha\right]^2(dx)^\alpha\right]\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f_\omega^{F,\alpha}(\omega)h_0\omega^\alpha\right]^2(d\omega)^\alpha\right].
\end{aligned} \tag{4.9}$$

Therefore, we deduce to

$$\begin{aligned}
&\Gamma^2(1+\alpha)\left|\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left|f(x)\right|^2(dx)^\alpha\right|^2 \\
&\quad\le 4\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f(x)x^\alpha\right]^2(dx)^\alpha\right]\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f_\omega^{F,\alpha}(\omega)h_0\omega^\alpha\right]^2(d\omega)^\alpha\right] \\
&\quad=4h_0^2\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f(x)x^\alpha\right]^2(dx)^\alpha\right]\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f_\omega^{F,\alpha}(\omega)\omega^\alpha\right]^2(d\omega)^\alpha\right].
\end{aligned} \tag{4.10}$$

Hence, this result is obtained. □

As a direct result, we have two equivalent forms as follows.

Theorem 5 *Suppose that* $f \in L_{2,\alpha}[\Re]$ *and* $f^{(\alpha)}(x) = \frac{d^\alpha f(x)}{dx^\alpha}$, *then we have*

$$\frac{\Gamma^2(1+\alpha)}{4} \le \left[\frac{\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}[f(x)x^\alpha]^2(dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}|f(x)|^2(dx)^\alpha}\right]\cdot\left[\frac{\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}[f^{(\alpha)}(x)]^2(dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}|f(x)|^2(dx)^\alpha}\right], \tag{4.11}$$

with equality only if $f(x)$ *is almost everywhere equal to a constant multiple of* $C_0E_\alpha(\frac{-x^{2\alpha}}{K})$, *with* $K > 0$ *and a constant* C_0.

Proof Applying Theorem 4, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f^{(\alpha)}(x)\right]^2(dx)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left[f_\omega^{F,\alpha}(\omega)h_0\omega^\alpha\right]^2(d\omega)^\alpha \tag{4.12}$$

such that

$$\frac{\Gamma^2(1+\alpha)}{4} \le \left[\frac{\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}[f(x)x^\alpha]^2(dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}|f(x)|^2(dx)^\alpha}\right]\cdot\left[\frac{\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}[f^{(\alpha)}(x)]^2(dx)^\alpha}{\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}|f(x)|^2(dx)^\alpha}\right]. \tag{4.13}$$

Hence, Theorem 5 is obtained. □

The above results [37, 38] are different from the results in fractional Fourier transform [36, 37] based on the fractional calculus theory.

5 The mathematical aspect of fractal quantum mechanics

5.1 Local fractional Schrödinger equation

We structure the non-differential phase of a fractal plane wave as a complex phase factor using the relations

$$\begin{aligned}\psi_\alpha &= AE_\alpha\left(i^\alpha\left(\overline{k}^\alpha\overline{r}^\alpha - \omega^\alpha t^\alpha\right)\right)\\ &= AE_\alpha\left(\frac{i^\alpha}{h_\alpha}\left(\overline{P}_\alpha\overline{r}^\alpha - E_\alpha t^\alpha\right)\right),\end{aligned} \tag{5.1}$$

where the Planck-Einstein and De Broglie relations are in fractal space

$$\begin{cases} E_\alpha = h_\alpha\omega^\alpha, \\ P_\alpha = h_\alpha k^\alpha. \end{cases} \tag{5.2}$$

We can realize the local fractional partial derivative with respect to fractal space

$$\begin{aligned}\nabla^\alpha\psi_\alpha &= \frac{i^\alpha}{h_\alpha}P_\alpha AE_\alpha\left(\frac{i^\alpha}{h_\alpha}\left(\overline{P}_\alpha\overline{r}^\alpha - E_\alpha t^\alpha\right)\right)\\ &= \frac{i^\alpha}{h_\alpha}P_\alpha\psi_\alpha\end{aligned} \tag{5.3}$$

and fractal time

$$\begin{aligned}\frac{\partial^\alpha\psi_\alpha}{\partial t^\alpha} &= -\frac{i^\alpha}{h_\alpha}E_\alpha AE_\alpha\left(\frac{i^\alpha}{h_\alpha}\left(\overline{P}_\alpha\overline{r}^\alpha - E_\alpha t^\alpha\right)\right)\\ &= -\frac{i^\alpha}{h_\alpha}E_\alpha\psi_\alpha,\end{aligned} \tag{5.4}$$

where $\nabla^\alpha = \frac{\partial^\alpha}{\partial x^\alpha}i^\alpha + \frac{\partial^\alpha}{\partial y^\alpha}j^\alpha + \frac{\partial^\alpha}{\partial z^\alpha}k^\alpha$ [26] with $\overline{r}^\alpha = x^\alpha i^\alpha + y^\alpha j^\alpha + z^\alpha k^\alpha$ [26].

From (5.3) we have

$$-i^\alpha h_\alpha\nabla^\alpha\psi_\alpha = P_\alpha\psi_\alpha \tag{5.5}$$

such that

$$-\frac{h_\alpha^2}{2m}\nabla^{2\alpha}\psi_\alpha = \frac{1}{2m}\overline{P}_\alpha \cdot \overline{P}_\alpha \psi_\alpha, \tag{5.6}$$

where $\nabla^{2\alpha} = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial z^{2\alpha}}$ with $\overline{r}^\alpha = x^\alpha i^\alpha + y^\alpha j^\alpha + z^\alpha k^\alpha$.

From (5.4) we have

$$i^\alpha h_\alpha \frac{\partial^\alpha \psi_\alpha}{\partial t^\alpha} = E_\alpha \psi_\alpha. \tag{5.7}$$

We have the energy equation

$$\begin{aligned} E_\alpha &= \frac{1}{2m}\overline{P}_\alpha \cdot \overline{P}_\alpha + V_\alpha \\ &= H_\alpha \end{aligned} \tag{5.8}$$

such that

$$E_\alpha \psi_\alpha = H_\alpha \psi_\alpha, \tag{5.9}$$

and

$$E_\alpha \psi_\alpha = -\frac{h_\alpha^2}{2m}\nabla^{2\alpha}\psi_\alpha + V_\alpha \psi_\alpha,$$

where H_α is the local fractional Hamiltonian in fractal mechanics.

Hence, we have that

$$i^\alpha h_\alpha \frac{\partial^\alpha \psi_\alpha}{\partial t^\alpha} = -\frac{h_\alpha^2}{2m}\nabla^{2\alpha}\psi_\alpha + V_\alpha \psi_\alpha. \tag{5.10}$$

Therefore, we can deduce that the local fractional energy operator is

$$\overline{E}_\alpha = i^\alpha h_\alpha \frac{\partial^\alpha}{\partial t^\alpha} \tag{5.11}$$

and that the local fractional momentum operator is

$$\overline{P}_\alpha = i^\alpha h_\alpha \nabla^\alpha. \tag{5.12}$$

Therefore, we get the local fractional Schrödinger equation in the form of local fractional energy and momentum operators

$$H_\alpha \psi_\alpha = \frac{1}{2m}\overline{P}_\alpha \cdot \overline{P}_\alpha \psi_\alpha + V_\alpha \psi_\alpha, \tag{5.13}$$

where the local fractional Hamiltonian is

$$H_\alpha = \frac{1}{2m}\overline{P}_\alpha \cdot \overline{P}_\alpha + V_\alpha. \tag{5.14}$$

We also deduce that the general time-independent local fractional Schrödinger equation is written in the form

$$i^{\alpha}h_{\alpha}\frac{\partial^{\alpha}\psi_{\alpha}}{\partial t^{\alpha}} = H_{\alpha}\psi_{\alpha}, \tag{5.15}$$

which is related to the following equation:

$$\frac{\partial^{\alpha}S_{\alpha}(q_i,t)}{\partial t^{\alpha}} = H_{\alpha}\left(q_i, \frac{\partial^{\alpha}S_{\alpha}}{\partial q_i^{\alpha}}, t\right), \tag{5.16}$$

where S_{α} is non-differential action, H_{α} is the local fractional Hamiltonian function, and q_i^{α} $(i = 1,2,3)$ are generalized fractal coordinates.

5.2 Solutions of the local fractional Schrödinger equation

5.2.1 General solutions of the local fractional Schrödinger equation

The general solution of the local fractional Schrödinger equation can be seen in the following. For discrete k, the sum is a superposition of fractal plane waves:

$$\begin{aligned}\psi_{\alpha}(r,t) &= \sum_{n=1}^{\infty}A_nE_{\alpha}\left(i^{\alpha}\left(k_n^{\alpha}r^{\alpha} - \omega_n^{\alpha}t^{\alpha}\right)\right)\\ &= \sum_{n=1}^{\infty}A_nE_{\alpha}\left(\frac{i^{\alpha}}{h_{\alpha}}\left(\overline{P}_{\alpha}\overline{r}^{\alpha} - E_{\alpha}t^{\alpha}\right)\right)\\ &= \sum_{n=1}^{\infty}A_nE_{\alpha}\left(\frac{i^{\alpha}}{h_{\alpha}}\left(\overline{P}_{\alpha}\overline{r}^{\alpha} - \frac{\overline{P}_{\alpha}^2}{2m}t^{\alpha}\right)\right)\end{aligned} \tag{5.17}$$

and

$$E_{\alpha} = \frac{P_{\alpha}^2}{2m}. \tag{5.18}$$

If we consider $\overline{P}_{\alpha} = p_{x\alpha}i^{\alpha} + p_{y\alpha}j^{\alpha} + p_{z\alpha}k^{\alpha} \equiv p_{x\alpha}i^{\alpha}$ and $\overline{r}^{\alpha} = x^{\alpha}i^{\alpha} + z^{\alpha}j^{\alpha} + z^{\alpha}k^{\alpha}$, we have fractal plane waves:

$$\begin{aligned}\psi_{\alpha}(x,t) &= \psi_{\alpha}(P_{x\alpha},t)\\ &= \sum_{n=1}^{\infty}A_nE_{\alpha}\left(\frac{i^{\alpha}}{h_{\alpha}}\left(p_{x\alpha}x^{\alpha} - \frac{p_{x\alpha}^2}{2m}t^{\alpha}\right)\right).\end{aligned} \tag{5.19}$$

5.2.2 Fractal complex wave functions

The meaning of this description can be seen in the following. Similar to the classical wave mechanics, we prepare N atoms independently, in the same state, so that when each of them is measured, they are described by the same wave function. Then the result of a position measurement is described as the fractal probability density, and we wish it is not the same for all. The set of impacts is distributed in space with the probability density

$$\phi_{\alpha}\left(\overline{P}(r),t\right) = \left|\psi_{\alpha}(r,t)\right|^2. \tag{5.20}$$

In view of (5.20), we have

$$\phi_\alpha(x,t) = \left|\psi_\alpha(x,t)\right|^2. \tag{5.21}$$

The set of N measurements is characterized by an expectation value $\langle r\rangle_\alpha$ and a root mean square dispersion $(\Delta r)^\alpha$,

$$\langle r\rangle_\alpha = \frac{1}{\Gamma^3(1+\alpha)}\int_{-\infty}^{\infty} r^\alpha \left|\psi_\alpha(r,t)\right|^2 (dr)^\alpha. \tag{5.22}$$

Similarly, the square of the dispersion $(\Delta r)^{2\alpha}$ is defined by

$$\begin{aligned}(\Delta r)^{2\alpha} &= \left\langle x^2\right\rangle_\alpha - \left(\langle x\rangle_\alpha\right)^2 \\ &= \left\langle\left(x^\alpha - \langle x\rangle_\alpha\right)^2\right\rangle_\alpha \\ &= \frac{1}{\Gamma^3(1+\alpha)}\int_{-\infty}^{\infty}\left(r^\alpha - \langle r\rangle_\alpha\right)^2\left|\psi_\alpha(r,t)\right|^2(dr)^\alpha.\end{aligned} \tag{5.23}$$

If the physical interpretation of a particle in fractal space is that the probability

$$dP(r) = \frac{1}{\Gamma^3(1+\alpha)}\left|\psi_\alpha(r,t)\right|^2(dr)^{3\alpha}, \tag{5.24}$$

the integral of this quantity over all fractal space is

$$\begin{aligned}P(r) &= \frac{1}{\Gamma^3(1+\alpha)}\int_{-\infty}^{\infty}\left|\psi_\alpha(r,t)\right|^2(dr)^{3\alpha} \\ &= 1.\end{aligned} \tag{5.25}$$

For (5.18) we have

$$\begin{aligned}\psi_\alpha(r,t) &= \sum_{n=1}^{\infty} A_n E_\alpha\left(i^\alpha\left(k_n^\alpha r^\alpha - \omega_n^\alpha t^\alpha\right)\right) \\ &= \sum_{n=1}^{\infty} A_n E_\alpha\left(\frac{i^\alpha}{h_\alpha}\left(\overline{P}_\alpha \overline{r}^\alpha - E_\alpha t^\alpha\right)\right) \\ &= \sum_{n=1}^{\infty} A_n E_\alpha\left(\frac{i^\alpha}{h_\alpha}\left(\overline{P}_\alpha \overline{r}^\alpha - \frac{\overline{P}_\alpha^2}{2m}t^\alpha\right)\right)\end{aligned} \tag{5.26}$$

such that

$$1 = \frac{1}{\Gamma^3(1+\alpha)}\int_{-\infty}^{\infty}\left|\psi_\alpha(r,t)\right|^2(dr)^{3\alpha}. \tag{5.27}$$

5.2.3 Probabilistic interpretation of fractal complex wave function of one variable

In (5.22), we have

$$\phi_\alpha(x,t) = \left|\psi_\alpha(x,t)\right|^2 \tag{5.28}$$

and

$$P(x) = 1 \tag{5.29}$$

such that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \left|\psi_\alpha(x,t)\right|^2 (dx)^\alpha \\
&\quad = \frac{1}{\Gamma(1+\alpha)} \int_{-L}^{L} \left| A E_\alpha \left(\frac{i^\alpha}{h_\alpha} \left(p_{x\alpha} x^\alpha - \frac{p_{x\alpha}^2}{2m} t^\alpha \right) \right) \right|^2 (dx)^\alpha \\
&\quad = \frac{2A^2 L^\alpha}{\Gamma(1+\alpha)} \\
&\quad = 1,
\end{aligned} \tag{5.30}$$

where

$$\psi_\alpha(x,t) = \begin{cases} A E_\alpha\left(\frac{i^\alpha}{h_\alpha}\left(p_{x\alpha} x^\alpha - \frac{p_{x\alpha}^2}{2m} t^\alpha\right)\right), & x \in L, \\ 0, & x \notin L. \end{cases} \tag{5.31}$$

We have an expectation value $\langle x \rangle_\alpha$ and a root mean square dispersion $(\Delta x)^\alpha$,

$$\langle x \rangle_\alpha = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} x^\alpha \left|\psi_\alpha(x,t)\right|^2 (dx)^\alpha \tag{5.32}$$

and

$$\begin{aligned}
(\Delta x)^{2\alpha} &= \left\langle x^2 \right\rangle_\alpha - \left(\langle x \rangle_\alpha\right)^2 \\
&= \left\langle \left(x^\alpha - \langle x \rangle_\alpha\right)^2 \right\rangle_\alpha \\
&= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \left(x^\alpha - \langle x \rangle_\alpha\right)^2 \left|\psi_\alpha(x,t)\right|^2 (dx)^\alpha.
\end{aligned} \tag{5.33}$$

For a given fractal mechanical operator$A$, we have an expectation value $\langle A \rangle_\alpha$ and a root mean square dispersion $(\Delta A)^\alpha$,

$$\langle A \rangle_\alpha = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} A \left|\psi_\alpha(x,t)\right|^2 (dx)^\alpha \tag{5.34}$$

and

$$\begin{aligned}
(\Delta A)^{2\alpha} &= \left\langle \left(A - \langle A \rangle_\alpha\right)^2 \right\rangle_\alpha \\
&= \left\langle A^2 \right\rangle_\alpha - \left(\langle A \rangle_\alpha\right)^2 \\
&= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \left(A - \langle A \rangle_\alpha\right)^2 \left|\psi_\alpha(x,t)\right|^2 (dx)^\alpha.
\end{aligned} \tag{5.35}$$

5.3 The Heisenberg uncertainty principle in fractal quantum mechanics

Suppose that

$$\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left|\psi_\alpha(x,t)\right|^2(dx)^\alpha = 1, \tag{5.36}$$

we have a fractal positional operator expectation value

$$\langle x\rangle_\alpha = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} i^\alpha x^\alpha \left|\psi_\alpha(x,t)\right|^2(dx)^\alpha = 0 \tag{5.37}$$

and a root mean square dispersion of positional operator

$$(\Delta x)^{2\alpha} = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} i^{2\alpha} x^{2\alpha} \left|\psi_\alpha(x,t)\right|^2(dx)^\alpha. \tag{5.38}$$

Similar to the fractal positional operator, we have a fractal momentum operator expectation value

$$\langle P_x\rangle_\alpha = \left\langle i^\alpha h_\alpha \frac{\partial^\alpha}{\partial x^\alpha}\right\rangle_\alpha = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} i^\alpha h_\alpha \frac{\partial^\alpha}{\partial x^\alpha}\left|\psi_\alpha(x,t)\right|^2(dx)^\alpha = 0 \tag{5.39}$$

and a root mean square dispersion of positional operator

$$(\Delta P_x)^{2\alpha} = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} i^{2\alpha} h_\alpha^2 \frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left|\psi_\alpha(x,t)\right|^2(dx)^\alpha. \tag{5.40}$$

Considering

$$(\Delta x)^{2\alpha} = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} i^{2\alpha} x^{2\alpha} \left|\psi_\alpha(x,t)\right|^2(dx)^\alpha, \tag{5.41}$$

$$(\Delta P_x)^{2\alpha} = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} i^{2\alpha} h_\alpha^2 \frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left|\psi_\alpha(x,t)\right|^2(dx)^\alpha, \tag{5.42}$$

and

$$\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\left|\psi_\alpha(x,t)\right|^2(dx)^\alpha = 1, \tag{5.43}$$

by using Theorem 5, we have that

$$\frac{\Gamma^2(1+\alpha)}{4} \le \left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} x^{2\alpha}\left|\psi_\alpha(x,t)\right|^2(dx)^\alpha\right]\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left|\psi_\alpha(x,t)\right|^2(dx)^\alpha\right]$$

such that

$$\frac{\Gamma^2(1+\alpha)}{4} \le (\Delta x)^{2\alpha}\frac{(\Delta P_x)^{2\alpha}}{h_\alpha^2}. \tag{5.44}$$

Hence, we have that

$$\frac{\Gamma^2(1+\alpha)h_\alpha^2}{4} \le (\Delta x)^{2\alpha}(\Delta P_x)^{2\alpha} \tag{5.45}$$

such that

$$\frac{\Gamma(1+\alpha)h_\alpha}{2} \le (\Delta x)^\alpha (\Delta P_x)^\alpha, \tag{5.46}$$

where

$$(\Delta x)^\alpha = \sqrt{(\Delta x)^{2\alpha}} = \sqrt{\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} i^{2\alpha} x^{2\alpha} \left|\psi_\alpha(x,t)\right|^2 (dx)^\alpha} \tag{5.47}$$

and

$$(\Delta P_x)^\alpha = \sqrt{(\Delta P_x)^{2\alpha}} = \sqrt{\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} i^{2\alpha} h_\alpha^2 \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \left|\psi_\alpha(x,t)\right|^2 (dx)^\alpha}. \tag{5.48}$$

Suppose that

$$h_\alpha = \left(\frac{h}{2\pi}\right)^\alpha, \tag{5.49}$$

then we have

$$\frac{\Gamma(1+\alpha)(\frac{h}{2\pi})^\alpha}{2} \le (\Delta x)^\alpha (\Delta P_x)^\alpha \tag{5.50}$$

and

$$i^\alpha \left(\frac{h}{2\pi}\right)^\alpha \frac{\partial^\alpha \psi_\alpha}{\partial t^\alpha} = -\frac{(\frac{h}{2\pi})^{2\alpha}}{2m} \nabla^{2\alpha} \psi_\alpha + V_\alpha \psi_\alpha, \tag{5.51}$$

where $\nabla^{2\alpha} = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial z^{2\alpha}}$ [24].

The above equation (5.50) differs from the results presented in [36, 37]. Also, Eq. (5.51) is different from the ones reported in [38–40, 54, 55].

Below we define the local fractional energy operator

$$\overline{E}_\alpha = i^\alpha \left(\frac{h}{2\pi}\right)^\alpha \frac{\partial^\alpha}{\partial t^\alpha} \tag{5.52}$$

and the local fractional momentum operator

$$\overline{P}_\alpha = i^\alpha \left(\frac{h}{2\pi}\right)^\alpha \nabla^\alpha, \tag{5.53}$$

where $\nabla^\alpha = \frac{\partial^\alpha}{\partial x^\alpha} i^\alpha + \frac{\partial^\alpha}{\partial y^\alpha} j^\alpha + \frac{\partial^\alpha}{\partial z^\alpha} k^\alpha$ [26].

Thus, we get the Planck-Einstein and de Broglie relations are in fractal space as

$$\begin{cases} E_\alpha = (\frac{h}{2\pi})^\alpha \omega^\alpha, \\ P_\alpha = (\frac{h}{2\pi})^\alpha k^\alpha, \end{cases} \tag{5.54}$$

where h is Planck's constant.

6 Conclusions

In this manuscript, the uncertainty principle in local fractional Fourier analysis is suggested. Since the local fractional calculus can be applied to deal with the non-differentiable functions defined on any fractional space, the local fractional Fourier transform is important to deal with fractal signal functions. The results on uncertainty principles could play an important role in time-frequency analysis in fractal space. From Eq. (A.7) we conclude that there is a semi-group property for the Mittag-Leffler function on fractal sets. Meanwhile, uncertainty principles derived from local fractional Fourier analysis are classical uncertainty principles in the case of $\alpha = 1$. We reported the structure the local fractional Schrödinger equation derived from Planck-Einstein and de Broglie relations in fractal time space.

Appendix

We have [13, 20]

$$\gamma^{\alpha}[F,a,b] + \gamma^{\alpha}[F,b,c] = \gamma^{\alpha}[F,a,c] \tag{A.1}$$

such that

$$S_F^{\alpha}(y) - S_F^{\alpha}(x) = \gamma^{\alpha}[F,x,y] = \frac{(y-x)^{\alpha}}{\Gamma(1+\alpha)}, \tag{A.2}$$

where $S_F^{\alpha}(y)$ is a fractal integral staircase function. We have the relation [18–20]

$$H^{\alpha}\big(F \cap (\gamma, 0)\big) = -\gamma^{\alpha} \tag{A.3}$$

such that

$$S_F^{\alpha}(y) - S_F^{\alpha}(x) = \gamma^{\alpha}[F,x,y] = H^{\alpha}\big(F \cap (x,y)\big) = \frac{(y-x)^{\alpha}}{\Gamma(1+\alpha)}. \tag{A.4}$$

Inversely we obtain

$$S_F^{\alpha}(x) - S_F^{\alpha}(y) = \gamma^{\alpha}[F,y,x] = H^{\alpha}\big(F \cap (y,x)\big) = -\frac{(y-x)^{\alpha}}{\Gamma(1+\alpha)}. \tag{A.5}$$

Hence, both $S_F^{\alpha}(x) = \frac{x^{\alpha}}{\Gamma(1+\alpha)}$ and $S_F^{\alpha}(y) = \frac{y^{\alpha}}{\Gamma(1+\alpha)}$ are seen in [20, 21].

In view of Eq. (A.4), we easily obtain that

$$E_{\alpha}\big(i^{\alpha}x^{\alpha}\big) = \cos_{\alpha} x^{\alpha} + i^{\alpha} \sin_{\alpha} x^{\alpha} \tag{A.6}$$

and

$$E_{\alpha}\big(x^{\alpha} + y^{\alpha}\big) = E_{\alpha}\big((x+y)^{\alpha}\big) = E_{\alpha}\big(x^{\alpha}\big)E_{\alpha}\big(y^{\alpha}\big), \tag{A.7}$$

where $E_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$, $\sin_{\alpha} x^{\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]}$, $\cos_{\alpha} x^{\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2\alpha k}}{\Gamma(1+2\alpha k)}$ and i^{α} is a fractal unit of imaginary number [18–20, 53].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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