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Sliding observer for synchronization of fractional order chaotic systems with mismatched parameter

Research Article

Hadi Delavari[1]*, Danial M. Senejohnny[2]†, Dumitru Baleanu[3,4]‡

1 Department of Electrical Engineering, Hamedan University of Technology, Hamedan 65155, Iran

2 Young Researchers club, Central Tehran Branch, Islamic Azad University, Tehran, Iran

3 Cankaya University Ogretmenler Cad. No: 14 06530, Balgat - Ankara, Turkey

4 Institute of Space Sciences P.O.BOX, MG-23, R76900, Magurele-Bucharest, Romania

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Abstract: In this paper, we propose an observer-based fractional order chaotic synchronization scheme. Our method concerns fractional order chaotic systems in Brunovsky canonical form. Using sliding mode theory, we achieve synchronization of fractional order response with fractional order drive system using a classical Lyapunov function, and also by fractional order differentiation and integration, i.e. differintegration formulas, state synchronization proved to be established in a finite time. To demonstrate the efficiency of the proposed scheme, fractional order version of a well-known chaotic system; Arnodo-Coullet system is considered as illustrative examples.

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Keywords: chaos synchronization • fractional calculus • sliding observer• finite hitting time

1. Introduction

Fractional dynamics is an attractive framework for understanding the complex phenomena that are likely to emerge beyond the energy range of the Standard Model for particle physics (SM) [1]. Until about 10 years ago, expressions involving fractional derivatives and integrals were pretty much restricted to the realm of mathematics, but over the past decade, many physicists have discovered that a number of systems, particularly those exhibiting anomalously slow diffusion or sub-diffusion, are usefully described by fractional calculus. Those systems include charge transport in underground water, relaxation in polymer systems, and tracer dynamics in polymer networks and in arrays of convection rolls. Fractional calculus application has only recently been applied to contemporary

*E-mail: delavari@hut.ac.ir (Corresponding author)
†E-mail: d.senejohnny@gmail.com
‡E-mail: dumitru@cankaya.edu

physics, and is mainly concerned with the complexity of media in classical and quantum treatments. Examples include fractional kinetics [2, 3], wave propagation in a fractal medium [4, 5], nonlinear optics [6], quantum mechanics [7], quantum field theory [8, 9]. Geometrical and physical interpretation of fractional integration and differentiation formulae can be found in [10]. Also, some notes about Hamilton-Jacobi formulation of systems within Caputo's fractional derivative, and Lagrangian formulation of classical fields within Riemann-Liouville fractional derivatives are presented in [11, 12] respectively.

Chaotic systems have received widespread renewed interest in the past few decades. Such nonlinear systems can occur in various natural and man-made systems, and are especially sensitive to initial conditions. I.e., two trajectories starting at arbitrarily nearby initial conditions in such systems could evolve in drastically different fashions, and soon become totally uncorrelated. This unpredictability has been applied, in the case of chaotic time signals, to communications security: the signal is encapsulated in a chaotic envelope (produced by a chaotic oscillator) at the transmitter end, and is transmitted over the communication channel as a chaotic signal. At the receiver end, the information-bearing signal is recovered using various techniques, e.g. synchronization [13, 14].

Synchronization of chaotic systems was first initiated and recorded by Pecora and Carroll in 1990 [15]. A variety of observer-based approaches have been proposed for the synchronization of chaotic systems, which include the exponential polynomial observer [16], sliding observer [17], adaptive sliding observer [18], higher order sliding mode observer [19], fuzzy disturbance observer [20], Thau observer [21]. Synchronization of fractional-order chaotic systems was studied by Deng and Li [22] who carried out synchronization in case of the fractional Lu system. Further, they have investigated synchronization of the fractional-order Chua system [23]. In [24], synchronization of a fractional Liu system with fractional Lorenz, fractional Chen and fractional Lu systems are presented.

Among the array of methods proposed for synchronization of chaotic dynamics, observer based methods [16–21] are the focal point of interest in the field of integer chaos synchronization, but a lack of observer schemes impedes designing and stability analysis under in fractional systems. This is why we developed a sliding observer scheme for synchronizing fractional-order chaotic dynamics. The observer scheme designed in this manuscript is a generalization of the work proposed for integer-order chaotic dynamics [17]. Stability analysis of the closed-loop fractional-order system is proved by a classical Lyapunov method. In order to show the robustness of the proposed synchronization scheme, a mismatch parameter rate is included in the response system. Using fractional-order differentiation and integration, i.e. differintegration, formula state synchronization proves to be established in a finite time.

The remainder of this paper is organized as follows. In section 2, we summarize some preliminaries about fractional calculus. In section 3, we describe fractional chaotic systems based on a sliding observer design. Simulation results are depicted in sections 4 and, finally, concluding remarks are presented in section 5.

2. About fractional calculus

Physicists are all familiar with the high-school calculus that introduces students to derivatives of integer order n, $d^n y/dx^n$. Those differentiations and their inverse operations, integrations, provide the language for formulating and analyzing many laws of physics. But physicists are generally not taught about fractional-order derivatives, which might be formally expressed as, for example, $d^{1/2}y/dx^{1/2}$. Recent books [25–27] provide good reference on fractional calculus. The calculus of fractional integral and derivatives is, in fact, almost as old as calculus itself. As early as 1695, Gottfried von Leibnitz, in a reply to Guillaume de l'Hôpital, wrote that "one day useful consequences will be drawn" from fractional derivatives. About three hundred years passed before what is now known as fractional calculus was slowly accepted as a practical instrument in physics [28]. For some recent publications in the realm of fractional calculus with application in observer design, one could name [39, 40].

The Riemann-Liouville definition of the αth-order fractional derivative operator ${}_0D_t^{\alpha}$, (i.e. $0 < \alpha < 1$), with zero lower terminal, is defined by [25–27]

$$f^{(\alpha)}(t) = {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau \quad (1)$$

where 0 and t are the limits of operation and $f(t)$ is an arbitrary differentiable function. $\Gamma(.)$ denotes the well-known Gamma function. For the sake of simplicity, the notations $f^{(\alpha)}(t)$ and $D^{\alpha} f(t)$ are used interchangeably throughout this paper.

The uniform formula of fractional integral with $\beta \in (0, 1)$ is defined by

$${}_0I_t^{\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau) d\tau \quad (2)$$

where $f(t)$ is an arbitrary integrable function, ${}_0I_t^{\beta}$ is the fractional integral of order β on $[0, t]$, [25–27].

It is evident from (2) that for $-1 < \alpha < 1$ [29, 30],

$$\mathrm{sgn}\left(I^{\alpha}\left(\mathrm{sgn}(f(t))\right)\right) = \mathrm{sgn}(f(t)) \tag{3}$$

where the signum (sign) function is defined as follows:

$$\mathrm{sgn}(f(t)) = \begin{cases} +1 & \text{if } f(t) > 0 \\ 0 & \text{if } f(t) = 0 \\ -1 & \text{if } f(t) < 0 \end{cases} \tag{4}$$

The material presented next is based on the aforementioned definitions of fractional differentiation and integration, the following integration rules describing the fractional order integral ${}_0I_t^{\beta}$, of $f^{(\beta)}(t)$ and constant c:

$${}_0I_t^{\beta} f^{(\beta)}(t) = f(t) - f^{(\beta-1)}(0)\frac{t^{\beta-1}}{\Gamma(\beta)} \tag{5}$$

$${}_0I_t^{\beta} c = c\frac{t^{\beta}}{\Gamma(1+\beta)} \tag{6}$$

An in-depth discussion about this topic can be found in [25–27].

3. Observer design

Let the fractional-order chaotic driven system be in Brunovsky canonical form [31] and given by the following equations

$$\begin{aligned} D^{\alpha}x_i &= x_{i+1} \\ D^{\alpha}x_n &= f(x, \mu), \\ y &= x_1 \end{aligned} \tag{7}$$

where $x \in R^n$ and $y \in R$ are the state vector and the output signal, respectively, and $f : R^n \times R^p \to R$ is any non-linear function. The parameter vector $\mu \in R^p$ is chosen such that (7) exhibits chaotic behavior. The index i runs from 1 to $n-1$ except where noted. The response system is designed as follows:

$$\begin{aligned} D^{\alpha}\hat{x}_i &= \hat{x}_{i+1} + w_i\mathrm{sgn}(\eta_i - \hat{x}_i) \\ D^{\alpha}\hat{x}_n &= f(\eta, \mu) + w_n\mathrm{sgn}(\eta_n - \hat{x}_n) \end{aligned} \tag{8}$$

where

$$\begin{aligned} \eta_1 &= y = x_1, \\ \eta_{i+1} &= \hat{x}_{i+1} + w_i\mathrm{sgn}(\eta_i - \hat{x}_i), \end{aligned} \tag{9}$$

and $\hat{x} \in R^n$ is the state vector of the response system, and $W = (w_i, ..., w_n)$ is a feedback gain vector to be determined such that the synchronization error $e(t) = x(t) - \hat{x}(t)$ decays to zero. Moreover, we associate the output equation (9) with the response system and its profit will be advocated in the sequel.

Theorem 1 gives a useful result about the stability of the error dynamic (10), so we can subsequently conclude (in Lemma 1) that state $\hat{x}(t)$ evolves to $x(t)$ in a finite time.

Theorem 1.

Given the driving and response systems (7) and (8), we can choose W such that the response system synchronizes with the driver.

Proof. From (7) and (8) the synchronization error dynamic is given by

$$\begin{aligned} D^{\alpha}e_i &= e_{i+1} - w_i\mathrm{sgn}(\eta_i - \hat{x}_i) \\ D^{\alpha}e_n &= f(x, \mu) - f(\eta, \mu) - w_n\mathrm{sgn}(\eta_n - \hat{x}_n) \end{aligned} \tag{10}$$

Let $|e_i|_{\max} = \max_{t\geq 0}|e_i(t)|$. Our aim is to show that the trajectory of (10) is confined to the sliding surface $s : e = 0$. To accomplish this we will proceed step-wise:
for $i = 1$ to $i = n-1$, consider the sliding surface $s_i : e_i = 0$ and the Lyapunov function

$$V_i = \frac{1}{2}e_i^2 \tag{11}$$

From (10) we know that $D^{\alpha}e_i = e_{i+1} - w_i\mathrm{sgn}(e_i)$ so $\dot{e}_i$ could be written as follows [29]:

$$e_i = I^{\alpha}\left(e_{i+1} - w_i\mathrm{sgn}(e_i)\right) \to \dot{e}_i = I^{\alpha-1}\left(e_{i+1} - w_i\mathrm{sgn}(e_i)\right) \tag{12}$$

Differentiating (11), using $\dot{e}_i$ in (12) and denoting $\beta = \alpha - 1$, we obtain

$$\begin{aligned} \dot{V}_i &= e_i\dot{e}_i = e_iI^{\beta}\left(e_{i+1} - w_i\mathrm{sgn}(e_i)\right) \\ &= -e_iI^{\beta}\left[\left(w_i - e_{i+1}\mathrm{sgn}(e_i)\right)\mathrm{sgn}(e_i)\right] \end{aligned} \tag{13}$$

If w_i be chosen such that $w_i > |e_{i+1}|_{\max}$, then $e_i^{(\alpha)} = -\sigma_i\mathrm{sgn}(e_i)$, where $\sigma_i = w_i - e_{i+1}\mathrm{sgn}(e_i) > 0$, and according to (3) (i.e. $\mathrm{sgn}\left(I^{\beta}\mathrm{sgn}(e_i)\right) = \mathrm{sgn}(e_i)$),

$$\dot{V}_i = -\sigma_ie_i\left(I^{\beta}\mathrm{sgn}(e_i)\right) = -\sigma_i|e_i| < 0 \tag{14}$$

where $\beta = \alpha - 1$.
Thus s_i is attractive. Moreover, we must prove that the switching function is reached in a finite hitting time, which we denote by ${}_ht_i$. For this purpose, we will develop Lemma 3 below. But, before proceeding, we will determine an upper bound for $|e_{i+1}|_{\max}$. □

Remark 2.

We note that the choice of the gain depends on the time evolution of the error (i.e., $w_i > |e_{i+1}|_{\max}$), but in designing the response system the gain vector should be available. To this end, we need to predict an upper bound of the maximum error magnitude. It is known that the time evolution of the chaotic system is confined to the chaotic attractor which can be estimated within a hyper-ellipsoid using the invariance theorem [32]. Let D_{i+1} be the orthogonal projection of the hyper-ellipsoid onto the $(i+1)$th axis of R^n, then D_{i+1} is a line segment of length L_{i+1}. It is clear that the maximum divergence of the error $e_{i+1}(t)$ is less than L_{i+1}. Therefore, $w_i = L_{i+1}$ is a suitable choice for the sliding observer gain.

In Theorem 1, we saw that the error dynamics are eventually confined to the sliding surface, but this confinement, from the standpoint of observer design, needs to be established in a finite time T, i.e. there must exist $T > 0$ such that, for $t > T$, $\hat{x}(t) = x(t)$.

Lemma 3.

Applying the fractional integration operator ${}_0I_t^\alpha$ described in (2) to both sides of $e_i^{(\alpha)} = e_{i+1} - w_i \mathrm{sgn}(e_i)$ and considering the particular result in (5) with final time $t = {}_h t_i$,

$$e_i({}_h t_i) - e_i^{(\alpha-1)}(0)\frac{{}_h t_i^{\alpha-1}}{\Gamma(\alpha)} = -\frac{w_i \mathrm{sgn}(e_i(0))}{\Gamma(1+\alpha)}{}_h t_i^\alpha + {}_0 I_{{}_h t_i}^\alpha e_{i+1}(t) \tag{15}$$

Noting that $e_i(t) = 0$ at $t = {}_h t_i$, multiplying both sides of (15) by $\mathrm{sgn}(e_i(0))$ gives

$$\begin{aligned} &- e_i^{(\alpha-1)}(0)\mathrm{sgn}(e_i(0))\frac{{}_h t_i^{\alpha-1}}{\Gamma(\alpha)} = \\ &- \frac{w_i}{\Gamma(1+\alpha)}{}_h t_i^\alpha + {}_0 I_{{}_h t_i}^\alpha (\mathrm{sgn}(e_i(0))e_{i+1}(t)) \end{aligned} \tag{16}$$

Due to the definition given in (2) and the result in (6),

$$\begin{aligned} &{}_0 I_{{}_h t_i}^\alpha (\mathrm{sgn}(e_i(0))e_{i+1}(t)) \le {}_0 I_{{}_h t_i}^\alpha |e_{i+1}(t)| \le {}_0 I_{{}_h t_i}^\alpha L_{i+1} = \\ &L_{i+1}\frac{{}_h t_i^\alpha}{\Gamma(1+\alpha)} \end{aligned} \tag{17}$$

Substituting the results in (17) into (16), we obtain an inequality:

$$-e_i^{(\alpha-1)}(0)\mathrm{sgn}(e_i(0))\frac{{}_h t_i^{\alpha-1}}{\Gamma(\alpha)} \le -\frac{w_i}{\Gamma(1+\alpha)}{}_h t_i^\alpha + L_{i+1}\frac{{}_h t_i^\alpha}{\Gamma(1+\alpha)} \tag{18}$$

Also, we know that

$$e_i^{(\alpha-1)}(0)\mathrm{sgn}(e_i(0))\frac{{}_h t_i^{\alpha-1}}{\Gamma(\alpha)} \le \left|e_i^{(\alpha-1)}(0)\right| \frac{{}_h t_i^{\alpha-1}}{\Gamma(\alpha)} \tag{19}$$

Straightforward manipulations in (18) and with sense of expression (19), will lead to

$$\frac{w_i - L_{i+1}}{\Gamma(1+\alpha)}{}_h t_i^\alpha \le \frac{\left|e_i^{(\alpha-1)}(0)\right|}{\Gamma(\alpha)}{}_h t_i^{\alpha-1} \tag{20}$$

Clearly, the left-hand side of the inequality in (20) is a non-converging and monotonically increasing function of ${}_h t_i$. On the other hand, the right-hand side of the inequality is a monotonically decreasing function of ${}_h t_i$. Given the powers of the arguments, an intersection is inevitable. Allowing for these observations, the inequality is satisfied on the interval ${}_h t_i \in (0, T]$ where T is the point of intersection of the two expressions lying on the left- and right-hand sides of (20). We can hence see that ${}_h t_i \le T$, for which we have the following value:

$$T = \left(\frac{\Gamma(1+\alpha)}{\Gamma(\alpha)}\right)\frac{\left|e_i^{(\alpha-1)}(0)\right|}{(w_i - L_{i+1})} \tag{21}$$

Hence, by choosing $w_i > |e_{i+1}|_{\max}$ and $e_i(t) = 0$ for $t > {}_h t_i$, we can ensure that $e_i^{(\alpha)} = 0$ for $t > {}_h t_i$.

Consequently, from (10) and (9) it follows

$$0 = e_{i+1} - w_i \mathrm{sgn}(e_i) \rightarrow x_{i+1} = \hat{x}_{i+1} + w_i \mathrm{sgn}(e_i) \tag{22}$$

$$\eta_{i+1} = \hat{x}_{i+1} + w_i \mathrm{sgn}(e_i) \rightarrow \eta_{i+1} = x_{i+1} \text{ for all } t > {}_h t_i \tag{23}$$

Up to this point, it is clear that the sliding surface $\sum s_i$ is attractive and becomes invariant for $t > {}_h t_{n-1}$. Hence, the response system (8), together with its output (9), correctly synchronizes with the drive system (7) in finite time ${}_h t_{n-1}$.

If e_{i+1} does not diverge to infinity in finite time $t_{div} < {}_h t_i$, then $\eta_{i+1} = x_{i+1}$ for $t > {}_h t_i$ and (10) becomes

$$\begin{aligned} D^\alpha e_i &= e_{i+1} - w_i \mathrm{sgn}(e_i) \\ D^\alpha e_n &= -w_n \mathrm{sgn}(e_n) \qquad \text{for } t > {}_h t_{n-1} \end{aligned} \tag{24}$$

thus, if one choose $w_n > 0$, where $\beta = \alpha - 1$, then the reaching condition is satisfied as in Theorem 1:

$$\begin{aligned} V_n &= \tfrac{1}{2}e_n^2, \\ \dot{V}_n &= e_n \dot{e}_n = -e_n I^\beta (w_n \mathrm{sgn}(e_n)) \\ &= -w_n |e_n| < 0 \end{aligned} \tag{25}$$

Also, as in Lemma 3, we can easily show that $e_n = 0$ for $t > {}_h t_n$, where ${}_h t_n \le T$,

$$T = \left(\frac{\Gamma(1+\alpha)}{\Gamma(\alpha)}\right)\frac{\left|e_n^{(\alpha-1)}(0)\right|}{(w_n)} \tag{26}$$

and we can show that (10) is stable, and thereby bounded. Eventually, $s : e = 0$ is reached and the synchronization is achieved in finite time $T = {}_h t_n$.

In view of the proof of Theorem 1, we have $\eta_{i+1} = x_{i+1}$ for $t > {}_h t_{n-1}$ if $e_{i+1}(t)$ are bounded and are judiciously

chosen. The existence of w_i depends on the boundedness of $e_{i+1}(t)$ (hence our Remark 2).
Theorem 1 holds in the ideal case when the parameter vector μ is exactly reproduced in the response system. Such perfection is, however, not realistic. We therefore now prove that the scheme proposed in this paper is robust with respect to parameter mismatch. In this case, the response system is given by

$$\begin{aligned} D^\alpha \hat{x}_i &= \hat{x}_{i+1} + w_i \mathrm{sgn}(\eta_i - \hat{x}_i) \\ D^\alpha \hat{x}_n &= f(\eta, \mu') + w_n \mathrm{sgn}(\eta_n - \hat{x}_n) \end{aligned} \tag{27}$$

where $\mu' = \mu - \Delta\mu$ and η_i is given by (9). We assume that $f(x, \mu)$ is Lipschitz with respect to μ, that is, for some positive constant k_μ, we have:

$$\left\| f(x,\mu) - f(x,\mu') \right\| \leq k_\mu \left\| \mu - \mu' \right\| \tag{28}$$

where $\|.\|$ represents the standard Euclidean norm. Then we have the following result.

Theorem 4.
Consider the chaotic drive and response systems given by (7), (9) and (27). We assume that $\Delta\mu = \mu - \mu'$ and (28) are satisfied. Then, if $w_n > k_\mu \left\|\Delta\mu\right\|$, the response system synchronizes with the drive system within a finite time, i.e., there exists $T > 0$ such that, for $t > T$, $\hat{x}(t) = x(t)$.

Proof. From (7) and (27) the error dynamic is

$$\begin{aligned} D^\alpha e_i &= e_{i+1} - w_i \mathrm{sgn}(\eta_i - \hat{x}_i) \\ D^\alpha e_n &= f(\eta, \mu) - f(x, \mu') - w_n \mathrm{sgn}(\eta_n - \hat{x}_n) \end{aligned} \tag{29}$$

In the view Theorem 1, $\eta_{i+1} = x_{i+1}$ for $t > {}_h t_{n-1}$ if $e_{i+1}(t)$ is bounded and w_i is chosen judiciously (i.e. $w_i = L_{i+1}$). Thus, from (29) and the fact that $\eta = x$, we have

$$D^\alpha e_n = f(x,\mu) - f(x,\mu') - w_n \mathrm{sgn}(e_n) \tag{30}$$

By considering the following Lyapunov function

$$V_n = \frac{1}{2} e_n^2 \tag{31}$$

the derivative of V_n along the trajectories of $\dot{e}_n$ (resulting from (30)), yields

$$\begin{aligned} \dot{V}_n &= e_n \dot{e}_n = e_n I^\beta \left((f(x,\mu) - f(x,\mu')) - w_n \mathrm{sgn}(e_n) \right) \\ &= -e_n I^\beta \left(w_n \mathrm{sgn}(e_n) - (f(x,\mu) - f(x,\mu')) \right) \end{aligned} \tag{32}$$

where, as in Theorem 1, $\beta = 1 - \alpha$.

So if w_n be chosen such that $w_n > k_\mu \left\|\Delta\mu\right\|$ (where, according to (28), $w_n > k_\mu \left\|\Delta\mu\right\|$ is the upper bound of $\left\| f(x,\mu) - f(x,\mu') \right\|$), then $\dot{V}_n < 0$ and $s_n : e_n = 0$ becomes attractive owing to the change in the signum function. This implies that $e_{i+1}(t)$ are bounded. Hence, by choosing $w_i = L_{i+1}$ and $w_n > k_\mu \left\|\Delta\mu\right\|$, we can make $s : e = 0$ attractive. By applying the fractional integration operator ${}_0 I_t^\alpha$ described in (2) to both sides of $e_n^{(\alpha)} = f(x,\mu) - f(x,\mu') - w_n \mathrm{sgn}(e_n)$, and considering the particular result in (5) with final time $t = {}_h \tilde{t}_n$, we get

$$\begin{aligned} e_n({}_h\tilde{t}_n) - e_n^{(\alpha-1)}(0) \frac{{}_h\tilde{t}_n^{\alpha-1}}{\Gamma(\alpha)} = &- \frac{w_n \mathrm{sgn}(e_n({}_h\tilde{t}_n))}{\Gamma(1+\alpha)} {}_h\tilde{t}_n^\alpha \\ &+ {}_0 I^\alpha_{{}_h\tilde{t}_n} \left(f(x,\mu) - f(x,\mu') \right) \end{aligned} \tag{33}$$

Pursuing the same steps in Lemma 3 and applying the definition in (2) and the result from (6), we have

$$\begin{aligned} {}_0 I^\alpha_{{}_h\tilde{t}_n} (\mathrm{sgn}\,(e_n(0))\,(f(x,\mu) - f(x,\mu'))) &\leq {}_0 I^\beta_{{}_h\tilde{t}_n} \left\| f(x,\mu) - f(x,\mu') \right\| \\ &\leq {}_0 I^\beta_{{}_h\tilde{t}_n} \left| k_\mu \left\|\Delta\mu\right\| \right| \\ &= \left| k_\mu \left\|\Delta\mu\right\| \right| \frac{{}_h\tilde{t}_n^\beta}{\Gamma(1+\beta)} \end{aligned} \tag{34}$$

We can hence deduce, with straightfoward manipulation, that $e_n = 0$ (i.e. $\hat{x}_n(t) = x_n(t)$) for $t > {}_h\tilde{t}_n$, where ${}_h\tilde{t}_n \leq T$.

$$T = \left(\frac{\Gamma(1+\alpha)}{\Gamma(\alpha)} \right) \frac{\left| e_n^{(\alpha-1)}(0) \right|}{(w_n - \left| k_\mu \left\|\Delta\mu\right\| \right|)} \tag{35}$$

And so the proposed synchronization scheme is robust with respect to parameter mismatch. □

Remark 5.
When designing the response systems, the difference between cases is the choice of w_n. Actually, it suffices to tune w_n according to the parameter mismatch, $w_n > k_\mu \left\|\Delta\mu\right\|$, to stir $\hat{x}_n(t)$ to follow $x_n(t)$.

4. Illustrative examples

In this section, we applied our proposed scheme to the fractional order version of chaotic systems. We consider the following fractional order Arnodo-Coullet system [33], whose chaotic attractor is depicted in Fig. 1.

$$\begin{aligned} D^\alpha x_1 &= x_2 \\ D^\alpha x_2 &= x_3 \\ D^\alpha x_3 &= 0.8x_1 - 1.1x_2 - 0.45x_3 - x_1^3 \end{aligned} \tag{36}$$

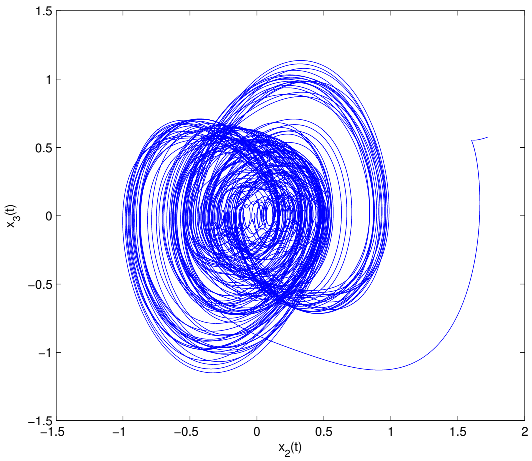

Figure 1. Fractional order chaotic system, α= 0.97.

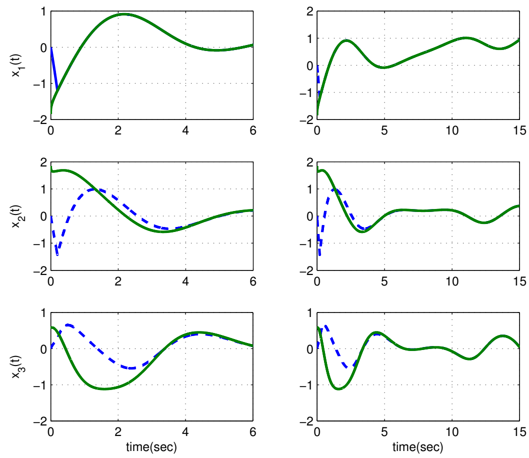

Figure 2. Drive system (solid-line) and response system (dot-line) behavior in ideal case. Figures on the left show a zoom on the time axis.

The system meets the specifications in (7). As a response system, we choose the following:

$$
\begin{aligned}
D^\alpha \hat{x}_1 &= \hat{x}_2 + 5\,\mathrm{sgn}(\eta_1 - \hat{x}_1)\ ,\\
D^\alpha \hat{x}_2 &= \hat{x}_3 + 7\,\mathrm{sgn}(\eta_2 - \hat{x}_2)\ ,\\
D^\alpha \hat{x}_3 &= 0.8\eta_1 - 1.1\eta_2 - 0.45\eta_3 - \eta_1^3 + 10\,\mathrm{sgn}(\eta_3 - \hat{x}_3)
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
\eta_1 &= x_1\\
\eta_2 &= \hat{x}_2 + 5\,\mathrm{sgn}(\eta_1 - \hat{x}_1)\\
\eta_3 &= \hat{x}_3 + 7\,\mathrm{sgn}(\eta_2 - \hat{x}_2)
\end{aligned}
\tag{38}
$$

Fractional differential equations in terms of the Riemann-Liouville derivatives require initial conditions expressed in terms of $D^{\alpha-1}x(0) = x_0$ and $D^{\alpha-1}\hat{x}(0) = \hat{x}_0$ [34], where the fractional order is $\alpha = 0.97$, $x_0 = (-1.2, 1.2, 0.4)^T$ and $\hat{x}_0 = (0, 0, 0)^T$.

Since most of the fractional-order differential equations do not have exact analytic solutions, we must turn to approximations and numerical techniques. Several analytical and numerical methods have been proposed to solve the fractional-order differential equations. According to [35], frequency-domain approximation methods are not suitable for recognizing chaos. Hence, in this paper we used a monotone iterative method for initial value problems involving Riemann-Liouville fractional derivatives [36]. The monotone iterative technique, combined with the method of upper and lower solutions, is a powerful tool for proving the existence of solutions of nonlinear differential equations [37, 38].

The obtained results are delineated in Fig. 2. Curves in the left column show that, ${}_ht_1 = 0.2$, ${}_ht_2 = 4.5$ and ${}_ht_3 = 5.8$. We clearly see that $\hat{x}_1(t) = x_1(t)$ so $\eta_2 = x_2$ for $t > {}_ht_1$. We also see that $\hat{x}_2(t) = x_2(t)$ so $\eta_3 = x_3$ for $t > {}_ht_2$ and $\hat{x}_3(t) = x_3(t)$ for $t > {}_ht_3$.

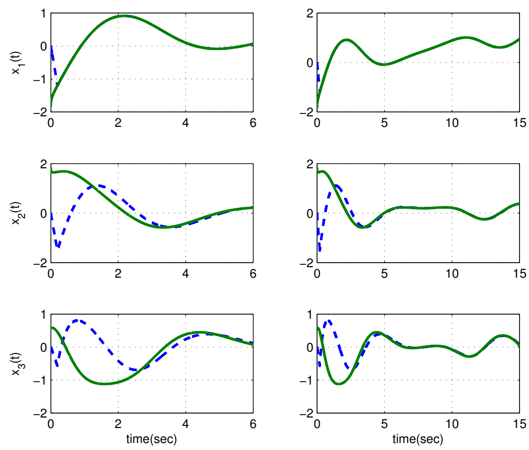

Figure 3. Drive system (solid line) and response system (dot-line) behavior with parameter mismatch. Figures on the left show a zoom on the time axis.

For the case with mismatched parameters, see Fig. 3. A mismatch of order 10% was considered in the following response system:

$$
\begin{aligned}
D^\alpha \hat{x}_1 &= \hat{x}_2 + 5\,\mathrm{sgn}(\eta_1 - \hat{x}_1)\ ,\\
D^\alpha \hat{x}_2 &= \hat{x}_3 + 7\,\mathrm{sgn}(\eta_2 - \hat{x}_2)\ ,\\
D^\alpha \hat{x}_3 &= 0.72\eta_1 - 0.99\eta_2 - 0.405\eta_3 - 0.9\eta_1^3 + 13\,\mathrm{sgn}(\eta_3 - \hat{x}_3)
\end{aligned}
\tag{39}
$$

In this case, $w_1 = 5$ and $w_2 = 7$ were unaltered, whereas we changed w_3 to 13. In fact, for the ideal case $w_3 = 10$

sufficed to guarantee synchronization, but it was necessary to tune $w_3 > 10$ in order to have $\hat{x}_3 = x_3$.

5. Conclusion

We proposed a sliding observer scheme for synchronization of fractional chaotic dynamics in Brunovsky canonical form. The proposed observer scheme is theoretically valid and numerically applicable for fractional systems in both ideal and mismatched cases. Using fractional differintegration formula, we showed that state synchronization will be achieved in a finite time. Also the proposed scheme showed robustness with respect to parameter mismatch in response system.

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