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Cosmological perturbations in FRW model with scalar field within Hamilton-Jacobi formalism and symplectic projector method

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Abstract: The Hamilton-Jacobi analysis is applied to the dynamics of the scalar fluctuations about the Friedmann-Robertson-Walker (FRW) metric. The gauge conditions are determined from the consistency conditions. The physical degrees of freedom of the model are obtained by the symplectic projector method. The role of the linearly dependent Hamiltonians and the gauge variables in the Hamilton-Jacobi formalism is discussed.

Keywords: FRW model, Hamilton-Jacobi formalism, symplectic projector method
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1 Introduction

The cosmological models which include the theory of a scalar field coupled to gravity have recently played an important role. A transformation from a reparametrization-invariant system to an ordinary gauge system was applied for deparametrizing cosmological models. In the path integral approach to false vacuum decay with the effect of gravity, there remains an unsolved problem, called the negative mode problem. A conjecture was proposed that there should be no supercritical supercurvature mode.This conjecture was verified for a wide variety of tunneling potentials [1]. For the monotonic potential no

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negative modes were found about the Hawking-Turok instanton. For a potential with a false vacuum the Hawking-Turok instanton was shown to have a negative mode for certain initial data [2]

It was shown that the cosmological perturbations in the Lorentzian regime are related to the cosmic microwave background radiation and large scale structure formation [3–5]. The unconstrained reduced action corresponding to the dynamics of scalar fluctuations about the FRW background was obtained by applying Dirac's method of singular Lagrangian systems [6, 7]. The results were applied to the negative mode problem in the description of tunneling transitions with gravity [8]. There are several known methods in obtaining and dealing with unconstrained quadratic action in terms of the physical variables [4, 5, 9] in the theory of scalar field coupled to gravity in non-spatially flat FRW Universe but the main problem appears at the quantum level [8]. For these reasons new quantization methods as Hamilton-Jacobi method and the symplectic projector method [10–13] should be applied on the theory mentioned above. By adding a surface term to the action functional the gauge invariance of the systems whose Hamilton-Jacobi equation is separable was improved [14].

The Hamilton-Jacobi formalism (HJ) based on Carathéodory's idea [15] has gained a considerable importance during the last decade due to its various applications to quantization of constrained systems [16].

However, some difficulties may occur for HJ in dealing with linear dependent constraints. The main problem comes from the construction of the canonical Hamiltonian. Let us assume that the canonical Hamiltonian is a linear combination of two terms and the second one is proportional to a given field having its momentum zero. After imposing the integrability condition for that momentum we obtain a new constraint, therefore the canonical Hamiltonian is a linear combination of two constraints. Therefore an interesting and as yet unsolved question is how to deal with the total differential equations within HJ in the above mentioned case. Another issue is related to the gauge fixing procedure within HJ formalism. Can we find inside of HJ a mechanism to obtain the gauge fixing condition? In order to analyze the above mentioned open problems we have to apply the HJ formalism to a constrained system possessing linearly dependent constraints. For these reasons the application of HJ formalism to cosmological perturbations in FRW model with scalar field is an interesting issue.

The paper is organized as follows:

Section 2 briefly presents the HJ formalism. In Section 3 the gauge fixing conditions of the investigated model are discussed inside the HJ formalism and the true degrees of freedom are obtained within the symplectic projector method. Finally, Section 4 is dedicated to our conclusions.

2 Hamilton-Jacobi formalism

HJ formalism presented in this paper is based on Carathéodory's idea of equivalent Lagrangians [15]. This approach can be considered an alternative method of quantization of constrained systems and it was subjected to an intense debate during the last decade (see Refs.[16–23] and the references therein). The starting point of this method is a singular Lagrangian $\mathcal{L}$. In this approach we use the initial canonical Hamiltonian H_0 and all primary constraints denoted by $H^{'}_\alpha$. Namely, the corresponding "Hamiltonians" are

$$H^{'}_\alpha = p_\alpha + H_\alpha(t_\beta, q_a, p_a), \tag{1}$$

where $\alpha, \beta = n-r+1, \cdots, na = 1, \cdots, n-r$ and the canonical one is given by

$$H_0 = p_a w_a + \dot{q}_\mu p_\mu \mid_{p_\nu = H_\nu} - L(t, q_i, \dot{q}_\nu, \dot{q}_a = w_a), \quad \nu = 0, n-r+1, \cdots, n. \tag{2}$$

Using (1) and (2) a set of total differential equations is obtained [15]

$$dq_a = \frac{\partial H^{'}_\alpha}{\partial p_a} dt_\alpha, \; dp_a = -\frac{\partial H^{'}_\alpha}{\partial q_a} dt_\alpha, \; dp_\mu = -\frac{\partial H^{'}_\alpha}{\partial t_\mu} dt_\alpha, \quad \mu = 1, \cdots, r \tag{3}$$

together with the HJ function z, which is defined by

$$dz = \left(-H_\alpha + p_\alpha \frac{\partial H^{'}_\alpha}{\partial p_\alpha} \right) dt_\alpha, \tag{4}$$

where t_α are gauge variables [15]. The next step is to investigate the integrability of the system (3). On the surface of constraints the system of differential equations is integrable if and only if the "Hamiltonians" $H^{'}_\alpha$ are in involution. If this condition is not fulfilled, then another set of "Hamiltonians" arises and we subject them to the integrability conditions. The process ends when no new "Hamiltonian" appears.

3 Gauge fixing conditions within HJ formalism

3.1 The model

The action of the system of scalar matter field coupled to gravity is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2k} - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right]. \tag{5}$$

Here $k = 8\pi G$ represents the reduced Newton's constant and the scalar potential field is denoted by $V(\phi)$. By making an expansion of both the metric and the scalar field in terms of a FRW type background we obtain [8]

$$\begin{aligned} ds^2 &= a(\eta)^2 \Big[-(1 + 2A(\eta)Y) d\eta^2 + 2\mathcal{B}(\eta) Y_{|i} d\eta dx^i \\ &\qquad + \{\gamma_{ij}(1 - 2\Psi(\eta)Y) + 2\mathcal{E}(\eta) Y_{|ij} dx^i dx^j\} \Big], \\ \phi &= \varphi(\eta) + \Phi(\eta) Y. \end{aligned} \tag{6}$$

In (6) γ_{ij} represents the three-dimensional metric on the constant curvature space sections, the background field values are denoted by a and ϕ respectively as well as $A, \Psi, \Phi, \mathcal{B}$.

$\mathcal{E}$ represents small perturbations. In addition, Y denotes a normalized function of three dimensional Laplacian, $\Delta Y = -k^2 Y$, and vertical line as a subscript denotes the covariant derivative with respect to γ_{ij}. The Lagrangian corresponding to the total action containing only the second order terms (see for more details Ref.[8]) is

$$\mathcal{L}=\frac{a^2\sqrt{\gamma}}{2k}\Big[-6\Psi'^2+2(k^2-3\mathcal{K})\Psi^2+k\Big\{\Phi'^2-(a^2\frac{\delta^2V}{\delta\phi\delta\phi}+k^2)\Phi^2+6\phi'\Psi'\Phi\Big\} -\Big\{2k\phi'\Phi'+2ka^2\frac{\delta V}{\delta\phi}\Phi+12\mathcal{H}\Phi'+4(k^2-3\mathcal{K})\Psi\Big\}A -2(\mathcal{H}'+2\mathcal{H}^2-\mathcal{K})A^2\Big], \tag{7}$$

where the prime denotes a derivative with respect to conformal time η, $\mathcal{H} = \frac{a'}{a}$ and $\mathcal{K}$ denotes the curvature parameter taking the values 1,0,-1, for closed, flat and open universes, respectively [8].

3.2 Hamilton-Jacobi analysis

From the Lagrangian density (7) we obtain the primary constraint as

$$\Pi_A = 0. \tag{8}$$

From (8) we conclude that A is a gauge variable and that $H_1' = \Pi_A$ represents a "Hamiltonian". In the HJ formalism the starting point for this model involves two "Hamiltonians", namely H_1' and $H_0' = p_0 + H_C$ which is given below

$$H_0' = p_0 - \frac{k}{12a^2\sqrt{\gamma}}\Pi_\Psi^2 + \frac{1}{2a^2\sqrt{\gamma}}\Pi_\Phi^2 + \frac{k}{2}\phi'\Pi_\Psi\Phi + a^2\sqrt{\gamma}[-\frac{k^2-3\mathcal{K}}{k}\Psi^2 + \frac{1}{2}(a^2\frac{\delta^2V}{\delta\phi\delta\phi} - \frac{3}{2}k\phi'^2 + k^2)\Phi^2] + A\{\phi'\Pi_\Phi - \mathcal{H}\Pi_\Psi + a^2\sqrt{\gamma}[(a^2\frac{\delta V}{\delta\phi} + 3\phi'\mathcal{H})\Phi + \frac{2(k^2-3\mathcal{K})}{k}\Psi]\}, \tag{9}$$

where H_C is the canonical Hamiltonian presented in [8].

The next step in HJ formalism is to obtain the total differential equations by using (8) and (9). In our case we obtain the following set of total differential equations

$$d\Psi = -\frac{k\Pi_\Psi d\tau}{6a^2\sqrt{\gamma}} + \frac{k}{2}\phi'\Phi d\tau - A\mathcal{H}d\tau, \tag{10}$$

$$d\Phi = (\frac{\Pi_\phi}{a^2\sqrt{\gamma}} + A\phi')d\tau, \tag{11}$$

$$d\Pi_\Psi = 2a^2\sqrt{\gamma}\frac{k^2-3\mathcal{K}}{k}\Psi d\tau - \frac{2a^2\sqrt{\gamma}A(k^2-3\mathcal{K})}{k}d\tau, \tag{12}$$

$$d\Pi_\Phi = -\frac{\Pi_\Phi}{a^2\sqrt{\gamma}}d\tau - Aa^2\sqrt{\gamma}(\frac{a^2\delta V}{\delta\phi} + 3\phi'\mathcal{H})d\tau -a^2(\frac{\delta^2V}{\delta\phi\delta\phi} - \frac{3}{2}k\phi'^2 + k^2)\Phi d\tau. \tag{13}$$

Taking into account (9) and the consistency condition

$$d\Pi_A = 0, \tag{14}$$

we get a new "Hamiltonian" denoted by H_2. Namely, the form of H_2 is given by

$$H_2 = \phi^{'}\Pi_\Phi - \mathcal{H}\Pi_\Psi + a^2\sqrt{\gamma}[(a^2\frac{\delta V}{\delta\phi} + 3\phi^{'}\mathcal{H})\Phi + \frac{2(k^2 - 3\mathcal{K})}{k}\Psi]. \tag{15}$$

In order to close the chain the variation of H_2 must be zero, otherwise a new constraint will appear. By using (10), (11), (12) and (13), and after tedious calculations, we find that if

$$A - \Psi = 0, \quad \Pi_\Psi = 0, \tag{16}$$

then $dH_2 = 0$ provided that A is given as a function of background fields,Φ and Π_Φ. We mention that (16) is in agreement with the results presented in [8].

3.3 Physical Hamiltonian

For a system admitting only second class constraints $\phi^m(\zeta^M) = 0$, where $\zeta^M = (x^a, p^a)$, $M = 1, 2, \ldots, 2N$ are the coordinates, the action of the symplectic projector (see Refs. [10–13] and the references therein) defined as follows

$$\Lambda^{MN} = \delta^{MN} - J^{ML}\frac{\delta\phi_m}{\delta\zeta^L}\Delta^{-1}_{mn}\frac{\delta\phi_n}{\delta\zeta^N} \tag{17}$$

is to project ζ^M onto local variables on the constraint surface defined as

$$\zeta^{\star M} = \Lambda^{MN}\zeta^N. \tag{18}$$

Here the matrix Δ^{-1}_{mn} is the inverse of the matrix $\Delta_{mn} = \{\phi_m, \phi_n\}$ and J^{MN} denotes the symplectic two form.

The set of second class constraints corresponding to our model is given by

$$\begin{aligned} &C_1 = \phi^{'}\Pi_\Phi + a^2\sqrt{\gamma}[(a^2\frac{\delta V}{\delta\phi} + 3\phi^{'}\mathcal{H})\Phi + \frac{2(k^2 - 3\mathcal{K})}{k}\Psi] \\ &C_2 = \Pi_A, C_3 = \Pi_\psi, C_4 = A - \Psi. \end{aligned} \tag{19}$$

By using (19) we obtain the form of the matrix Δ as follows

$$\Delta = \begin{pmatrix} 0 & 0 & 2\frac{a^2\sqrt{\gamma}(k^2-3K)}{k} & 0 \\ 0 & 0 & 0 & -1 \\ -2\frac{a^2\sqrt{\gamma}(k^2-3K)}{k} & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}\delta(\vec{x} - \vec{y}). \tag{20}$$

The form of the matrix projector becomes

$$\Lambda = \begin{pmatrix} 0 & -k\frac{a^2\frac{\delta V}{\delta\phi}+3\phi'\mathcal{H}}{2(k^2-3K)} & 0 & 0 & -\frac{k\phi'}{2a^2\sqrt{\gamma}(k^2-3K)} & 0 \\ 0 & 1 & 0 & \frac{k\phi'}{2a^2\sqrt{\gamma}(k^2-3K)} & 0 & \frac{k\phi'}{2a^2\sqrt{\gamma}(k^2-3K)} \\ 0 & -k\frac{a^2\frac{\delta V}{\delta\phi}+3\phi'\mathcal{H}}{2(k^2-3K)} & 0 & 0 & -\frac{k\phi'}{2a^2\sqrt{\gamma}(k^2-3K)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k\frac{a^2\frac{\delta V}{\delta\phi}+3\phi'\mathcal{H}}{2(k^2-3K)} & 1 & -k\frac{a^2\frac{\delta V}{\delta\phi}+3\phi'\mathcal{H}}{2(k^2-3K)} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \delta(\vec{x}-\vec{y}). \tag{21}$$

We observed that $Tr\Lambda = 2$, therefore we have only two true physical degrees of freedom. Let us introduce the phase space vector ξ with the following components

$$(\xi^1, \xi^2, \xi^3, \xi^4, \xi^5, \xi^6) = (A, \Phi, \Psi, \Pi_A, \Pi_\Phi, \Pi_\Psi). \tag{22}$$

By using (18) we obtain

$$\begin{aligned} \zeta_1^\star &= -k\frac{(a^2\frac{\delta V}{\delta\phi}+3\phi'\mathcal{H})}{2(k^2-3K)}\xi^2 - \frac{k\phi'}{2a^2\sqrt{\gamma}(k^2-3K)}\xi^5, \\ \zeta_2^\star &= \xi^2 + \frac{k\phi'}{2a^2\sqrt{\gamma}(k^2-3K)}(\xi^4+\xi^6), \\ \zeta_3^\star &= -k\frac{(a^2\frac{\delta V}{\delta\phi}+3\phi'\mathcal{H})}{2(k^2-3K)}\xi^2 - \frac{k\phi'}{2a^2\sqrt{\gamma}(k^2-3K)}\xi^5, \\ \zeta_4^\star &= 0, \\ \zeta_5^\star &= -k\frac{(a^2\frac{\delta V}{\delta\phi}+3\phi'\mathcal{H})}{2(k^2-3K)}(\xi^4+\xi^6)+\xi^5, \\ \zeta_6^\star &= 0\ . \end{aligned} \tag{23}$$

We observed that

$$\xi_1^\star = \xi_3^\star \tag{24}$$

and

$$\zeta_5^\star = -\frac{a^2\sqrt{\gamma}}{\varphi'}\{\zeta_2^\star(a^2\frac{\delta V}{\delta\phi}+3\phi'\mathcal{H}) + \frac{1}{2(k^2-3K)}\zeta_1^\star\}, \tag{25}$$

therefore only two physical variables $\zeta_1^\star, \zeta_5^\star$ can be used as a starting point for the quantization of the system. As it can be seen from (24) and (25) we obtain the same degrees of freedom as in [8].

4 Conclusions

The integrability of HJ total differential equations is an open and attractive issue. In our study we obtained the gauge conditions directly from the consistency conditions within the HJ formalism. This result is based on the fact that if the canonical Hamiltonian represents a sum of two terms, the second one becomes another "Hamiltonian" in the HJ formalism. In other words the canonical Hamiltonian represents a case of an irregular Hamiltonian. If we set $H_3 = A - \Psi$ and $H_4 = \Pi_\Psi$ we obtain four "Hamiltonians" in our case. As it can be seen, the obtained "Hamiltonians" are not in involution, therefore the systems corresponding to these "Hamiltonians" is not integrable. To make it integrable we work on the surface of constraints and this way leads us to the same canonical Hamiltonian from up to a constant.

The above result can be generalized to the case when the canonical Hamiltonian has the form $H_c = H_0 + \phi_1 H_1 + \cdots, \phi_n H_n$, where the fields $\phi_1, \cdots, \phi_n$ do not appear in any "Hamiltonians" $H_1, \cdots, H_n$. In this case all $\phi_1, \cdots, \phi_n$ are gauge variables and they can be fixed after imposing the integrability conditions. In order to calculate the action we have to find the linearly independent "Hamiltonians" possessing the physical significance from the HJ point of view.

Since the set of four "Hamiltonians" is a second class typed in Dirac classifications the symplectic projector method was used to obtain the true degrees of freedom of the model under investigation. The results were found to be in agreement with those from [8].

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