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Global attractivity for fractional order delay partial integro-differential equations

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Abstract

Our aim in this work is to study the existence and the attractivity of solutions for a system of delay partial integro-differential equations of fractional order. We use the Schauder fixed point theorem for the existence of solutions, and we prove that all solutions are locally asymptotically stable.

AMS (MOS) Subject Classifications: 26A33.

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1. Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus and it has been developed up to nowadays (see Kilbas et al. [1], Hilfer [2]). Fractional differential and integral equations have recently been applied in various areas of Engineering, Mathematics, Physics and Bio-engineering and so on. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Baleanu et al. [3], Hilfer [2], Kilbas et al. [1], Lakshmikantham et al. [4], Podlubny [5], and the articles by Abbas et al. [6-8], Vityuk and Golushkov [9]. Recently interesting results of the stability of the solutions of various classes of integral equations of fractional order have obtained by Banaś et al. [10,11], Darwish et al. [12], Dhage [13,14] and the references therein.

In this article, we established sufficient conditions for the existence and the attractivity of solutions of the following system of delay integro-differential equations of fractional order of the form

$$^{c}D_{\theta}^{r}u(t,x)=f(t,x,I_{0,x}^{r_2}u(t,x),u(t-\tau_1,x-\xi_1),\ \ldots,\ u(t-\tau_m,\ x-\xi_m));$$

$$\text{for } (t,\ x)\in J:=\mathbb{R}_+\times[0,\ b], \tag{1}$$

$$u(t,\ x)=\Phi(t,\ x);\ \text{for } (t,\ x)\in\tilde{J}:=[-T,\ \infty)\times[-\xi,\ b]\backslash(0,\ \infty)\times(0,\ b], \tag{2}$$

$$\begin{cases} u(t,\ 0) = \varphi(t);\ t \in [0,\ \infty), \\ u(0,\ x) = \psi(x);\ x \in [0,\ b], \end{cases} \tag{3}$$

Where $b > 0$, $\theta = (0, 0)$ $\mathbb{R}_+ = [0, \infty)$, τ_i, $\xi_i \geq 0$; $i = 1\ldots, m$, $T = \max_{i=1,\ldots m}\{\tau_i\}$, ${}^cD_\theta^r$, ${}^cD_\theta^r$ is the Caputo fractional derivative of order $r = (r_1,r_2) \in (0, \infty)\times(0, \infty)$, $I_{0,x}^{r_2}$ is the partial Riemann-Liouville integral of order r_2 with respect to x, $f : J \times \mathbb{R}^m \to \mathbb{R}$ is a given continuous function, $\phi : \mathbb{R}_+ \to \mathbb{R}$, $\psi : [0, b] \to \mathbb{R}$ are absolutely continuous functions with $\lim_{t\to\infty} \phi(t) = 0$, and $\psi(x) = \phi(0)$ for each $x \in [0, b]$, and $\Phi : \tilde{J} \to \mathbb{R}^n$ is continuous with $\phi(t) = \Phi(t, 0)$ for each $t \in \mathbb{R}+$, and $\psi(x) = \Phi(0, x)$ for each $x \in [0, b]$.

This article initiates the question of local attractivity of the solution of problem (1)-(3).

2. Preliminaries

In the following, we present briefly notations, definitions, and preliminary facts which are used throughout this article. By $L^1([0, a] \times [0, b])$; $a, b > 0$, we denote the space of Lebesgue-integrable functions $u : [0, a] \times [0, b] \to \mathbb{R}$ with the norm

$$||u||_1 = \int_0^a \int_0^b |u(t,\ x)|dxdt.$$

By $BC := BC([-T, \infty)\times[-\xi, b])$ we denote the Banach space of all bounded and continuous functions from $[-T, \infty) \times [-\xi, b]$ into $\mathbb{R}$ equipped with the standard norm

$$||u||_{BC} = \sup_{(t,x)\in[-T,\infty)\times[-\xi,b]} |u(t,\ x)|.$$

For $u_0 \in BC$ and $\eta \in (0, \infty)$, we denote by $B(u_0, \eta)$, the closed ball in BC centered at u_0 with radius η.

Definition 2.1 [15]*Let* $\rho \in (0, \infty)$ *and* $u \in L^1([0, a]\times[0, b])$,$a, b > 0$. *The partial Riemann-Liouville integral of order ρ of u(t, x) with respect to t is defined by the expression*

$$I_{0,t}^r u(t,\ x) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} u(s,\ x)ds, \textit{ for almost all } (t,\ x) \in [0,\ a] \times [0,\ b].$$

Analogously, we define the integral

$$I_{0,x}^r u(t,\ x) = \frac{1}{\Gamma(\rho)} \int_0^x (x-s)^{\rho-1} u(t,\ s)ds, \text{for almost all } (t,\ x) \in [0,\ a] \times [0,\ b].$$

Definition 2.2 [9]*Let* $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, $\theta = (0, 0)$ *and* $u \in L^1([0, a] \times [0, b])$. *The left-sided mixed Riemann-Liouville integral of order r of u is defined by*

$$(I_\theta^r u)(t,\ x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1} u(s,\ \tau)dsd\tau.$$

In particular,

$$(I_\theta^\theta u)(t,\ x) = u(t,\ x), (I_\theta^\sigma u)(t,\ x) = \int_0^t \int_0^x u(\tau,\ s) ds d\tau; \text{for almost all} (t,\ x) \in [0,\ a] \times [0,\ b],$$

where $\sigma = (1,\ 1)$. For instance, $I_\theta^r u$ exists for all $r_1,\ r_2 \in\ (0,\ \infty)$, when $u \in\ L^1([0,\ a]\ \times$ $[0,\ b])$.

Note also that when $u \in\ C([0,\ a]\ \times\ [0,\ b])$, then $(I_\theta^r u) \in C([0, a] \times [0, b])$, moreover

$$(I_\theta^r u)(t,\ 0) = (I_\theta^r u)(0,\ x) = 0; t \in [0,\ a],\ x \in [0,\ b].$$

Example 2.3 *Let* $\lambda,\ \omega \in\ (-1,\ \infty)$ *and* $r = (r_1,\ r_2) \in\ (0,\ \infty)\ \times\ (0,\ \infty)$, *then*

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2}, \textit{for almost all} (t,\ x) \in [0,\ a] \times [0,\ b].$$

By $1\ -\ r$ we mean $(1\ -\ r_1,\ 1\ -\ r_2) \in\ (0,\ 1]\ \times\ (0,\ 1]$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

Definition 2.4 [9]*Let* $r \in\ (0,\ 1]\ \times\ (0,\ 1]$ *and* $u \in\ L^1([0,\ a]\ \times\ [0,\ b])$. *The Caputo fractional-order derivative of order* r *of* u *is defined by the expression* ${}^cD_\theta^r u(t,\ x) = (I_\theta^{1-r} D_{tx}^2 u)(t,\ x)$.

The case $\sigma = (1,\ 1)$ is included and we have

$$({}^cD_\theta^\sigma u)(t,\ x) = (D_{tx}^2 u)(t,\ x), \text{for almost all } (t,\ x) \in [0,\ a] \times [0,\ b].$$

Example 2.5 *Let* $\lambda,\ \omega \in\ (-1,\ \infty)$ *and* $r = (r_1,\ r_2) \in\ (0,\ 1]\ \times\ (0,\ 1]$, *then*

$${}^cD_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2}, \textit{ for almost all} (t,\ x) \in [0,\ a] \times [0,\ b].$$

Let $\emptyset \neq \Omega \subset BC$, and let $G : \Omega \to \Omega$, and consider the solutions of equation

$$(Gu)(t,x) = u(t,x). \tag{4}$$

Inspired by the definition of the attractivity of solutions of integral equations (see for instance [10]), we introduce the following concept of attractivity of solutions for Equation (4).

Definition 2.6 *Solutions of Equation (4) are locally attractive if there exists a ball* $B(u_0,\ \eta)$ *in the space BC such that, for arbitrary solutions* $v = v(t,\ x)$ *and* $w = w(t,\ x)$ *of Equation (4) belonging to* $B(u_0,\ \eta) \cap \Omega$, *we have that, for each* $\times \in\ [0,\ b]$,

$$\lim_{t\to\infty} (v(t,\ x) - w(t,\ x)) = 0. \tag{5}$$

When the limit (5) is uniform with respect to $B(u_0,\ \eta) \cap \Omega$, *solutions of Equation (4) are said to be uniformly locally attractive (or equivalently that solutions of (4) are asymptotically stable).*

Lemma 2.7 [16]*Let* $D \subset BC$. *Then* D *is relatively compact in BC if the following conditions hold:*

(a) D is uniformly bounded in BC,

(b) The functions belonging to D are almost equicontinuous on $\mathbb{R}_+ \times [0, b]$*, i.e., equicontinuous on every compact of* $\mathbb{R}_+ \times [0, b]$,

(c) The functions from D are equiconvergent, that is, given $\varepsilon > 0$, $x \in [0, b]$ *there corresponds* $T(\varepsilon, x) > 0$ *such that* $|u(t, x)\text{-}\lim_{t\to\infty} u(t, x)| < \varepsilon$ *for any* $t \geq T(\varepsilon, x)$ *and* $u \in D$.

3. Main results

Let us start by defining what we mean by a solution of the problem (1)-(3).

Definition 3.1 *A function* $u \in BC$ *is said to be a solution of (1)-(3) if u satisfies Equation*

(1) on J, Equation (2) on $\tilde{J}$*and condition (3) is satisfied.*

Lemma 3.2 [6]*Let* $f \in L^1([0, a] \times [0, b])$; $a, b > 0$. *A function* $u \in AC([0, a] \times [0, b])$ *is a solution of problem*

$$\begin{cases} ({}^cD^r_\theta u)(t, x) = f(t, x); (t, x) \in [0, a] \times [0, b], \\ u(t, 0) = \varphi(t);\ t \in [0, a],\ u(0, x) = \psi(x);\ x \in [0, b], \\ \varphi(0) = \psi(0), \end{cases}$$

if and only if $u(t, x)$ *satisfies*

$$u(t, x) = \mu(t, x) + (I^r_\theta f)(t, x); (t, x) \in [0, a] \times [0, b],$$

where

$$\mu(t, x) = \varphi(t) + \psi(x) - \varphi(0).$$

Now, we shall prove the following theorem concerning the existence and the attractivity of a solution of problem (1)-(3).

Theorem 3.3 *Assume that the function f satisfying the following hypothesis*

(H) There exists continuous functions $p_i : \mathbb{R}_+ \times [0, b] \to \mathbb{R}_+$ *such that*

$$\left(1 + \sum_{i=0}^m |u_i|\right) \left|f(t, x, u_0, u_1, u_2, \dots, u_m)\right| \leq \sum_{i=0}^m |u_i| p_i(t, x);$$

for $(t, x) \in \mathbb{R}_+ \times [0, b]$ *and for* $u_i \in \mathbb{R}$; $i = 0,\dots, m$. *Moreover, assume that*

$$\lim_{t\to\infty} I^r_\theta p_i(t, x) = 0; x \in [0, b]; i = 0, \dots, m.$$

Then the problem (1)-(3) has at least one solution in the space BC. Moreover, solutions of problem (1)-(3) are uniformly locally attractive.

Proof. Set

$$\Phi^* := \sup_{(t,x)\in\tilde{J}} \Phi(t, x), \varphi^* := \sup_{t\in\mathbb{R}_+} \varphi(t)\, and\, p_i^* := \sup_{(t,x)\in\mathbb{R}_+\times[0,b]} I^r_\theta p_i(t, x); i = 0, \dots, m.$$

From (H), we infer that $p_i^*; i = 0, \dots, m$ are finite. Let us define the operator N such that, for any $u \in BC$,

$$(Nu)(t, x) = \begin{cases} \Phi(t, x); & (t, x) \in \tilde{J}, \\ \varphi(t) & \\ +I^r_\theta f(t, x, I^{r_2}_{0,x} u(t, x), u(t - \tau_1, x - \xi_1), \dots, u(t - \tau_m, x - \xi_m)); & (t, x) \in J. \end{cases} \quad (6)$$

The operator N maps BC into BC; Indeed the map $N(u)$ is continuous on $[-T, \infty) \times [-\xi, b]$ for any $u \in BC$, and for each $(t, x) \in J$ we have

$$\begin{aligned}
&|(Nu)(t,\ x)| \le |\varphi(t)| \\
&+ |I_\theta^r f(t,\ x,\ I_{0,x}^{r_2}u(t,\ x),\ u(t-\tau_1,\ x-\xi_1),\ \ldots,\ u(t-\tau_m,\ x-\xi_m))| \\
&\le |\varphi(t)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1} \\
&\times \left(|I_{0,s}^{r_2}u(\tau,\ s)|p_0(\tau,\ s) + \sum_{i=1}^m |u(\tau-\tau_i,\ s-\xi_i)|p_i(\tau,\ s) \right) \\
&\times \left(1 + |I_{0,s}^{r_2}u(\tau,\ s)| + \sum_{i=1}^m |u(\tau-\tau_i,\ s-\xi_i)| \right)^{-1} dsd\tau. \\
&\le \varphi^* + \sum_{i=0}^m p_i^*,
\end{aligned}$$

and for $(t, x) \in \tilde{J}$ we have

$$|(Nu)(t,\ x)| = |\Phi(t,\ x)| \le \Phi^*.$$

Thus,

$$||N(u)||_{BC} \le \max\left\{\Phi^*, \varphi^* + \sum_{i=0}^m p_i^*\right\} := \eta. \tag{7}$$

Hence, $N(u) \in BC$. This proves that the operator N maps BC into itself.

By Lemma 3.2, the problem of finding the solutions of the problem (1)-(3) is reduced to finding the solutions of the operator equation $N(u)= u$. Equation (7) yields that N transforms the ball $B_\eta := B(0, \eta)$ into itself. We shall show that $N : B_\eta \to B_\eta$ satisfies the assumptions of Schauder's fixed point theorem [17]. The proof will be given in several steps and cases.

Step 1: *N is continuous.*

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_η. Then, for each $(t, x) \in [-T, \infty) \times[-\xi, b]$, we have

$$\begin{aligned}
&|(Nu_n)(t, x) - (Nu)(t, x)| \\
&\le \tfrac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1} \\
&\times |f(\tau,\ s,\ I_{0,s}^{r_2}u_n(\tau,\ s),\ u_n(\tau-\tau_1,\ s-\xi_1),\ \ldots,\ u_n(\tau-\tau_m,\ s-\xi_m)) \\
&- f(\tau,\ s,\ I_{0,s}^{r_2}u(\tau,\ s),\ u(\tau-\tau_1,\ s-\xi_1),\ \ldots,\ u(\tau-\tau_m,\ s-\xi_m)))|dsd\tau.
\end{aligned} \tag{8}$$

Case 1. If $(t,\ x) \in \tilde{J} \cup ([0,\ T_0] \times [0,\ b]), T_0 > 0$, then, since $u_n \to u$ as $n \to\infty$ and f, $I_{0,\xi}^{r_2}$, are continuous, (8) gives

$$||N(u_n) - N(u)||_{BC} \to 0 \,\text{as}\, n \to \infty.$$

Case 2. If $(t, x) \in (T_0, \infty) \times [0, b]$, $T_0 > 0$, then from (H) and (8), we get

$$\begin{aligned}
&|(Nu_n)(t, x) - (Nu)(t, x)| \\
&\leq \frac{2}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1} \\
&\times \left(\left| I_{0,s}^{r_2} u(\tau, s) \right| p_0(\tau, s) + \sum_{i=1}^m |u(\tau-\tau_i, s-\xi_i)| p_i(\tau, s) \right) \\
&\times \left(1 + |I_{0,s}^{r_2} u(\tau, s)| + \sum_{i=1}^m |u(\tau-\tau_i, s-\xi_i)| \right)^{-1} ds d\tau \\
&\leq \sum_{i=1}^m \frac{2}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1} p_i(\tau, s) ds d\tau \\
&\leq 2 \sum_{i=1}^m I_\theta^\tau p_i(t, x).
\end{aligned} \tag{9}$$

Since $u_n \to u$ as $n \to\infty$ and $t \to\infty$, then (9) gives

$$||N(u_n) - N(u)||_{BC} \to 0 \text{ as } n \to \infty.$$

Step 2: *$N(B_\eta)$ is uniformly bounded.*

This is clear since $N(B_\eta) \subset B_\eta$ and B_η is bounded.

Step 3: *$N(B_\eta)$ is equicontinuous on every compact subset $[-T, a] \times [-\xi, b]$ of $[-T, a] \times [-\xi, \infty)$, $a > 0$.*

Let $(t_1, x_1), (t_2, x_2) \in [0, a] \times [0, b]$, $t_1 < t_2$, $x_1 < x_2$ and let $u \in B_\eta$. Thus we have

$$\begin{aligned}
&|(Nu)(t_2, x_2) - (Nu)(t_1, x_1)| \leq |\varphi(t_2) - \varphi(t_1)| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_0^{x_1} [(t_2-\tau)^{r_1-1}(x_2-s)^{r_2-1} - (t_1-\tau)^{r_1-1}(x_1-s)^{r_2-1}] \\
&\quad \times |f(\tau, s, I_{0,s}^{r_2} u(\tau, s), u(\tau-\tau_1, s-\xi_1), \ldots, u(\tau-\tau_m, s-\xi_m))| ds d\tau \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_{x_1}^{x_2} (t_2-\tau)^{r_1-1}(x_2-s)^{r_2-1} \\
&\quad \times |f(\tau, s, I_{0,s}^{r_2} u(\tau, s), u(\tau-\tau_1, s-\xi_1), \ldots, u(\tau-\tau_m, s-\xi_m))| ds d\tau \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_{x_1}^{x_2} (t_2-\tau)^{r_1-1}(x_2-s)^{r_2-1} \\
&\quad \times |f(\tau, s, I_{0,s}^{r_2} u(\tau, s), u(\tau-\tau_1, s-\xi_1), \ldots, u(\tau-\tau_m, s-\xi_m))| ds d\tau \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_0^{x_1} (t_2-\tau)^{r_1-1}(x_2-s)^{r_2-1} \\
&\quad \times |f(\tau, s, I_{0,s}^{r_2} u(\tau, s), u(\tau-\tau_1, s-\xi_1), \ldots, u(\tau-\tau_m, s-\xi_m))| ds d\tau.
\end{aligned}$$

Thus

$$
\begin{aligned}
&|(Nu)(t_2,\ x_2) - (Nu)(t_1, x_1)| \le |\varphi(t_2) - \varphi(t_1)| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{t_1} \int_0^{x_1} \left[(t_2 - \tau)^{r_1 - 1}(x_2 - s)^{r_2 - 1} - (t_1 - \tau)^{r_1 - 1}(x_1 - s)^{r_2 - 1} \right] \\
&\times \sum_{i=0}^{m} p_i(\tau,\ s) ds d\tau + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_{1}^{x_2} (t_2 - \tau)^{r_1 - 1}(x_2 - s)^{r_2 - 1} \sum_{i=0}^{m} p_i(\tau,\ s) ds d\tau \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_1} \int_{x_1}^{x_2} (t_2 - \tau)^{r_1 - 1}(x_2 - s)^{r_2 - 1} \sum_{i=0}^{m} p_i(\tau,\ s) ds d\tau \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2} \int_{0}^{x_1} (t_2 - \tau)^{r_1 - 1}(x_2 - s)^{r_2 - 1} \sum_{i=0}^{m} p_i(\tau,\ s) ds d\tau.
\end{aligned}
$$

From continuity of ϕ, p_i; $i = 0,...,m$ and as $t_1 \to t_2$ and $x_1 \to x_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 < 0$, $x_1 < x_2 < 0$ and $t_1 \le 0 \le t_2$, $x_1 \le 0 \le x_2$ is obvious.

Step 4: *$N(B_\eta)$ is equiconvergent.*

Let $(t, x) \in \mathbb{R}_+ \times [0, b]$ and $u \in B_\eta$, then we have

$$
\begin{aligned}
|(Nu)(t,\ x)| &\le |\varphi(t)| + \Bigg| \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t - \tau)^{r_1 - 1}(x - s)^{r_2 - 1} \\
&\qquad \times f(\tau,\ s,\ I_{0,s}^{r_2} u(\tau,\ s),\ u(\tau - \tau_1,\ s - \xi_1),\ \ldots,\ u(\tau - \tau_m,\ s - \xi_m)) ds d\tau \Bigg| \\
&\le |\varphi(t)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t - \tau)^{r_1 - 1}(x - s)^{r_2 - 1} \sum_{i=0}^{m} p_i(\tau,\ s) ds d\tau \\
&\le |\varphi(t)| + \sum_{i=0}^{m} I_\theta^r p_i(t,\ x).
\end{aligned}
$$

Thus, for each $x \in [0, b]$, we get

$$|(Nu)(t,\ x)| \to 0,\ as\ t \to +\infty.$$

Hence,

$$|(Nu)(t,\ x) - (Nu)(+\infty,\ x)| \to 0,\ as\ t \to +\infty.$$

As a consequence of Steps 1-4 together with the Lemma 2.7, we can conclude that $N : B_\eta \to B_\eta$ is continuous and compact. From an application of Schauder's theorem [17], we deduce that N has a fixed point u which is a solution of the problem (1)-(3).

Now we investigate the uniform local attractivity for solutions of problem (1)-(3). Let us assume that u_0 is a solution of problem (1)-(3) with the conditions of this theorem. Consider the ball $B(u_0, \eta^*)$, where

$$
\begin{aligned}
\eta^* := \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sup_{(t,x)\in J} \Bigg\{ & \int_0^t \int_0^x (t - \tau)^{r_1 - 1}(x - s)^{r_2 - 1} \\
& \times |f(\tau,\ s,\ I_{0,s}^{r_2} u(\tau,\ s),\ u(\tau - \tau_1, s - \xi_1),\ \ldots,\ u(\tau - \tau_m,\ s - \xi_m)) \\
& - f(\tau,\ s,\ I_{0,s}^{r_2} u_0(\tau,\ s),\ u_0(\tau - \tau_1,\ s - \xi_1),\ \ldots,\ u_0(\tau - \tau_m,\ s - \xi_m))| ds d\tau; u \in BC \Bigg\}.
\end{aligned}
$$

Taking $u \in B(u_0, \eta^*)$, we have

$$\begin{aligned}|(Nu)(t,x) - u_0(t,x)| &= |(Nu)(t,x) - (Nu_0)(t,x)| \\ &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1} \\ &\quad \times |f(\tau, s, I_{0,s}^{r_2}u(\tau,s), u(\tau-\tau_1, s-\xi_1), \ldots, u(\tau-\tau_m, s-\xi_m)) \\ &\quad - f(\tau, s, I_{0,s}^{r_2}u_0(\tau,s), u_0(\tau-\tau_1, s-\xi_1), \ldots, u_0(\tau-\tau_m, s-\xi_m))|dsd\tau \\ &\le \eta^*.\end{aligned}$$

Thus we observe that N is a continuous function such that $N(B(u_0, \eta^*)) \subset B(u_0, \eta^*)$. Moreover, if u is a solution of problem (1)-(3), then

$$\begin{aligned}|u(t,x) - u_0(t,x)| &= |(Nu)(t,x) - (Nu_0)(t,x)| \\ &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1} \\ &\quad \times |f(\tau, s, I_{0,s}^{r_2}u(\tau,s), u(\tau-\tau_1, s-\xi_1), \ldots, u(\tau-\tau_m, s-\xi_m)) \\ &\quad - f(\tau, s, I_{0,s}^{r_2}u_0(\tau,s), u_0(\tau-\tau_1, s-\xi_1), \ldots, u_0(\tau-\tau_m, s-\xi_m))|dsd\tau \\ &\le \frac{2}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1}\sum_{i=0}^m p_i(\tau,s)dsd\tau \\ &\le 2\sum_{i=0}^m I_\theta^r p_i(t,x).\end{aligned} \tag{10}$$

By using (10) and the fact that $I_\theta^r p_i(t,x) \to 0$as $t \to \infty$; $i = 0,\ldots, m$ we deduce that

$$\lim_{t\to\infty}\left|u(t,x) - u_0(t,x)\right| = 0.$$

Consequently, all solutions of problem (1)-(3) are uniformly locally attractive.

4. An example

As an application and to illustrate our results, we consider the following system of delay integro-differential equations of fractional order

$$^cD_\theta^r u(t,x) = f\left(t, x, I_{0,x}^{r_2}u(t,x), u\left(t-1, x-\frac{1}{4}\right), u\left(t-\frac{2}{3}, x-\frac{1}{5}\right)\right);$$

$$\text{for } (t,x) \in J := [0, \infty) \times [0,1], \tag{11}$$

$$u(t,x) = e^{-t}; \text{for } (t,x) \in \tilde{J} := [-1, \infty) \times \left[-\frac{1}{4}, 1\right] \backslash (0, \infty) \times (0,1], \tag{12}$$

$$\begin{cases} u(t,0) = e^{-t}; \ t \in [0, \infty), \\ u(0,x) = 1; \quad x \in [0,1], \end{cases} \tag{13}$$

where $r = (r_1, r_2) = \left(\frac{1}{4}, \frac{1}{2}\right)$ and

$$\begin{cases} f(t,x,u,v,w) = \dfrac{xt^{\frac{-3}{4}}\left(|v|\sin t + |w|e^{-\frac{1}{t}}\right)}{2+|u|+|v|+|w|}; & (t,x) \in (0,\infty)\times[0,1] \ and \ u, v, w \in \mathbb{R}, \\ f(t,x,u,v,w) = 0; & (t,x) \in \{0\}\times[0,1] \ and \ u, v, w \in \mathbb{R}. \end{cases}$$

We have for each $x \in [0, \infty)$, $\mu(t) = e^{-t} \to 0 \ as \ t \to \infty$.

Let us notice that the function f satisfies assumption (H), where

$$p_0(t,x) = 0; (t,x) \in \mathbb{R}_+ \times [0,1],$$

$$\begin{cases} p_1(t,\ x) = xt^{-\frac{3}{4}}|\sin t|; & (t,\ x) \in (0,\ \infty) \times [0, 1], \\ p_1(0,\ x) = 0; & x \in [0, 1], \end{cases}$$

and

$$\begin{cases} p_2(t,\ x) = xt^{-\frac{3}{4}}e^{-\frac{1}{t}}; & (t,\ x) \in (0,\ \infty) \times [0, 1], \\ p_2(0,\ x) = 0; & x \in [0, 1]. \end{cases}$$

Also, for each $x \in\ 0[1]$, we get

$$I_\theta^r p_0(t,\ x) = 0,$$

$$\begin{aligned} I_\theta^r p_1(t,\ x) &= \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1} p_1(\tau,\ s) ds d\tau \\ &= \frac{1}{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})} \int_0^t \int_0^x (t-\tau)^{-\frac{3}{4}}(x-s)^{-\frac{1}{2}} s\tau^{-\frac{3}{4}}|\sin \tau| ds d\tau \\ &\le \frac{1}{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})} \int_0^t \int_0^x (t-\tau)^{-\frac{3}{4}}(x-s)^{-\frac{1}{2}} s\tau^{-\frac{3}{4}} ds d\tau \\ &= \frac{\Gamma(\frac{1}{4})\Gamma(2)}{\Gamma(\frac{5}{4})\Gamma(\frac{5}{2})} t^{-\frac{1}{2}} x^{\frac{3}{2}} \to 0 \ as\ t \to \infty, \end{aligned}$$

and

$$\begin{aligned} I_\theta^r p_2(t,\ x) &= \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-s)^{r_2-1} p_2(\tau,\ s) ds d\tau \\ &= \frac{1}{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})} \int_0^t \int_0^x (t-\tau)^{\frac{-3}{4}}(x-s)^{\frac{-1}{2}} s\tau^{\frac{-3}{4}} e^{-\frac{1}{\tau}} ds d\tau \\ &\le \frac{1}{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})} \int_0^t \int_0^x (t-\tau)^{\frac{-3}{4}}(x-s)^{\frac{-1}{2}} s\tau^{\frac{-3}{4}} ds d\tau \\ &= \frac{\Gamma(\frac{1}{4})\Gamma(2)}{\Gamma(\frac{5}{4})\Gamma(\frac{5}{2})} t^{\frac{-1}{2}} x^{\frac{3}{2}} \to 0 \ as\ t \to \infty. \end{aligned}$$

Thus

$$\lim_{t\to\infty} I_\theta^r p_i(t,\ x) = 0; x \in [0, 1]; i = 0, 1, 2.$$

Hence by Theorem 3.3, the problem (11)-(13) has a solution defined on $[-1,\ \infty) \times [-\frac{1}{4}, 1]$ and all solutions are uniformly locally attractive on $[-1,\ \infty) \times [-\frac{1}{4}, 1]$

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Authors' contributions

SA wrote the first draft of the article, MB has corrected this draft, DB had prepared the final version of the study and enhanced the revised version of the manuscript. The authors read carefully and approved the final version of the manuscript.

Competing interests

The authors declare that they have no competing interests.

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