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Article

A Discussion on Random Meir-Keeler Contractions

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Abstract: The aim of this paper is to enrich random fixed point theory, which is one of the cornerstones of probabilistic functional analysis. In this paper, we introduce the notions of random, comparable MT-γ contraction and random, comparable Meir-Keeler contraction in the framework of complete random metric spaces. We investigate the existence of a random fixed point for these contractions. We express illustrative examples to support the presented results.

Keywords: random fixed point; random metric space; probabilistic functional analysis; random comparable MT-γ contraction; random, comparable Meir-Keeler contraction

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1. Introduction and Preliminaries

One of the most interesting and useful research fields is "probabilistic functional analysis," due to its wide application potential for probabilistic models in applied problems. Random fixed point theory is one of the cornerstones of probabilistic functional analysis. Random fixed point theory is the extension of standard fixed point theory within the framework of random analysis. Putting it differently, random fixed point theory emerges at the intersection of functional analysis, topology and stochastic analysis. That is why in some sources, the results in this direction are known as probabilistic functional analysis. Given the advances in fixed point theorems, the objective of random fixed point theory is to channel the gains of classical fixed point theorems into the field of random analysis. The initial results in random fixed point theory were reported by Spacek [1] and Hans [2,3]. Roughly speaking, the authors expressed the characterization of Banach's fixed point theorem in the setting of separable metric space. After that, the analogs of several well-known, standard-metric fixed point theorems have been reported by distinct authors; see, e.g., [4–14], and corresponding references therein.

In this paper, we also focus on one of the outstanding generalizations of Banach's contraction principle: the Meir-Keeler contraction [15]. Roughly speaking, Meir-Keeler considered "uniform contraction." Recently, Chen and Chang [16] introduced the notions of the "weaker Meir-Keeler function" and the "strong Meir-Keeler function," which were observed from the abstraction of the original idea of Meir-Keeler [15]. These approaches have been investigated heavily by several authors; see, e.g., [17–23].

In what follows, we state some basic definitions and set out our terminology needed in the sequel. Throughout the paper, we assume that all considered sets are non-empty. We set $\mathbb{R}_0^+ = [0, \infty)$, and the letter $\mathbb{N}$ is reserved for positive integers. Let Σ be a sigma-algebra of subsets of Ω. Under this assumption, the pair (Ω, Σ) is called measurable space. Let $\mathcal{P}(X)$ and $CL(X)$ denote the family of all subsets of X, and the family of all closed subset of X, respectively. For a Banach space X, multivalued operator $F : \Omega \to \mathcal{P}(X)$ is named (Σ)-measurable if, for any closed subset B of X, we have

$$F^{-1}(B) = \{v \in \Omega; \;\; F(v) \cap B \neq \emptyset\} \in \Sigma. \tag{1}$$

Here, an operator F is called weakly measurable if the openness condition of the subset B of X in the expression (1) is replaced by "closeness."

For a subset Y of a Banach space X, we say that $f : \Omega \times Y \to X$ is a random operator if $f(\cdot, x)$ is measurable for each fixed $x \in Y$. Furthermore, we say that $F : \Omega \times Y \to \mathcal{P}(X)$ is a random multivalued operator if, for each fixed $x \in Y$, the operator $F(\cdot, x) : \Omega \to \mathcal{P}(X)$ is measurable. Further, we say that $\mu : \Omega \to X$ is a measurable selector of a measurable multivalued operator $F : \Omega \to \mathcal{P}(X)$, in the case that μ is measurable and $\mu(v) \in F(v)$ for any $v \in \Omega$. An operator $\mu : \Omega \to X$ is said to be a random fixed point of the operator $f : \Omega \times Y \to X$ (or, $T : \Omega \times Y \to \mathcal{P}(X)$) if $\mu(v) = f(v, \mu(v))$ (or, $\mu(v) \in T(v, \mu(v))$).

The following result related to the concept of measurability has a crucial role in the sequel.

Lemma 1. *[24] For a complete separable metric space (X, d), if a multivalued operator $T : \Omega \to CL(X)$ is measurable, then T has a measurable selector.*

We introduce the random metric as follows:

Definition 1. *Let M be a nonempty set; let $v \in \Omega$ be a selector; and let the mapping $d : (\Omega \times M)^2 \to \mathbb{R}_0^+$, satisfy the following conditions:*

(1) $d(\xi(v), \eta(v)) \geq 0$ *for all* $\xi(v), \eta(v) \in \Omega \times M$, *where we denote* $\xi(v) := (v, \xi)$;
(2) $d(\xi(v), \eta(v)) = 0$ *if and only if* $\xi(v) = \eta(v)$, *for all* $\xi(v), \eta(v) \in \Omega \times M$;
(3) $d(\xi(v), \eta(v)) = d(\eta(v), \xi(v))$;
(4) $d(\xi(v), \zeta(v)) \leq d(\xi(v), \eta(v)) + d(\eta(v), \zeta(v))$ *for all* $\xi, \eta, \zeta \in M$, *and let* v *be a selector;*
(5) *for any* $\xi, \eta \in M$, $v \in \Omega$, $\xi(v), \eta(v)$ *is nonincreasing and left continuous.*

Then, we say that d is called random metric on M. Furthermore, the pair (M, d) is called random metric space.

In what follows, we state the definition of the $\mathcal{MT}$ function.

Definition 2. *[25] Let ψ be a function that is defined from non-negative reals into the interval $[0, 1)$. If the following is fulfilled,*

$$\limsup_{s \to t^+} \psi(s) = \inf_{\alpha > 0} \sup_{0 < s - t < \alpha} \psi(s) < 1 \;\; \textit{for all} \;\; t \in \mathbb{R}^+,$$

then ψ is called the $\mathcal{MT}$ function.

Theorem 1. *[25] For a mapping $\psi : \mathbb{R}^+ \to [0, 1)$, the following are equivalent.*

(a) *ψ is an $\mathcal{MT}$ function.*
(b) *For any non-increasing sequence $\{\delta_n\}_{n \in \mathbb{N}}$ in $\mathbb{R}^+$, we have*

$$0 \leq \sup_{n \in \mathbb{N}} \psi(\delta_n) < 1.$$

Remark 1. *[25] Notice that in the case that $\psi : \mathbb{R}^+ \to [0, 1)$ is non-increasing or non-decreasing, then ψ is a $\mathcal{MT}$ function.*

2. Main Results

The mapping $\gamma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called a comparable function, if the following three axioms are fulfilled:

(1) γ is a non-decreasing, continuous function in each coordinate;
(2) for all $r > 0$, $\gamma(r, r, r, r) \leq r$, $\gamma(0, r, 0, r) \leq r$ and $\gamma(0, 0, r, r) \leq r$;
(3) $\gamma(r_1, r_2, r_3, r_4) = 0$ if and only if $r_1 = r_2 = r_3 = r_4 = 0$.

Definition 3. *Let M be a nonempty subset of a random metric space (X, d), ψ be a $\mathcal{MT}$ function and $T : \Omega \times M \to M$ be a random operator. Then, for $v \in \Omega$, $T(v, \cdot)$ is called a random, comparable $\mathcal{MT}$-γ contraction if the following condition holds:*

$$d(T(\xi(v)), T(\eta(v))) \leq \psi(d(\xi(v), \eta(v))) \cdot \Gamma(\xi(v), \eta(v)),$$

where

$$\begin{aligned}\Gamma(\xi(v), \eta(v)) = \gamma(&d(\xi(v), \eta(v)), d(\xi(v), T(\xi(v))), d(\eta(v), T(\eta(v))),\\ &\frac{d(\xi(v), T(\eta(v))) + d(\eta(v), T(\xi(v)))}{2}),\end{aligned}$$

for all $\xi, \eta \in M$.

Theorem 2. *Suppose (X, d) is a complete random metric space and $M \subset X$. If $T(v, \cdot) : \Omega \times M \to M$ is a continuous, random, comparable $\mathcal{MT}$-γ contraction, then T possesses a random fixed point in X.*

Proof. Given $\xi_0(v) \in \Omega \times X$ and defining $\xi_1(v) = T(\xi_0(v))$ and $\xi_{n+1}(v) = T(\xi_n(v)) = T^{n+1}(\xi_0(v))$ for each $n \in \mathbb{N}$, since $T(v, \cdot) : \Omega \times M \to M$ is a random, comparable $\mathcal{MT}$-γ contraction, we have

$$\begin{aligned}d(\xi_n(v), \xi_{n+1}(v)) =& d(T(\xi_{n-1}(v)), T(\xi_n(v)))\\ \leq& \psi(d(\xi_{n-1}(v), \xi_n(v))) \cdot \Gamma(\xi_{n-1}(v), \xi_n(v)),\end{aligned}$$

and

$$\begin{aligned}\Gamma(\xi_{n-1}(v), \xi_n(v)) =& \gamma(d(\xi_{n-1}(v), \xi_n(v)), d(\xi_{n-1}(v), T(\xi_{n-1}(v))), d(\xi_n(v), T(\xi_n(v))),\\ &\frac{d(\xi_{n-1}(v), T(\xi_n(v))) + d(\xi_n(v), T(\xi_{n-1}(v)))}{2})\\ =& \gamma(d(\xi_{n-1}(v), \xi_n(v)), d(\xi_{n-1}(v), \xi_n(v)), d(\xi_n(v), \xi_{n+1}(v)),\\ &\frac{d(\xi_{n-1}(v), \xi_{n+1}(v)) + d(\xi_n(v), \xi_n(v))}{2}).\end{aligned}$$

If $d(\xi_n(v), \xi_{n+1}(v)) > d(\xi_{n-1}(v), \xi_n(v))$ for some n, then by the conditions of the function γ we have that

$$\begin{aligned}\Gamma(\xi_{n-1}(v), \xi_n(v)) =& \gamma(d(\xi_{n-1}(v), \xi_n(v)), d(\xi_{n-1}(v), \xi_n(v)), d(\xi_n(v), \xi_{n+1}(v)),\\ &\frac{d(\xi_{n-1}(v), \xi_{n+1}(v)) + d(\xi_n(v), \xi_n(v))}{2})\\ \leq& \gamma(d(\xi_n(v), \xi_{n+1}(v)), d(\xi_n(v), \xi_{n+1}(v), d(\xi_n(v), \xi_{n+1}(v)),\\ &d(\xi_n(v), \xi_{n+1}(v)))\\ \leq& d(\xi_n(v), \xi_{n+1}(v)).\end{aligned}$$

In a different order pair of γ

$$\begin{aligned}d(\xi_{n+1}(v), \xi_n(v)) =& d(T(\xi_n(v)), T(\xi_{n-1}(v)))\\ \leq& \psi(d(\xi_n(v), \xi_{n-1}(v))) \cdot \Gamma(\xi_n(v), \xi_{n-1}(v)),\end{aligned}$$

and

$$\begin{aligned}\Gamma(\xi_n(v),\xi_{n-1}(v)) =&\gamma(d(\xi_n(v),\xi_{n-1}(v)),d(\xi_n(v),T(\xi_n(v))),d(\xi_{n-1}(v),T(\xi_{n-1}(v))),\\ &\frac{d(\xi_n(v),T(\xi_{n-1}(v)))+d(\xi_{n-1}(v),T(\xi_n(v)))}{2})\\ =&\gamma(d(\xi_n(v),\xi_{n-1}(v)),d(\xi_n(v),\xi_{n+1}(v)),d(\xi_{n-1}(v),\xi_n(v)),\\ &\frac{d(\xi_n(v),\xi_n(v))+d(\xi_{n-1}(v),\xi_{n+1}(v))}{2}).\end{aligned}$$

If $d(\xi_n(v),\xi_{n+1}(v)) > d(\xi_{n-1}(v),\xi_n(v))$ for some n, then by the conditions of the comparable function γ we have that

$$\begin{aligned}\Gamma(\xi_n(v),\xi_{n-1}(v)) =&\gamma(d(\xi_n(v),\xi_{n-1}(v)),d(\xi_n(v),\xi_{n+1}(v)),d(\xi_{n-1}(v),\xi_n(v)),\\ &\frac{d(\xi_n(v),\xi_n(v))+d(\xi_{n-1}(v),\xi_{n+1}(v))}{2})\\ \leq&\gamma(d(\xi_n(v),\xi_{n+1}(v)),d(\xi_n(v),\xi_{n+1}(v),d(\xi_n(v),\xi_{n+1}(v)),\\ &d(\xi_n(v),\xi_{n+1}(v)))\\ \leq&d(\xi_n(v),\xi_{n+1}(v)).\end{aligned}$$

Since ψ is a $\mathcal{MT}$ function, we conclude that

$$d(\xi_n(v),\xi_{n+1}(v)) \leq \psi(d(\xi_{n-1}(v),\xi_n(v)))\cdot d(\xi_n(v),\xi_{n+1}(v)) < d(\xi_n(v),\xi_{n+1}(v)),$$

which implies a contradiction. So, we conclude that

$$d(\xi_n(v),\xi_{n+1}(v)) \leq d(\xi_{n-1}(v),\xi_n(v)), \quad \text{for each} \;\; n\in\mathbb{N}.$$

From above argument, the sequence $\{d(\xi_n(v),\xi_{n+1}(v))\}_{n\in\mathbb{N}\cup\{0\}}$ is non-increasing in $\mathbb{R}_0^+$. Since ψ is an $\mathcal{MT}$ function, by Theorem 1 we conclude that

$$0 \leq \sup_{n\in\mathbb{N}} \psi(d(\xi_n(v),\xi_{n+1}(v))) < 1.$$

Let $\lambda = sup_{n\in\mathbb{N}}\psi(d(\xi_n(v),\xi_{n+1}(v))) < 1$; then

$$0 \leq \psi(d(\xi_n(v)),\xi_{n+1}(v))) \leq \lambda, \;\; \text{for all} \;\; n\in\mathbb{N}.$$

Following from the above argument and by T being a random, comparable $\mathcal{MT}$ contraction, we conclude that for each n

$$\begin{aligned}&d(\xi_n(v),\xi_{n+1}(v))\\ \leq&\psi(d(\xi_{n-1}(v),\xi_n(v)))\cdot d(\xi_{n-1}(v),\xi_n(v))\\ \leq&\lambda\cdot d(\xi_{n-1}(v),\xi_n(v)).\end{aligned}$$

Therefore, we also conclude that

$$\begin{aligned}
&d(\xi_n(v),\xi_{n+1}(v))\\
=&d(T\xi_{n-1}(v),T\xi_n(v))\\
\leq&\lambda\cdot d(\xi_{n-1}(v),\xi_n(v))\\
\leq&\lambda^2\cdot d(\xi_{n-2}(v),\xi_{n-1}(v))\\
\leq&\cdots\\
\leq&\lambda^n\cdot d(\xi_0(v),\xi_1(v)).
\end{aligned}$$

So we have that $\lim_{n\to\infty} d(\xi_n(v),\xi_{n+1}(v))=0$, since $\lambda<1$, and for $n>m$,

$$\begin{aligned}
&d(\xi_m(v),\xi_n(v))\\
\leq&(\lambda^m+\lambda^{m+1}+\cdots+\lambda^{n-1})\cdot d(\xi_0(v),\xi_1(v))\\
\leq&\frac{\lambda^m}{1-\lambda}\cdot d(\xi_0(v),\xi_1(v)).
\end{aligned}$$

Let $0\leq\delta$ be given. Then we can choose a natural number M such that

$$\frac{\lambda^m}{1-\lambda}\cdot d(\xi_0(v),\xi_1(v))<\delta,\ \ \text{for all}\ \ m\geq M,$$

and we also conclude that

$$d(\xi_m(v),\xi_n(v))<\delta,\ \ \text{for all}\ \ m\geq M.$$

So $\{\xi_n(v)\}$ is a Cauchy sequence in $\Omega\times X$. On account of the fact that (X,d) is complete, there exists a $\xi^*(v)\in\Omega\times X$ such that $\xi_n(v)$ converges to $\xi^*(v)$; that is,

$$\lim_{n\to\infty}\xi_n(v)=\xi^*(v).$$

Thus, we have

$$\begin{aligned}
&d(\xi^*(v),T(\xi^*(v)))\\
\leq&d(\xi^*(v),\xi_{n+1}(v))+d(\xi_{n+1}(v),T(\xi^*(v)))\\
\leq&d(\xi^*(v),\xi_{n+1}(v))+d(T(\xi_n(v)),T(\xi^*(v)))\\
\leq&d(\xi^*(v),\xi_{n+1}(v))+\psi(d(\xi_n(v),\xi^*(v)))\cdot\Gamma(\xi_n(v),\xi^*(v))\\
<&d(\xi^*(v),\xi_{n+1}(v))+\Gamma(\xi_n(v),\xi^*(v)),
\end{aligned}$$

and

$$\begin{aligned}
&\Gamma(\xi_n(v),\xi^*(v))\\
=&\gamma(d(\xi_n(v),\xi^*(v)),d(\xi_n(v),T(\xi_n(v))),d(\xi^*(v),T(\xi^*(v))),\\
&\quad\frac{d(\xi_n(v),T(\xi^*(v)))+d(\xi^*(v),T(\xi_n(v)))}{2})\\
\leq&\gamma(d(\xi_n(v),\xi^*(v)),d(\xi_n(v),\xi_{n+1}(v)),d(\xi^*(v),T(\xi^*(v))),\\
&\quad\frac{d(\xi_n(v),\xi^*(v))+d(\xi^*(v),T(\xi^*(v)))+d(\xi^*(v),\xi_{n+1}(v))}{2}).
\end{aligned}$$

Taking $n\to\infty$, we have

$$\lim_{n\to\infty}\Gamma(\xi_n(v),\xi^*(v))=\gamma(0,0,d(\xi^*(v),T(\xi^*(v))),\frac{d(\xi^*(v),T(\xi^*(v)))}{2}).$$

In a different order pair of γ

$$\begin{aligned}
&\Gamma(\xi^*(v),\xi_n(v))\\
=&\gamma(d(\xi^*(v),\xi_n(v)),d(\xi^*(v),T(\xi^*(v))),d(\xi_n(v),T(\xi_n(v))),\\
&\frac{d(\xi^*(v),T(\xi_n(v))+d(\xi_n(v),T(\xi^*(v))))}{2})\\
\leq&\gamma(d(\xi^*(v),\xi_n(v)),d(\xi^*(v),T(\xi^*(v))),d(\xi_n(v),\xi_{n+1}(v)),\\
&\frac{d(\xi^*(v),\xi_{n+1}(v))+d(\xi_n(v),\xi^*(v))+d(\xi^*(v),T(\xi^*(v)))}{2}).
\end{aligned}$$

Taking $n\to\infty$, we have

$$\lim_{n\to\infty}\Gamma(\xi^*(v),\xi_n(v)) =\gamma(0,d(\xi^*(v),T(\xi^*(v)),0,\frac{d(\xi^*(v),T(\xi^*(v)))}{2}).$$

By the condition of the mapping γ, we conclude that

$$d(\xi^*(v),T(\xi^*(v)))<d(\xi^*(v),T(\xi^*(v))),$$

and this is a contraction unless $d(\xi^*(v),T(\xi^*(v)))=0$.

Therefore, $\xi^*(v)=T(\xi^*(v))$, that is $\xi^*(v)$ is a random fixed point of T in X. $\square$

Example 1. *Let $X=M=\mathbb{R}_0^+$, $\Omega=[0,1]$ and Σ be the sigma algebra of Lebegue's measurable subset of [0,1]. We define mapping as $d:(\Omega\times X)\times(\Omega\times X)\to M$ by $d(\xi(v),\eta(v))=\mid\xi(v)-\eta(v)\mid$. Then (X,d) is a random metric space. Define random operator $T:\Omega\times X\to X$ as*

$$T(\xi(v))=\frac{\xi(v)+1-v^2}{2}.$$

Let $\psi(t)=\frac{t}{2t+1}+\frac{1}{2}$ and $\gamma(t_1,t_2,t_3,t_4)=max\{t_1,t_2,t_3,t_4\}$; then

$$\begin{aligned}
d(T(\xi(v)),T(\eta(v))) =&d(\frac{\xi(v)+1-v^2}{2},\frac{\eta(v)+1-v^2}{2})\\
=&\mid\frac{\xi(v)+1-v^2}{2}-\frac{\eta(v)+1-v^2}{2}\mid\\
=&\mid\frac{\xi(v)-\eta(v)}{2}\mid\\
=&\frac{1}{2}d(\xi(v),\eta(v)),
\end{aligned}$$

and

$$\begin{aligned}
&\psi(t)\cdot\gamma(d(\xi(v),\eta(v)),d(\xi(v),T(\xi(v))),d(\eta(v),T(\eta(v))),\\
&\quad\frac{d(\xi(v),T(\eta(v)))+d(\eta(v),T(\xi(v)))}{2})\\
\geq&\frac{1}{2}\cdot max\{\mid\xi(v)-\eta(v)\mid,\mid\xi(v)-\frac{\xi(v)+1-v^2}{2}\mid,\mid\eta(v)-\frac{\eta(v)+1-v^2}{2}\mid\\
&,\frac{\mid\xi(v)-\frac{\eta(v)+1-v^2}{2}\mid+\mid\eta(v)-\frac{\xi(v)+1-v^2}{2}\mid}{2}\}\\
\geq&\frac{1}{2}\cdot\mid\xi(v)-\eta(v)\mid=\frac{1}{2}d(\xi(v),\eta(v)),
\end{aligned}$$

and then T is continuous, random, comparable $\mathcal{MT}$-γ contraction.

Take the measurable mapping $x : \Omega \to X$ *as* $\xi(v) = \{1 - v^2\}$*; then, for every* $v \in \Omega$*,*

$$T(\xi(v)) = \frac{\xi(v) + 1 - v^2}{2} = \frac{1 - v^2 + 1 - v^2}{2} = 1 - v^2 = \xi(v).$$

$(1 - v^2)$ *is a random fixed point of* T*.*

Definition 4. *Let* M *be a nonempty subset of a random metric space* (X, d)*, and let* $T : \Omega \times M \to M$ *be a random operator. Then, for* $v \in \Omega$*,* $T(v, \cdot)$ *is called a random Meir-Keeler contraction if for any real number* $\eta > 0$*, there exists* $\delta > 0$ *such that for each* $\xi(v), \eta(v) \in \Omega \times M$*,*

$$\eta \leq d(\xi(v), \eta(v)) < \eta + \delta \Longrightarrow d(T(\xi(v)), T(\eta(v))) < \eta.$$

Remark 2. *Note that if* T *is a random Meir-Keeler contraction, then we have*

$$d(T(\xi(v)), T(\eta(v))) < d(\xi(v), \eta(v)).$$

Further, if $d(\xi(v), \eta(v)) = 0$*, then* $d(T(\xi(v)), T(\eta(v))) = 0$*. On the other hand, if* $d(\xi(v), \eta(v)) > 0$*, then*

$$d(T(\xi(v)), T(\eta(v))) < d(\xi(v), \eta(v)).$$

Theorem 3. *Suppose* (X, d) *is a complete random metric space and* $M \subset X$*. If* $T(v, \cdot) : \Omega \times M \to M$ *is a continuous, random, comparable Meir-Keeler contraction, then* T *possesses a random fixed point in* X*.*

Proof. Given $\xi_0(v) \in \Omega \times X$ and defining $\xi_1(v) = T(\xi_0(v))$, and $\xi_{n+1}(v) = T(\xi_n(v)) = T^n(\xi_0(v))$ for each $n \in \mathbb{N}$, since $T(v, \cdot) : \Omega \times M \to M$ is a random Meir-Keeler contraction, by Remark 2, we have

$$d(\xi_n(v), \xi_{n+1}(v)) = d(T(\xi_{n-1}(v)), T(\xi_n(v))) < d(\xi_{n-1}(v), \xi_n(v)).$$

Therefore, $\{d(\xi_n(v), \xi_{n+1}(v))\}_{n \in \mathbb{N} \cup \{0\}}$ is decreasing and bounded below; it must converge to some real number $\eta \geq 0$; that is,

$$d(\xi_n(v), \xi_{n+1}(v)) \searrow \eta, \quad \text{as} \quad n \to \infty.$$

Note that

$$\eta = inf\{d(\xi_n(v), \xi_{n+1}(v))\}_{n \in \mathbb{N} \cup \{0\}}. \tag{*}$$

We assert that $\eta = 0$. Suppose, on the contrary, that $\eta > 0$. Since $T(v, \cdot) : \Omega \times M \to M$ is a continuous, random Meir-Keeler contraction, corresponding to this η, there exist $\delta > 0$ and $m \in \mathbb{N}$ such that

$$\eta \leq d(\xi_m(v), \xi_{m+1}(v)) < \eta + \delta$$
$$\Longrightarrow d(\xi_{m+1}(v), \xi_{m+2}(v)) = d(T(\xi_m(v)), T(\xi_{m+1}(v))) < \eta,$$

a contradiction. Attendantly, we find that $\eta = 0$.

We next show that $\{\xi_n(w)\}$ is a Cauchy sequence in (X, d). We shall use the method of reductio ad absurdum. Suppose, on the contrary, that there exists a real number $\varepsilon > 0$ such that for any $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$ with $n_k \geq m_k > k$ satisfying

$$d(\xi_{m_k}(v), \xi_{n_k}(v)) \geq \varepsilon.$$

In addition, comparable to $m_k \geq k$, we can choose n_k so that $n_k > m_k \geq k$ and $d(\xi_{m_k}(v), \xi_{n_k}(v)) \geq \varepsilon$.

Therefore, we also have $d(\xi_{m_k}(v), \xi_{n_k - 2}(v)) < \varepsilon$.

So, we have that for all $k \in \mathbb{N}$,

$$\begin{aligned}\varepsilon \leq & d(\xi_{m_k}(v), \xi_{n_k}(v)) \\ \leq & d(\xi_{m_k}(v), \xi_{n_k-2}(v)) + d(\xi_{n_k-2}(v), \xi_{n_k-1}(v)) + d(\xi_{n_k-1}(v), \xi_{n_k}(v)) \\ < & \varepsilon + d(\xi_{n_k-2}(v), \xi_{n_k-1}(v)) + d(\xi_{n_k-1}(v), \xi_{n_k}(v)).\end{aligned}$$

Letting $k \to \infty$, we have that

$$\lim_{k\to\infty} d(\xi_{m_k}(v), \xi_{n_k}(v)) = \varepsilon.$$

On the other hand, we have that

$$\begin{aligned}\varepsilon \leq & d(\xi_{m_k}(v), \xi_{n_k}(v)) \\ \leq & d(\xi_{m_k}(v), \xi_{m_k+1}(v)) + d(\xi_{m_k+1}(v), \xi_{n_k+1}(v)) + d(\xi_{n_k+1}(v), \xi_{n_k}(v)) \\ \leq & d(\xi_{m_k}(v), \xi_{m_k+1}(v)) + d(\xi_{m_k+1}(v), \xi_{m_k}(v)) + d(\xi_{m_k}(v), \xi_{n_k}(v)) + \\ & d(\xi_{n_k}(v), \xi_{n_k+1}(v)) + d(\xi_{n_k+1}(v), \xi_{n_k}(v)).\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} d(\xi_{m_k+1}(v), \xi_{n_k+1}(v)) = \varepsilon.$$

Since T is a continuous, random Meir-Keeler contraction, we have

$$d(\xi_{m_k+1}(v), \xi_{n_k+1}(v)) = d(T(\xi_{m_k}(v)), T(\xi_{n_k}(v))) < d(\xi_{m_k}(v), \xi_{n_k}(v)).$$

Letting $k \to \infty$, we have that $\varepsilon < \varepsilon$, which implies a contradiction. So $\xi_n(v)$ is a Cauchy sequence. Since (X, d) is complete and $\xi_n(v)$ is Cauchy, there exists $\xi^*(v) \in X$ such that

$$\lim_{n\to\infty} \xi_n(v) = \xi^*(v),$$

thus

$$\begin{aligned} & d(T(\xi^*(v)), \xi^*(v)) \\ \leq & d(T(\xi^*(v)), T(\xi_n(v))) + d(T(\xi_n(v)), \xi^*(v)) \\ < & d(\xi^*(v), \xi_n(v)) + d(\xi_{n+1}(v), \xi^*(v)).\end{aligned}$$

Letting $n \to \infty$, we have

$$d(T(\xi^*(v)), \xi^*(v)) = 0.$$

This implies that $T(\xi^*(v)) = \xi^*(v)$; that is, $\xi^*(v)$ is a random fixed point of T. $\square$

Example 2. *Let $X = M = \mathbb{R}_0^+ \cup \{0\}$,also $\Omega = [0,1]$ and Σ be the sigma algebra of Lebegue's measurable subset of [0,1]. Let $X = [0, \infty)$, and define mapping as $d : (\Omega \times X) \times (\Omega \times X) \to M$ by*

$$d(\xi(v), \eta(v)) =\mid \xi(v) - \eta(v) \mid .$$

Then, (X, d) is a cone random metric space. Define random operator $T : \Omega \times X \to X$ as

$$T(\xi(v)) = \frac{\xi(v) + 1 - v^2}{2}.$$

For any $\eta > 0$, take $\delta = \eta$, if

$$\eta \leq d(\xi(v), \eta(v)) < \eta + \delta,$$

then

$$
\begin{aligned}
d(T(\xi(v)),T(\eta(v))) =& d(\frac{\xi(v)+1-v^2}{2},\frac{\eta(v)+1-v^2}{2})\\
=& \mid \frac{\xi(v)+1-v^2}{2}-\frac{\eta(v)+1-v^2}{2}\mid\\
=& \mid \frac{\xi(v)-\eta(v)}{2}\mid\\
\leq& \frac{1}{2}d(\xi(v),\eta(v))\\
<& \frac{1}{2}(\eta+\delta)=\eta.
\end{aligned}
$$

This implies that T is a continuous, random Meir-Keeler contraction.
Take the measurable mapping $x:\Omega\to X$ as $\xi(v)=\{1-v^2\}$,then for every $v\in\Omega$,

$$T(\xi(v))=\frac{\xi(v)+1-v^2}{2}=\frac{1-v^2+1-v^2}{2}=1-v^2=\xi(v).$$

$(1-v^2)$ *is a random fixed point of T.*

3. Conclusions

One of the most interesting and useful research fields is probabilistic functional analysis due to its wide application potential to probabilistic models in applied problems. Several distinct classes of random equations has been investigated in random operator theory. More precisely, in describing several distinct phenomena in various quantitative disciplines (such as engineering, physics and biology) we need some mathematical models or equations. These models and equations contain certain parameters or coefficients whose values are unknown, but they have specific interpretations. At this point, the most useful and realistic models require random operator equations, random differential/integral equations. Random fixed point results play an interesting role in the solution of the random differential equations; see, e.g., [26]. In addition, random fixed point theorems have been used in finding the general existence principle for the random operator equation [27]. Consequently, the random fixed point theory is an important tool in solutions to real-world problems whenever they are modeled in a realistic way. Our results are aimed to enrich the random fixed point theory.

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