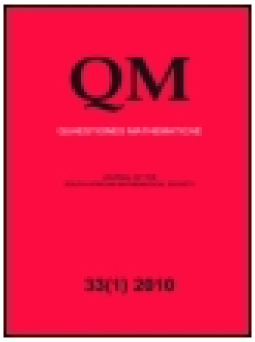

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INVESTIGATION OF THE EIGENVALUES AND ROOT FUNCTIONS OF THE BOUNDARY VALUE PROBLEM TOGETHER WITH A TRANSMISSION MATRIX

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Abstract. In this paper, we consider a singular even-order Hamiltonian system on the union of two intervals together with appropriate boundary and transmission conditions. For investigating the spectral properties of the problem we pass to the inverse operator with an explicit form and we prove some completeness theorems.

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Key words: Hamiltonian system, spectral analysis, transmission condition.

1. Introduction. Many real world applications may be identified with the aid of a differential equation and appropriate boundary conditions. For instance, deformation of beams and plate deflection theory, viscoelastic and inelastic flows, control and optimization theory, stellor interiors, astrophysics, mass and heat transfer, oxygen diffusion in cells may be studied by modelling suitable boundary value problems [2], [3], [6], [15], [18], [21], [23]. However, for detailed analysis one has to impose some additional conditions to the solutions satisfying some conditions called transmission conditions. Transmission conditions occur in a natural way in scientific problems. To be more pricise we should note that bursting rhythm models in medicine, pharmacokinetics and frequency modulated systems exhibit transmission effects.

In this paper we consider the following first-order system called Hamiltonian system

$$JY' = [\lambda A + B]\, Y,\ x \in [a, c) \cup (c, b), \tag{1.1}$$

where J, A, B are $2n \times 2n$ matrices, Y is a $2n \times 1$ vector such that A and B are real-valued, $A^* = A \geq 0$, $B^* = B$ and

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

Here I denotes the $n \times n$ identity matrix and it is better to note that $J^* = -J$. Throughout the paper we assume that the points a, c are regular and b is singular for the Equation (1.1). The first fundamental results for the system (1.1) has been introduced by Atkinson [1]. The system (1.1) with the assumptions on J, A and B contain even-order differential equations and Dirac systems. In fact, in 1974 Walker [24] showed that any formally symmetric ordinary differential equation of order m may be considered as an equivalent system (1.1) of the same order m. However, in this paper we only consider even-order Hamiltonian systems.

According to the results of Atkinson, the system (1.1) has at least $k-$independent solutions of square integrable, where $n \leq k \leq 2n$. If $k = 2n$ it is called limit-circle case. This case has been studied by many authors (for example see, [7]–[14], [19]. In this paper we assume that the system (1.1) is in limit-circle case at the singular point b.

An efficient approximation to the problems containing transmission conditions has been given by Mukhtarov and his colleguaes [16], [17], [22]. This new approximation allows one to understand the geometrical meaning of the Hilbert space with the special inner product. Each problem needs to be considered a special inner product. However, in this work using this idea of Mukhtarov and his colleagues we introduce a new inner product for the Hamiltonian system with transmission conditions that contains several even-order Sturm-Liouville equations as well as Dirac type equations and we investigate the spectral properties of the corresponding boundary value transmission problem. At the end of the work we give a specific example to make the results clear.

Finally we should note that to indicate the solution of the Equation (1.1) we will use the notation $Y(x, \lambda)$. However, when $\lambda = 0$ we will use the notation $Y(x) := Y(x, 0)$ for brevity.

2. Boundary value transmission problem. We denote by $H = L^2_{A_1}(a, c) \oplus L^2_{A_2}(c, b)$ the Hilbert space consisting of all functions Y such that

$$\int_a^b Y^* A Y dx < \infty$$

with the usual inner product

$$(Y, Z) = \int_a^b Z^* A Y dx,$$

where Z^* denotes the conjugate transpose of the vector Z and

$$A = \begin{cases} A_1, & x \in [a, c), \\ A_2, & x \in (c, b). \end{cases}$$

We assume for all $\lambda \in \mathbb{C}$ and nontrivial solution $Y(x, \lambda)$ of (1.1) that the inequality

$$\int_a^b Y^* A Y dx > 0$$

holds [1], p. 253.

Let ℓ denote the differential expression with the rule

$$\ell(Y) = JY' - BY.$$

Now we consider the subspace D of H that consists of those vectors $Y \in H$ such that Y is locally absolutely continuous on $[a, c)$ and (c, b) satisfying

$$\ell(Y) = AF,\ x \in [a, c) \cup (c, b),$$

where $F \in H$. We define the maximal operator L on D as

$$LY = F,\ Y \in D,\ F \in H.$$

For the vectors $Y, Z \in D$ we obtain the following Green's formula

$$\int_a^b [Z^* A(LY) - (LZ)^* AY]\, dx = Z^* JY \mid_a^{c-} + Z^* JY \mid_{c+}^b, \tag{2.1}$$

where $Z^* JY \mid_\eta^\xi = (Z^* JY)(\xi) - (Z^* JY)(\eta)$.

(2.1) particularly implies for $Y, Z \in D$ that the value $(Z^* JY)(b)$ exists and is finite.

Since b is a singular point for (1.1) there must exist $k-$independent solutions which are square integrable on the interval $[a, c) \cup (c, b)$, where $n \leq k \leq 2n$. We assume that $k = 2n$. In other words, we assume that limit-circle case holds at b for (1.1).

The assumptions on J, A and B and the system (1.1) allow us to consider a fundamental matrix-function. Indeed, let

$$\mathcal{Y}_1(x, \lambda) = \begin{cases} \mathcal{Y}_1(x, \lambda), & x \in [a, c), \\ \mathcal{Y}_2(x, \lambda), & x \in (c, b), \end{cases}$$

be an $2n \times 2n$ matrix solution of (1.1) satisfying

$$\mathcal{Y}_1(a, \lambda) = \begin{bmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{bmatrix} \tag{2.2}$$

and

$$\mathcal{Y}_2(c+, \lambda) = M^{-1} \mathcal{Y}_1(c-, \lambda),$$

where α_1, α_2 are real $n \times n$ matrices satisfying $rank(\alpha_1, \alpha_2) = n$ and

$$\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^* = I,$$
$$\alpha_1 \alpha_2^* - \alpha_2 \alpha_1^* = 0,$$

and M is an $2n \times 2n$ matrix of the form

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

Here M_{ij} $(1 \leq i, j \leq 2)$ are real $n \times n$ matrices such that $M_{ij}^* = M_{ij}$ with $M_{11} M_{21} = M_{21} M_{11}$ and $M_{12} M_{22} = M_{22} M_{12}$. Finally we assume that $\delta := \det M > 0$. Now we may introduce the following.

Hypothesis 2.1. *We have the following:*
(*i*) $M^*JM = \delta J$,
(*ii*) $M_{22}M_{11} - M_{12}M_{21} = \delta I$,
(*iii*) $M_{11}M_{22} - M_{21}M_{12} = \delta I$.

This hypothesis is meaningful. In fact, consider the expression

$$M^*JM \tag{2.3}$$

or

$$\begin{bmatrix} -M_{11}^*M_{21} + M_{21}^*M_{11} & -M_{11}^*M_{22} + M_{21}^*M_{12} \\ -M_{12}^*M_{21} + M_{22}^*M_{11} & -M_{12}^*M_{22} + M_{22}^*M_{12} \end{bmatrix}. \tag{2.4}$$

(2.4) is equivalent to

$$\begin{bmatrix} 0 & -M_{11}M_{22} + M_{21}M_{12} \\ -M_{12}M_{21} + M_{22}M_{11} & 0 \end{bmatrix}. \tag{2.5}$$

Our assertion is that (2.5) can be regarded as

$$\delta \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = \delta J. \tag{2.6}$$

This equivalence comes from the results of Silvester [20]. In fact, since $M_{11}M_{21} = M_{21}M_{11}$ we have

$$\det M = \det(M_{11}M_{22} - M_{21}M_{12}) \tag{2.7}$$

and since $M_{12}M_{22} = M_{22}M_{12}$ we get

$$\det M = \det(M_{22}M_{11} - M_{12}M_{21}). \tag{2.8}$$

Using (2.7) and (2.8) it is seen that (2.5) can be regarded as (2.6). Therefore (2.3) is identical with (2.6).

We shall set the following

$$\mathcal{Y}_r(x,\lambda) = \begin{bmatrix} \theta_r(x,\lambda) & \phi_r(x,\lambda) \end{bmatrix}, \ r = 1, 2,$$

where

$$\theta(x,\lambda) = \begin{cases} \theta_1(x,\lambda), \ x \in [a,c), \\ \theta_2(x,\lambda), \ x \in (c,b), \end{cases} \quad \phi(x,\lambda) = \begin{cases} \phi_1(x,\lambda), \ x \in [a,c), \\ \phi_2(x,\lambda), \ x \in (c,b). \end{cases}$$

Taking $\lambda = 0$ we obtain the real solution

$$\mathcal{Y}_r := \mathcal{Y}_r(x) := \mathcal{Y}_r(x,0) = \begin{bmatrix} \theta_r(x) & \phi_r(x) \end{bmatrix}, \ r = 1, 2.$$

Lemma 2.2. *For $x \in [a,c)$ we have*

$$\mathcal{Y}_1^* J \mathcal{Y}_1 = J,$$

and for $x \in (c,b)$ we have

$$\mathcal{Y}_2^* J \mathcal{Y}_2 = \frac{J}{\delta}.$$

Proof. $\mathcal{Y}_1(x)$ satisfies the equation

$$J\mathcal{Y}_1' = B\mathcal{Y}_1,\ x \in [a,c), \tag{2.9}$$

while $\mathcal{Y}_2(x)$ satisfies the equation

$$J\mathcal{Y}_2' = B\mathcal{Y}_2,\ x \in (c,b). \tag{2.10}$$

From (2.9) we obtain

$$-\mathcal{Y}_1^{*\prime} J = \mathcal{Y}_1^* B,\ x \in [a,c). \tag{2.11}$$

Then (2.9) and (2.11) imply

$$\mathcal{Y}_1^* J\mathcal{Y}_1 = C_1,\ x \in [a,c), \tag{2.12}$$

where C_1 is a constant. Using the initial condition (2.2) and (2.12) we obtain $\mathcal{Y}_1^* J\mathcal{Y}_1 = J$.

Now from (2.10) we obtain

$$-\mathcal{Y}_2^{*\prime} J = \mathcal{Y}_2^* B,\ x \in (c,b). \tag{2.13}$$

(2.10) and (2.13) give

$$\mathcal{Y}_2^* J\mathcal{Y}_2 = C_2,\ x \in (c,b), \tag{2.14}$$

where C_2 is a constant. The condition

$$\mathcal{Y}_2(c+) = M^{-1}\mathcal{Y}_1(c-)$$

gives

$$\mathcal{Y}_2^*(c+)J\mathcal{Y}_2(c+) = \mathcal{Y}_1^*(c-)(M^{-1})^* JM^{-1}\mathcal{Y}_1(c-). \tag{2.15}$$

On the other side of the equation given in Hypothesis 2.1. (i) we obtain

$$(M^*)^{-1} JM^{-1} = \frac{J}{\delta}. \tag{2.16}$$

(2.14)-(2.16) show $\mathcal{Y}_2^* J\mathcal{Y}_2 = J/\delta$ and this completes the proof. □

Note that $\theta(x)$ and $\phi(x)$ are the solutions of

$$JY' = BY,\ x \in [a,c) \cup (c,b),$$

and they belong to H and D. This implies for arbitrary vector-function $Y \in D$ that the values $(\theta^* JY)(b)$, $(\phi^* JY)(b)$ exist and are finite.

We set $2n \times n$ vector function $V(x)$ as follows

$$V(x) = \mathcal{Y}(x) \begin{bmatrix} I \\ H^* \end{bmatrix},$$

where H is a complex $n \times n$ matrix such that the entries h_{ii} are complex numbers and h_{ij} are real numbers $(i \neq j)$, $1 \leq i, j \leq n$, $H^t = H$ and $\operatorname{Im} H > 0$. Here t denotes the transpose of the matrix.

Now we may introduce the boundary-value-transmission problem as follows

$$LY = \lambda Y,\ Y \in D,\ x \in [a, c) \cup (c, b), \tag{2.17}$$

$$\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} Y(a) = 0, \tag{2.18}$$

$$Y(c-) = MY(c+), \tag{2.19}$$

$$(V^* JY)(b) = 0. \tag{2.20}$$

One of our aim is to investigate the spectral properties of the problem (2.17)-(2.20). Before investigating the problem we shall introduce another representation of (2.20). For this purpose we shall define the following $S-$transformation

$$(SY)(x) = \mathcal{Y}^{-1}(x) Y(x) \tag{2.21}$$

where

$$SY = \begin{bmatrix} S_1 Y \\ S_2 Y \end{bmatrix}.$$

Here $S_1 Y$ is an $n \times 1$ vector and $S_2 Y$ is an $n \times 1$ vector.

LEMMA 2.3. *The condition (2.20) may be replaced with the following*

$$(S_2 Y)(b) - H (S_1 Y)(b) = 0,$$

where $Y \in D$.

Proof. One has

$$\mathcal{Y}^{-1} = -\delta J \mathcal{Y}^* J,\ x \in (c, b). \tag{2.22}$$

Using (2.21) and (2.22) we obtain

$$\frac{1}{\delta} JSY = \mathcal{Y}^* JY,\ x \in (c, b). \tag{2.23}$$

Therefore (2.23) implies that

$$\frac{1}{\delta} \begin{bmatrix} I & H \end{bmatrix} J \begin{bmatrix} S_1 Y \\ S_2 Y \end{bmatrix} = \frac{1}{\delta} \left(-(S_2 Y)(x) + H (S_1 Y)(x) \right) = (V^* JY)(x). \tag{2.24}$$

Since $\delta > 0$ passing to the limit as $x \to b$ (2.24) implies the result. □

3. Dissipative operator. In this section, first of all, we shall introduce a suitable inner product for the problem (2.17)-(2.20).

Now let us consider the following inner product in the Hilbert space $H = L^2_{A_1}(a,c) \oplus L^2_{A_2}(c,b)$

$$\langle Y, Z\rangle_H = \int_a^c Z_1^* A_1 Y_1 dx + \delta \int_c^b Z_2^* A_2 Y_2 dx.$$

Consider the subspace $D(\mathcal{L})$ consisting of all vector-functions $Y \in D$ such that $\mathcal{L}Y \in H$ and Y satisfies (2.18)-(2.20). We construct the operator $\mathcal{L}$ on $D(\mathcal{L})$ as

$$\mathcal{L}Y = LY,\ x \in [a,c) \cup (c,b).$$

Then the equation

$$\mathcal{L}Y = \lambda Y,\ x \in [a,c) \cup (c,b)$$

coincides with the problem (2.17)-(2.20).

Then we have the following.

THEOREM 3.1. *T is dissipative in H.*

Proof. Let $Y \in D(\mathcal{L})$. Then

$$\langle \mathcal{L}Y, Y\rangle_H - \langle Y, \mathcal{L}Y\rangle_H = Y^* JY \mid_a^{c-} + \delta Y^* JY \mid_{c+}^b . \tag{3.1}$$

Since $Y \in D(\mathcal{L})$ one may regard that Y satisfies the following condition at a

$$Y(a) = \begin{bmatrix} \alpha_2^* & 0 \\ -\alpha_1^* & 0 \end{bmatrix} \nu,$$

where ν is a $2n \times 1$ vector. Therefore we have

$$(Y^* JY)(a) = 0. \tag{3.2}$$

Moreover from transmission conditions we get

$$(Y^* JY)(c-) = (Y^* M^* JMY)(c+) = \delta (Y^* JY)(c+). \tag{3.3}$$

Since

$$(Y^* JY)(b) = \delta^{-1} \left[(SY)^* J (SY)\right](b)$$

we have

$$(Y^* JY)(b) = \delta^{-1} \left[(S_2 Y)^* (b) (S_1 Y)(b) - (S_1 Y)^* (b) (S_2 Y)(b)\right]. \tag{3.4}$$

Therefore Lemma 2.3 implies

$$(Y^* JY)(b) = \delta^{-1} (S_1 Y)^* (b) 2i \operatorname{Im} H^* (S_1 Y)(b). \tag{3.5}$$

(3.1)-(3.5) give

(3.6)

$$\operatorname{Im} \langle \mathcal{L}Y, Y\rangle_H = (S_1 Y)^* (b) \operatorname{Im} H^* (S_1 Y)(b) = \langle \operatorname{Im} H (S_1 Y)(b), (S_1 Y)(b)\rangle_{\mathbb{C}^n},$$

where $\langle .,.\rangle_{\mathbb{C}^n}$ denotes the inner product in $\mathbb{C}^n$. Therefore (3.6) completes the proof. □

COROLLARY 3.2. *All eigenvalues of $\mathcal{L}$ belong to the open upper half-plane.*

Proof. This follows from Theorem 3.1 and (3.6). □

COROLLARY 3.3. *Zero is not an eigenvalues of $\mathcal{L}$.*

Now we shall find the inverse of $\mathcal{L}$. In other words, we shall solve the equation

$$\mathcal{L}Y = F,\ x \in [a,c) \cup (c,b), \tag{3.7}$$

where $Y \in D(\mathcal{L})$, $F \in H$ and

$$Y = \begin{cases} Y_1, & x \in [a,c), \\ Y_2, & x \in (c,b), \end{cases} \qquad F = \begin{cases} F_1, & x \in [a,c), \\ F_2, & x \in (c,b). \end{cases}$$

Let $Y_1 = \mathcal{Y}_1 D_1$ ($x \in [a,c)$) and $Y_2 = \mathcal{Y}_2 D_2$ ($x \in (c,b)$), where D_1 and D_2 are constants. Using the method of variation of constants we have

$$J\mathcal{Y}_1 D_1' = A_1 F_1,\ x \in [a,c), \tag{3.8}$$

and

$$J\mathcal{Y}_2 D_2' = A_2 F_2,\ x \in (c,b). \tag{3.9}$$

Using Lemma 2.2 we get

$$-J\mathcal{Y}_1^* J\mathcal{Y}_1 = I_{2n},\ x \in [a,c), \tag{3.10}$$

and

$$-\delta J\mathcal{Y}_2^* J\mathcal{Y}_2 = I_{2n},\ x \in (c,b), \tag{3.11}$$

where I_{2n} denotes the $2n \times 2n$ identity matrix. Therefore (3.8), (3.10) and (3.9), (3.11) imply that

$$D_1' = -J\mathcal{Y}_1^* A_1 F_1,\ x \in [a,c),$$
$$D_2' = -\delta J\mathcal{Y}_2^* A_2 F_2,\ x \in (c,b).$$

Hence we obtain

$$Y_1 = -\mathcal{Y}_1 \int_a^x J\mathcal{Y}_1^* A_1 F_1 ds + \mathcal{Y}_1 K_1,\ x \in [a,c), \tag{3.12}$$

and

$$Y_2 = -\delta\mathcal{Y}_2 \int_c^x J\mathcal{Y}_2^* A_2 F_2 ds + \mathcal{Y}_2 K_2,\ x \in (c,b). \tag{3.13}$$

Since

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ 0 & 0 \end{bmatrix} Y(a) = 0$$

we have

$$\left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array}\right] K_1 = 0. \tag{3.14}$$

Now we construct

$$\left[\begin{array}{cc} 0 & 0 \\ I & H \end{array}\right] \mathcal{Y}_2^* J Y_2 = -\delta \left[\begin{array}{cc} 0 & 0 \\ I & H \end{array}\right] \mathcal{Y}_2^* J \mathcal{Y}_2 \int_c^x J \mathcal{Y}_2^* A_2 F_2 ds$$
$$+ \left[\begin{array}{cc} 0 & 0 \\ I & H \end{array}\right] \mathcal{Y}_2^* J \mathcal{Y}_2 K_2, \ x \in (c, b).$$

Now passing to the limit as $x \to b$ we obtain

$$0 = \left[\begin{array}{cc} 0 & 0 \\ I & H \end{array}\right] \int_c^b \mathcal{Y}_2^* A_2 F_2 dx + \frac{1}{\delta} \left[\begin{array}{cc} 0 & 0 \\ H & -I \end{array}\right] K_2. \tag{3.15}$$

Transmission conditions give

$$-\mathcal{Y}_1(c-) \int_a^c J \mathcal{Y}_1^* A_1 F_1 dx + \mathcal{Y}_1(c-) K_1 = M \mathcal{Y}_2(c+) K_2$$

and therefore

$$K_2 = K_1 - \int_a^c J \mathcal{Y}_1^* A_1 F_1 dx. \tag{3.16}$$

Using (3.15) and (3.16) we obtain

(3.17)
$$\left[\begin{array}{cc} 0 & 0 \\ H & -I \end{array}\right] K_1 = \left[\begin{array}{cc} 0 & 0 \\ H & -I \end{array}\right] \int_a^c J \mathcal{Y}_1^* A_1 F_1 dx - \delta \left[\begin{array}{cc} 0 & 0 \\ I & H \end{array}\right] \int_c^b \mathcal{Y}_2^* A_2 F_2 dx.$$

Substituting (3.14) into (3.17) we get

(3.18)
$$\left[\begin{array}{cc} I & 0 \\ H & -I \end{array}\right] K_1 = \left[\begin{array}{cc} 0 & 0 \\ H & -I \end{array}\right] \int_a^c J \mathcal{Y}_1^t A_1 F_1 dx - \delta \left[\begin{array}{cc} 0 & 0 \\ I & H \end{array}\right] \int_c^b \mathcal{Y}_2^* A_2 F_2 dx.$$

Note that the inverse of the coefficient of K_1 is itself. Therefore (3.18) gives

$$K_1 = \int_a^c \left[\begin{array}{cc} 0 & 0 \\ I & H \end{array}\right] \mathcal{Y}_1^* A_1 F_1 dx + \delta \int_c^b \left[\begin{array}{cc} 0 & 0 \\ I & H \end{array}\right] \mathcal{Y}_2^* A_2 F_2 dx. \tag{3.19}$$

Using (3.19) in (3.16) we get

$$K_2 = \int_a^c \left[\begin{array}{cc} 0 & I \\ 0 & H \end{array}\right] \mathcal{Y}_1^* A_1 F_1 dx + \delta \int_c^b \left[\begin{array}{cc} 0 & 0 \\ I & H \end{array}\right] \mathcal{Y}_2^* A_2 F_2 dx. \tag{3.20}$$

Consequently, (3.12), (3.13), (3.19), (3.20) give

$$Y_1 = V_1 \int_a^x \phi_1^* A_1 F_1 ds + \phi_1 \int_x^c V_1^* A_1 F_1 ds + \delta \phi_1 \int_c^b V_2^* A_2 F_2 dx, \ x \in [a, c),$$

and

$$Y_2 = \delta V_2 \int_c^x \phi_2^* A_2 F_2 ds + \delta \phi_2 \int_x^b \mathcal{V}_2^* A_2 F_2 ds + V_2 \int_c^b \phi_1^* A_1 F_1 dx, \ x \in (c,b).$$

Therefore the solution $Y(x)$ of (3.7) is obtained as

$$Y(x) = \left\langle G(x,s), \overline{F}(s) \right\rangle_H ,$$

where

$$G(x,s) = \begin{cases} V(x)\phi^*(s), & a \le s \le x \le b, \\ \phi(x)V^*(s), & a \le x \le s \le b. \end{cases}$$

Let us construct the operator

$$RF = \left\langle G(x,s), \overline{F}(s) \right\rangle_H ,$$

where $F \in H$. Then R is the inverse of $\mathcal{L}$.

4. Completeness theorems. In this section we shall give some definitions and results that we will use.

A complex parameter μ is called a regular point of the operator T if $T - \mu I$ is invertible. Consider that T is a closed linear operator that has at least one regular point. A linear operator T_1 is said to be $T-$completely continuous if $D(T) \subseteq D(T_1)$ and if for some regular point μ_0 of the operator T the operator $T_1(T - \mu_0 I)$ is completely continuous. In particular, if $\mu = 0$ is a regular point of the operator T then the operator T_1 is $T-$completely continuous if and only if $D(T) \subseteq D(T_1)$ and $T_1 T^{-1}$ is completely continuous.

We denote by $\mathfrak{S}_p$ $(1 \le p < \infty)$ the set consisting of all completely continuous operators T for which

$$\sum_{j=1}^{\infty} \mu_j^p (T^* T) < \infty,$$

where $\mu_j(T^*T)$ denotes the jth eigenvalue of T^*T.

THEOREM 4.1. ([5] (p. 276)) *Let $A = L + T$, where L is selfadjoint operator with a discrete spectrum, and T is an $L-$ completely continuous operator such that*

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{\ln \left[1/\mu_n ((L^{-1}TL^{-1})^* \left(L^{-1}TL^{-1}\right)) \right]} < \infty.$$

Then the entire spectrum of the operator A consists of normal eigenvalues. For any $\epsilon > 0$ all of them, with the possible exception of a finite number, lie in the sectors

$$-\epsilon < \arg \mu < \epsilon, \ \pi - \epsilon < \arg \mu < \pi + \epsilon.$$

The system of root functions of the operator A is complete in the Hilbert space.

COROLLARY 4.2. ([5] (p. 277)) *The system of root functions of the operator $A = L + K$ is complete whenever L is a selfadjoint operator with a discrete spectrum and K belongs to some $\mathfrak{S}_p$.*

THEOREM 4.3. ([25]) *Assume that a densely defined operator T is invertible and has a dense range. If E and F are linear complements of*

$$\{y \in D(T) \cap D(T^*) : Ty = T^*y\}$$

in $D(T)$ and $D(T^*)$, respectively, then the range of the imaginary part of inverse of T is contained in $E \oplus F$.

Now let us consider the operator

$$-R = -R_1 - iR_2,$$

where R_1 is the real part of R and R_2 is the imaginary part of R. We should note that R_1 is the inverse of the operator T_1 which is generated by

$$\mathcal{L}Y = \lambda Y, \ x \in [a,c) \cup (c,b),$$

$$\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} Y(a) = 0,$$
$$Y(c-) = MY(c+),$$
$$(V_R^* JY)(b) = 0,$$

where

$$V_R = \mathcal{Y} \begin{bmatrix} I \\ \operatorname{Re} H^* \end{bmatrix}.$$

Note that R and R_1 are Hilbert-Schmidt operators. Moreover, R_1 is selfadjoint operator and R_2 is a finite rank operator. Therefore from Corollary 4.2 we may introduce the following.

THEOREM 4.4. *All root functions of $-R$ (also R) are complete in H.*

Consequently we may introduce the following results.

THEOREM 4.5. *All eigenvalues of the operator $\mathcal{L}$ having finite multiplicity belong to the open upper half-plane without any finite point of accumulation. Moreover, all eigenfunctions and associated functions of $\mathcal{L}$ span the Hilbert space H.*

5. Conclusion and remarks. In this paper, we consider general abstract first-order differential equation together with suitable boundary and transmission conditions which contains several ordinary even-order differential equations as well as even-order matrix differential equations and it seems that the construction of such a transmission condition and boundary condition at the singular point is new.

To be more precise we shall consider the following fourth-order equation

$$(5.1) \qquad (q_2 y'')'' - (q_1 y')' + q_0 y = \lambda w y, \ x \in [a,c) \cup (c,b),$$

where q_0, q_1, q_2, w are real-valued, locally integrable functions on each $[a,c)$ and (c,b) and λ is a complex parameter. We assume that a, c are regular and b is singular for (5.1) and (5.1) is in limit-circle case at b. This assumption is meaningful because there are several sufficient conditions for which the equation (5.1) is in limit-circle case at a singular point. For example, Devinatz showed that [4] the following equation

$$y^{iv} - m(x^{\gamma}y')' + nx^{\xi}y = \lambda y$$

has four linearly independent solutions of square integrable on a semi-infinite interval provided that $\xi = 2\gamma$, $\gamma > 2/3$ and $m \pm (m^2 - 4n)^{1/2} < 0$.

Here $H = L^2_{w_1}(a, c) \oplus L^2_{w_2}(c, b)$, where

$$w = \begin{cases} w_1, & x \in [a, c), \\ w_2, & x \in (c, b). \end{cases}$$

Equation (5.1) can be embedded into (1.1). In fact, (5.1) is equivalent to the following

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ y^{[3]} \\ y^{[2]} \end{pmatrix}' = \tag{5.2}$$
$$\left[\lambda \begin{pmatrix} w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -q_0 & 0 & 0 & 0 \\ 0 & -q_1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q_2^{-1} \end{pmatrix} \right] \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ y^{[3]} \\ y^{[2]} \end{pmatrix}$$

or

$$JY' = [\lambda A + B]\, Y, \;\; Y = \left(y^{[0]}, y^{[1]}, y^{[3]}, y^{[2]} \right)^t,$$

where t stands for the transpose of the vector, $y^{[0]} = y$, $y^{[1]} = y'$, $y^{[2]} = q_2 y''$ and $y^{[3]} = -(q_2 y'')' + q_1 y'$.

Let

$$\mathcal{Y}(x, \lambda) = \begin{bmatrix} \theta_1(x,\lambda) & \theta_2(x,\lambda) & \phi_1(x,\lambda) & \phi_2(x,\lambda) \\ \theta_1^{[1]}(x,\lambda) & \theta_2^{[1]}(x,\lambda) & \phi_1^{[1]}(x,\lambda) & \phi_2^{[1]}(x,\lambda) \\ \theta_1^{[3]}(x,\lambda) & \theta_2^{[3]}(x,\lambda) & \phi_1^{[3]}(x,\lambda) & \phi_2^{[3]}(x,\lambda) \\ \theta_1^{[2]}(x,\lambda) & \theta_2^{[2]}(x,\lambda) & \phi_1^{[2]}(x,\lambda) & \phi_2^{[2]}(x,\lambda) \end{bmatrix}$$

be the fundamental matrix for (5.2) satisfying the condition

$$\mathcal{Y}(a, \lambda) = \begin{bmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{bmatrix},$$

where α_1, α_2 are real 2×2 matrices satisfying

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I,$$
$$\alpha_1\alpha_2^* - \alpha_2\alpha_1^* = 0,$$

I is the 2×2 identity matrix.

Let

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

be a real 4×4 matrix such that

$$M_{11} = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{11} \end{bmatrix}, \qquad M_{12} = \begin{bmatrix} m_{13} & m_{14} \\ m_{14} & m_{13} \end{bmatrix},$$
$$M_{21} = \begin{bmatrix} m_{12} & -m_{11} \\ -m_{11} & m_{12} \end{bmatrix}, \quad M_{22} = \begin{bmatrix} m_{14} & -m_{13} \\ -m_{13} & m_{14} \end{bmatrix},$$

where $\det M > 0$. Since

$$M_{22}M_{11} - M_{12}M_{21} = \begin{bmatrix} 2(m_{14}m_{11} - m_{13}m_{12}) & 0 \\ 0 & 2(m_{14}m_{11} - m_{13}m_{12}) \end{bmatrix}$$

we set $\delta = 4(m_{14}m_{11} - m_{13}m_{12})^2$.

Adopting the notation $\mathcal{Y}(x) := \mathcal{Y}(x, 0)$ we set

$$V(x) = \mathcal{Y}(x) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \overline{h}_1 & k \\ k & \overline{h}_2 \end{bmatrix} = \begin{bmatrix} \theta_1 + \overline{h}_1\phi_1 + k\phi_2 & \theta_2 + k\phi_1 + \overline{h}_2\phi_2 \\ \theta_1^{[1]} + \overline{h}_1\phi_1^{[1]} + k\phi_2^{[1]} & \theta_2^{[1]} + k\phi_1^{[1]} + \overline{h}_2\phi_2^{[1]} \\ \theta_1^{[3]} + \overline{h}_1\phi_1^{[3]} + k\phi_2^{[3]} & \theta_2^{[3]} + k\phi_1^{[3]} + \overline{h}_2\phi_2^{[3]} \\ \theta_1^{[2]} + \overline{h}_1\phi_1^{[2]} + k\phi_2^{[2]} & \theta_2^{[2]} + k\phi_1^{[2]} + \overline{h}_2\phi_2^{[2]} \end{bmatrix}$$

where h_1, h_2 are complex numbers such that $\operatorname{Im} h_1 > 0$, $\operatorname{Im} h_2 > 0$ and k is a real number.

Consequently the boundary-transmission conditions subject to the equation (5.1) can be given as follows

$$\begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix} \begin{bmatrix} y(a) \\ y^{[1]}(a) \\ y^{[3]}(a) \\ y^{[2]}(a) \end{bmatrix} = 0, \tag{5.3}$$

$$\begin{aligned} y(c-) &= m_{11}y(c+) + m_{12}y^{[1]}(c+) + m_{14}y^{[2]}(c+) + m_{13}y^{[3]}(c+), \\ y^{[1]}(c-) &= m_{12}y(c+) + m_{11}y^{[1]}(c+) + m_{13}y^{[2]}(c+) + m_{14}y^{[3]}(c+), \\ y^{[2]}(c-) &= -m_{11}y(c+) + m_{12}y^{[1]}(c+) + m_{14}y^{[2]}(c+) - m_{13}y^{[3]}(c+), \\ y^{[3]}(c-) &= m_{12}y(c+) - m_{11}y^{[1]}(c+) - m_{13}y^{[2]}(c+) + m_{14}y^{[3]}(c+), \end{aligned} \tag{5.4}$$

$$\lim_{x\to b} V^*(x) \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y(x) \\ y^{[1]}(x) \\ y^{[3]}(x) \\ y^{[2]}(x) \end{bmatrix} \tag{5.5}$$

$$= \lim_{x\to b} \begin{bmatrix} [y, \theta_1](x) + h_1[y, \phi_1](x) + k[y, \phi_2](x) \\ [y, \theta_2](x) + k[y, \phi_1](x) + h_2[y, \phi_2](x) \end{bmatrix} = 0, \tag{5.6}$$

where h_1, h_2 are complex numbers such that $\operatorname{Im} h_1 > 0$, $\operatorname{Im} h_2 > 0$ and k is a real number. Then we may introduce the following results.

THEOREM 5.1. *All eigenvalues of the problem (5.1), (5.3)–(5.5) having finite multiplicity belong to the open upper half-plane without any finite point of accumulation. Moreover, all eigenfunctions and associated functions of the problem (5.1), (5.3)–(5.5) span the Hilbert space H.*

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