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Research Article

The Existence of Positive Solutions for a New Coupled System of Multiterm Singular Fractional Integrodifferential Boundary Value Problems

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We discuss the existence of positive solutions for the coupled system of multiterm singular fractional integrodifferential boundary value problems $D^{\alpha}_{0^+}u(t) + f_1(t,u(t),v(t),(\phi_1 u)(t),(\psi_1 v)(t),D^{p}_{0^+}u(t),D^{\mu_1}_{0^+}v(t),D^{\mu_2}_{0^+}v(t),\dots,D^{\mu_m}_{0^+}v(t)) = 0$, $D^{\beta}_{0^+}v(t) + f_2(t,u(t),v(t),(\phi_2 u)(t),(\psi_2 v)(t),D^{q}_{0^+}v(t),D^{\nu_1}_{0^+}u(t),D^{\nu_2}_{0^+}u(t),\dots,D^{\nu_m}_{0^+}u(t)) = 0$, $u^{(i)}(0) = 0$ and $v^{(i)}(0) = 0$ for all $0 \le i \le n-2$, $[D^{\delta_1}_{0^+}u(t)]_{t=1} = 0$ for $2 < \delta_1 < n-1$ and $\alpha - \delta_1 \ge 1$, $[D^{\delta_2}_{0^+}v(t)]_{t=1} = 0$ for $2 < \delta_2 < n-1$ and $\beta - \delta_2 \ge 1$, where $n \ge 4$, $n-1 < \alpha, \beta < n$, $0 < p, q < 1$, $1 < \mu_i, \nu_i < 2$ $(i = 1,2,\dots,m)$, $\gamma_j, \lambda_j : [0,1]\times[0,1] \to (0,\infty)$ are continuous functions $(j = 1,2)$ and $(\phi_j u)(t) = \int_0^t \gamma_j(t,s)u(s)ds$, $(\psi_j v)(t) = \int_0^t \lambda_j(t,s)v(s)ds$. Here D is the standard Riemann-Liouville fractional derivative, f_j $(j = 1,2)$ is a Caratheodory function, and $f_j(t,x,y,z,w,v,u_1,u_2,\dots,u_m)$ is singular at the value 0 of its variables.

1. Introduction

During the last decade, there were a lot of manuscripts on fractional differential equations (see, e.g., [1–19] and the references therein). Fractional equations have been discussed extensively as valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in fluid mechanics, viscoelasticity, edge detection, porous media, and electromagnetism, as well as in various other areas. For more examples and details, see [3, 13–15] and references therein. On the other hand, there are many works about the existence of positive solutions of fractional differential equations (see, e.g., [1, 2, 6–8, 10, 17, 19] and the references therein).

2. The Problem

In this paper, we investigate the following coupled system of multiterm singular fractional integrodifferential boundary value problem:

$$
\begin{aligned}
&D^{\alpha}_{0^+}u(t) + f_1\Big(t, u(t), v(t), (\phi_1 u)(t),\\
&\qquad (\psi_1 v)(t), D^{p}_{0^+}u(t), D^{\mu_1}_{0^+}v(t),\\
&\qquad D^{\mu_2}_{0^+}v(t), \dots, D^{\mu_m}_{0^+}v(t)\Big) = 0,\\
&D^{\beta}_{0^+}v(t) + f_2\Big(t, u(t), v(t), (\phi_2 u)(t),
\end{aligned}
$$

$$\left(\psi_2 v\right)(t), D_{0^+}^{q} v(t), D_{0^+}^{\nu_1} u(t), \\ D_{0^+}^{\nu_2} u(t), \ldots, D_{0^+}^{\nu_m} u(t)\Big) = 0, \tag{1}$$

$$u^{(i)}(0) = 0 \quad \forall 0 \le i \le n-2, \\ \left[D_{0^+}^{\delta_1} u(t)\right]_{t=1} = 0 \quad \text{for } 2 < \delta_1 < n-1,\ \alpha - \delta_1 \ge 1, \tag{2}$$

$$v^{(i)}(0) = 0 \quad \forall 0 \le i \le n-2, \\ \left[D_{0^+}^{\delta_2} v(t)\right]_{t=1} = 0 \quad \text{for } 2 < \delta_2 < n-1,\ \beta - \delta_2 \ge 1, \tag{3}$$

where $n \ge 4$, $n-1 < \alpha, \beta < n$, $0 < p, q < 1$, $1 < \mu_i, \nu_i < 2$ $(i = 1, 2, \ldots, m)$, γ_j and λ_j are positive-valued continuous functions on $[0,1] \times [0,1]$ $(j = 1, 2)$,

$$\left(\phi_j u\right)(t) = \int_0^t \gamma_j(t,s) u(s) ds, \\ \left(\psi_j v\right)(t) = \int_0^t \lambda_j(t,s) v(s) ds, \tag{4}$$

$\gamma_0^j = \sup_{t\in[0,1]} \left| \int_0^t \gamma_j(t,s) ds \right| < \infty$ and $\lambda_0^j = \sup_{t\in[0,1]} \left| \int_0^t \lambda_j(t,s) ds \right| < \infty$. Also, f_1 and f_2 satisfy the local Caratheodory condition on $[0,1] \times \mathfrak{D}$($f_1, f_2 \in \operatorname{Car}([0,1] \times \mathfrak{D})$), where $\mathfrak{D}$ is a subset of $\mathbb{R}^{m+5}$ and $f_j(t, x, y, z, w, v, u_1, u_2, \ldots, u_m)$ may be singular at the value 0 of all its variables. Here, D is the standard Riemann-Liouville fractional derivative. In fact, a function f satisfies the local Caratheodory condition on $[0,1] \times \mathfrak{D}$ whenever

(i) $f(\cdot; x, y, z, w, v, u_1, u_2, \ldots, u_m) : [0,1] \to \mathbb{R}$ is measurable for all $(x, y, z, w, v, u_1, u_2, \ldots, u_m) \in \mathfrak{D}$

(ii) $f(t; \cdot, \cdot, \ldots, \cdot) : \mathfrak{D} \to \mathbb{R}$ is continuous for almost all $t \in [0,1]$,

(iii) for each compact subset $\kappa \subset \mathfrak{D}$, there is a function $\varphi_\kappa \in L^1[0,1]$ such that

$$\left| f(t; x, y, z, w, v, u_1, u_2, \ldots, u_m) \right| \le \varphi_\kappa(t), \tag{5}$$

for almost all $t \in [0,1]$ and all $(x, y, z, w, v, u_1, u_2, \ldots, u_m) \in \kappa$. The functions $u, v \in C^2[0,1]$ are called positive solutions of the problem (1), (2), and (3) whenever $u > 0$ and $v > 0$ on $(0,1]$, $D_{0^+}^{\alpha} u, D_{0^+}^{\beta} v \in L^1[0,1]$, u, v satisfy boundary conditions (2), (3), and (1) holds for almost all $t \in [0,1]$. In 2010, Agarwal et al. reviewed positive solutions of the singular Dirichlet problem

$$D_{0^+}^{\alpha} u(t) + f\left(t, u(t), D_{0^+}^{\mu} u(t)\right) = 0, \tag{6}$$

$u(0) = u(1) = 0$, where $1 < \alpha < 2$, $0 < \mu \le \alpha - 1$ and f satisfies the local Caratheodory condition on $[0,1] \times (0,\infty) \times \mathbb{R}$ [1]. In 2011, Staněk reviewed the singular problem

$$D_{0^+}^{\alpha} u(t) + f\left(t, u(t), u'(t), D_{0^+}^{\mu} u(t)\right) = 0, \tag{7}$$

$u(0) = 0$, $u'(0) = u'(1) = 0$, where $2 < \alpha < 3$, $0 < \mu < 1$ and f satisfies the local Caratheodory condition on $[0,1] \times \mathfrak{D}$ [19]. Here, $\mathfrak{D}$ is a subset of $\mathbb{R}^3$. In 2012, Bai and Sun reviewed the singular problem

$$D_{0^+}^{\alpha} u(t) + f\left(t, u(t), D_{0^+}^{\nu} u(t), D_{0^+}^{\mu} u(t)\right) = 0, \tag{8}$$

$u(0) = u'(0) = u''(0) = u''(1) = 0$, where $3 < \alpha \le 4$, $0 < \nu \le 1$, $1 < \mu \le 2$ and f satisfies the local Caratheodory condition on $[0,1] \times \mathfrak{D}$ [8]. Again, $\mathfrak{D}$ is a subset of $\mathbb{R}^3$. In this year, Agarwal et al. reviewed the singular problem

$$^{c}D_{0^+}^{\alpha} u(t) + f\left(t, u(t), u'(t), {}^{c}D_{0^+}^{\mu} u(t)\right) = 0, \tag{9}$$

$u'(0) = 0$, $u(1) = 0$, where $1 < \alpha < 2$, $0 < \mu < 1$ and f is a L^q-Caratheodory function on $[0,1] \times \mathfrak{B}$ [2]. Here, $\mathfrak{B} = (0,\infty) \times (-\infty, 0) \times (-\infty, 0)$, $q > 1/(\alpha - 1)$ and ^{c}D is the Caputo fractional derivative. Many researchers have established the existence and uniqueness of solutions for some systems of nonlinear fractional differential equations, but there are few works about coupled system of multiterm nonlinear fractional differential equations (see, for example, [4, 5, 20, 21]). Also, there are a few papers discussing singular system of fractional differential equations (see, e.g., [6, 16]).

Let $\|x\|_1 = \int_0^1 |x(t)| dt$ be the norm of $L^1[0,1]$, $\|x\| = \max\{|x(t)| : t \in [0,1]\}$ the norm of $C[0,1]$, and $\|u\|_* = \max\{\|u\|, \|u'\|, \|u''\|\}$ the norm of $X = C^2[0,1]$. As you know, $(X, \|\cdot\|_*)$ and $(X \times X, \|\cdot\|_{**})$ are Banach spaces, where $\|(u,v)\|_{**} = \max\{\|u\|_*, \|v\|_*\}$. Suppose that $AC[0,1]$ and $AC^k[0,1]$ are the spaces of absolutely continuous functions and functions having absolutely continuous kth derivatives on $[0,1]$, respectively. In this paper, we suppose that the functions f_1 and f_2 in (1) satisfy the following conditions:

(H_1) $f_1, f_2 \in \operatorname{Car}([0,1] \times \mathfrak{D})$, where $\mathfrak{D} = (0,\infty)^{m+5}$. Also, there exists a positive constant m_j such that $f_j(t, x, y, z, w, v, u_1, u_2, \ldots, u_m) \ge m_j$ for almost all $t \in [0,1]$ and $(x, y, z, w, v, u_1, u_2, \ldots, u_m) \in \mathfrak{D}$;

(H_2) there exist positive mappings $k_1, k_2 \in L^1[0,1]$, $P_1, P_2 \in C(\mathfrak{D})$, and $h_1, h_2 \in C([0,\infty)^{m+5})$ such that P_1 and P_2 are nonincreasing and h_1 and h_2 are nondecreasing in all their variables,

$$M_1 = \frac{m_1}{(\alpha - \delta_1)\Gamma(\alpha+1)}, \qquad M_2 = \frac{m_2}{(\beta - \delta_2)\Gamma(\beta+1)},$$

$$\lim_{x\to\infty} \frac{h_j(x, x, \ldots, x)}{x} = 0, \quad (j = 1, 2),$$

$$\int_0^1 P_1\Bigg(M_1 s^{\alpha-1}, M_2 s^{\beta-1}, \\ M_1 \int_0^s \gamma_1(s,\tau) \tau^{\alpha-1} d\tau, M_2 \int_0^s \lambda_1(s,\tau) \tau^{\beta-1} d\tau, \\ \frac{(\delta_1 - 2) M_1}{\Gamma(\alpha - p)} s^{\alpha-p}, \frac{(\delta_2 - 2) M_2}{\Gamma(\beta - \mu_1)} s^{\beta-\mu_1},$$

$$\left.\frac{(\delta_2-2)\,M_2}{\Gamma(\beta-\mu_2)}s^{\beta-\mu_2},\ldots,\frac{(\delta_2-2)\,M_2}{\Gamma(\beta-\mu_m)}s^{\beta-\mu_m}\right)ds<\infty,$$

$$\int_0^1 P_2\left(M_1 s^{\alpha-1}, M_2 s^{\beta-1},\right.$$
$$M_1\int_0^s \gamma_2(s,\tau)\,\tau^{\alpha-1}d\tau, M_2\int_0^s \lambda_2(s,\tau)\,\tau^{\beta-1}d\tau,$$
$$\frac{(\delta_2-2)\,M_2}{\Gamma(\beta-q)}s^{\beta-q},\frac{(\delta_1-2)\,M_1}{\Gamma(\alpha-\nu_1)}s^{\alpha-\nu_1},$$
$$\left.\frac{(\delta_1-2)\,M_1}{\Gamma(\alpha-\nu_2)}s^{\alpha-\nu_2},\ldots,\frac{(\delta_1-2)\,M_1}{\Gamma(\alpha-\nu_m)}s^{\alpha-\nu_m}\right)ds<\infty,$$

$$f_j(t,x,y,z,w,v,u_1,u_2,\ldots,u_m)$$
$$\leq P_j(x,y,z,w,v,u_1,u_2,\ldots,u_m)$$
$$+k_j(t)\,h_j(x,y,z,w,v,u_1,u_2,\ldots,u_m) \tag{10}$$

for almost all $t \in [0,1]$ and $(x,y,z,w,v,u_1,u_2,\ldots,u_m) \in \mathfrak{D}$. Since we suppose that the problem (1) is singular, we use regularization and sequential techniques for the existence of positive solutions of the problem. In this way, for each natural number n define the function f_n^j $(j=1,2)$ by

$$f_n^j(t,x,y,z,w,v,u_1,u_2,\ldots,u_m)$$
$$=f_j(t,\chi_n(x),\chi_n(y),\chi_n(z),\chi_n(w),$$
$$\chi_n(v),\chi_n(u_1),\chi_n(u_2),\ldots,\chi_n(u_m)) \tag{11}$$

for almost all $t \in [0,1]$ and $(x,y,z,w,v,u_1,u_2,\ldots,u_m) \in \mathbb{R}^{m+5}$, where

$$\chi_n(x)=\begin{cases} x & \text{if } x\geq \dfrac{1}{n},\\ \dfrac{1}{n} & \text{if } x<\dfrac{1}{n}.\end{cases} \tag{12}$$

Then condition (H_1) implies that $f_n^j \in \mathrm{Car}([0,1]\times\mathbb{R}^{m+5})$ and $f_n^j(t,x,y,z,w,v,u_1,u_2,\ldots,u_m) \geq m_j$ for almost all $t\in[0,1]$ and $(x,y,z,w,v,u_1,u_2,\ldots,u_m) \in \mathbb{R}^{m+5}$, $(j=1,2)$. Also, the condition (H_2) implies

$$f_n^j(t,x,y,z,w,v,u_1,u_2,\ldots,u_m)$$
$$\leq P_j\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)$$
$$+k_j(t)\,h_j(1+x,1+y,1+z,1+w,$$
$$1+v,1+u_1,1+u_2,\ldots,1+u_m) \tag{13}$$

for almost all $t \in [0,1]$ and $(x,y,z,w,v,u_1,u_2,\ldots,u_m) \in [0,\infty)^{m+5}$ and

$$f_n^j(t,x,y,z,w,v,u_1,u_2,\ldots,u_m)$$
$$\leq P_j(x,y,z,w,v,u_1,u_2,\ldots,u_m)$$
$$+k_j(t)\,h_j(1+x,1+y,1+z,1+w,$$
$$1+v,1+u_1,1+u_2,\ldots,1+u_m) \tag{14}$$

for almost all $t\in[0,1]$ and $(x,y,z,w,v,u_1,u_2,\ldots,u_m)\in\mathfrak{D}$. For obtaining solutions of the system (1), (2), and (3), for each natural number n, we will obtain solutions of the system

$$D_{0^+}^{\alpha}u(t)+f_n^1\left(t,u(t),v(t),(\phi_1 u)(t),(\psi_1 v)(t),\right.$$
$$\left.D_{0^+}^{p}u(t),D_{0^+}^{\mu_1}v(t),D_{0^+}^{\mu_2}v(t),\ldots,D_{0^+}^{\mu_m}v(t)\right)=0,$$
$$D_{0^+}^{\beta}v(t)+f_n^2\left(t,u(t),v(t),(\phi_2 u)(t),(\psi_2 v)(t),\right.$$
$$\left.D_{0^+}^{q}v(t),D_{0^+}^{\nu_1}u(t),D_{0^+}^{\nu_2}u(t),\ldots,D_{0^+}^{\nu_m}u(t)\right)=0, \tag{15}$$

via the boundary conditions (2) and (3) and by using solutions of this system, we will obtain solution of the system (1), (2), and (3). It has been proved that the fractional integral $I_{0^+}^{\alpha}$ maps $L^1[0,1]$ into $L^1[0,1]$ whenever $\alpha\in(0,1)$ and maps $L^1[0,1]$ into $AC^{[\alpha]-1}[0,1]$ whenever $\alpha\geq 1$ [19]. Here, $[\alpha]$ means the integral part of α and $AC^0[0,1]=AC[0,1]$. Suppose that $\alpha>0$ and α is not a natural number. If $x\in C(0,1]$ and $D_{0^+}^{\alpha}x \in L^1[0,1]$, then $x(t)=I_{0^+}^{\alpha}D_{0^+}^{\alpha}x(t)+\sum_{k=1}^{n}c_k t^{\alpha-k}$ for all $t\in(0,1]$, where $n=[\alpha]+1$ and $c_k\in\mathbb{R}$ for $k=1,2,\ldots,n$ [7]. Suppose that $\nu\in(0,1)$, $\mu\in(1,2)$, $x\in C^2[0,1]$, and $x(0)=x'(0)=0$. Then, $D_{0^+}^{\mu}x\in C[0,1]$, $D_{0^+}^{\nu}x\in C[0,1]$, $D_{0^+}^{\mu}x(t)=(1/\Gamma(2-\mu))\int_0^t(t-s)^{1-\mu}x''(s)ds$, and $D_{0^+}^{\nu}x(t)=(1/\Gamma(2-\nu))\int_0^t(t-s)^{1-\nu}x''(s)ds$ [2].

3. Main Results

Now, we are ready to state and prove our main results. One can find main idea of next result in [17].

Lemma 1. *Let $n\geq 4$ and $n-1<\alpha<n$. For each $g\in L^1[0,1]$, $u(t)=\int_0^1 G_\alpha(t,s)g(s)ds$ is the unique solution of the equation $D_{0^+}^{\alpha}u(t)+g(t)=0$ in $C^2[0,1]$ which satisfies the boundary condition* (2)*, where*

$$G_\alpha(t,s)=\begin{cases}\dfrac{t^{\alpha-1}(1-s)^{\alpha-\delta_1-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)} & 0\leq s\leq t\leq 1,\\[2ex] \dfrac{t^{\alpha-1}(1-s)^{\alpha-\delta_1-1}}{\Gamma(\alpha)} & 0\leq t\leq s\leq 1.\end{cases} \tag{16}$$

Proof. It is easy to see that the functions $u(t) = -I_{0^+}^{\alpha} g(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$ are solutions of $D_{0^+}^{\alpha} u(t) + g(t) = 0$ in $C(0,1]$ for all $c_i \in \mathbb{R}$. Since $[\alpha] \geq 3$, $I_{0^+}^{\alpha} g \in AC^{[\alpha]-1}[0,1]$ and so $u(t) = -I_{0^+}^{\alpha} g(t) + c_1 t^{\alpha-1}$ are solutions of $D_{0^+}^{\alpha} u(t) + g(t) = 0$ in $C^2[0,1]$, where $c_1 \in \mathbb{R}$. The boundary condition (2) implies that $c_2 = \cdots = c_{n-1} = 0$ and

$$0 = c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha - \delta_1)} (1)^{\alpha-\delta_1-1} - \frac{1}{\Gamma(\alpha - \delta_1)} \int_0^1 (1-s)^{\alpha-\delta_1-1} g(s)\, ds. \tag{17}$$

Thus, $u(t) = (t^{\alpha-1}/\Gamma(\alpha)) \int_0^1 (1-s)^{\alpha-\delta_1-1} g(s) ds - (1/\Gamma(\alpha)) \int_0^t (t-s)^{\alpha-1} g(s) ds = \int_0^1 G_\alpha(t,s) g(s) ds$ is the unique solution of the problem in $C^2[0,1]$. □

Note that the Green function G_α in Lemma 1 has some properties. For example, G_α, $(\partial/\partial t) G_\alpha$ and $(\partial^2/\partial t^2) G_\alpha$ are continuous functions on $[0,1] \times [0,1]$, $G_\alpha(t,s) > 0$ on $(0,1) \times (0,1)$, $G_\alpha(t,s) \leq 1/\Gamma(\alpha)$ for all $(t,s) \in [0,1] \times [0,1]$, $\int_0^1 G_\alpha(t,s) ds \geq t^{\alpha-1}/(\alpha - \delta_1)\Gamma(\alpha+1)$ for all $t \in [0,1]$, $(\partial/\partial t) G_\alpha(t,s) > 0$ on $(0,1) \times (0,1)$, $(\partial/\partial t) G_\alpha(t,s) \leq 1/\Gamma(\alpha-1)$ for all $(t,s) \in [0,1] \times [0,1]$, $\int_0^1 (\partial/\partial t) G_\alpha(t,s) ds \geq t^{\alpha-2}/(\alpha - \delta_1)\Gamma(\alpha)$ for all $t \in [0,1]$, $(\partial^2/\partial t^2) G_\alpha(t,s) > 0$ on $(0,1) \times (0,1)$, $(\partial^2/\partial t^2) G_\alpha(t,s) \leq 1/\Gamma(\alpha-2)$ for all $(t,s) \in [0,1] \times [0,1]$, and $\int_0^1 (\partial^2/\partial t^2) G_\alpha(t,s) ds \geq (\delta_1 - 2) t^{\alpha-3}/(\alpha - \delta_1)\Gamma(\alpha-1)$ for all $t \in [0,1]$. Let $X = C^2[0,1]$. Define the cone P on $X \times X$ by

$$P = \Big\{ (x,y) \in X \times X : x(0) = x'(0) = y(0) = y'(0) = 0, \; x(t), x'(t), x''(t), y(t), y'(t), y''(t) \geq 0, \; \forall t \in [0,1] \Big\}. \tag{18}$$

It is easy to see that for each $(x,y) \in P$, $i = 1, 2, \ldots, m$ and $t \in [0,1]$ we have $D_{0^+}^{p} x, D_{0^+}^{\nu_i} x, D_{0^+}^{q} y, D_{0^+}^{\mu_i} y \in C[0,1]$ and $D_{0^+}^{p} x(t) \geq 0$, $D_{0^+}^{\nu_i} x(t) \geq 0$, $D_{0^+}^{q} y(t) \geq 0$, $D_{0^+}^{\mu_i} y(t) \geq 0$. Now for each natural number n, define the operator T_n on P by

$$\begin{aligned} T_n(x,y)(t) &= \begin{pmatrix} T_n^1(x,y)(t) \\ T_n^2(x,y)(t) \end{pmatrix} \\ &= \Bigg(\int_0^1 G_\alpha(t,s) f_n^1 \times \big(s, x(s), y(s), (\phi_1 x)(s), (\psi_1 y)(s), D_{0^+}^{p} x(s), D_{0^+}^{\mu_1} y(s), D_{0^+}^{\mu_2} y(s), \ldots, D_{0^+}^{\mu_m} y(s)\big) ds \\ &\quad \int_0^1 G_\beta(t,s) f_n^2 \times \big(s, x(s), y(s), (\phi_2 x)(s), (\psi_2 y)(s), D_{0^+}^{q} y(s), D_{0^+}^{\nu_1} x(s), D_{0^+}^{\nu_2} x(s), \ldots, D_{0^+}^{\nu_m} x(s)\big) ds \Bigg). \end{aligned} \tag{19}$$

Lemma 2. *For each natural number n, T_n is a completely continuous operator on P.*

Proof. Let n be a natural number, $(x,y) \in P$,

$$\rho^1(t) = f_n^1\big(t, x(t), y(t), (\phi_1 x)(t), (\psi_1 y)(t), D_{0^+}^{p} x(t), D_{0^+}^{\mu_1} y(t), D_{0^+}^{\mu_2} y(t), \ldots, D_{0^+}^{\mu_m} y(t)\big) \tag{20}$$

and $\rho^2(t) = f_n^2(t, x(t), y(t), (\phi_2 x)(t), (\psi_2 y)(t), D_{0^+}^{q} y(t), D_{0^+}^{\nu_1} x(t), D_{0^+}^{\nu_2} x(t), \ldots, D_{0^+}^{\nu_m} x(t))$. Then, $\rho^1, \rho^2 \in L^1[0,1]$ and there exist positive constants m_1 and m_2 such that $\rho^1(t) \geq m_1$ and $\rho^2(t) \geq m_2$ for almost all $t \in [0,1]$. Since G_α, G_β, $(\partial/\partial t) G_\alpha$, $(\partial/\partial t) G_\beta$, $(\partial^2/\partial t^2) G_\alpha$, and $(\partial^2/\partial t^2) G_\beta$ are nonnegative and continuous functions on $[0,1] \times [0,1]$, $G_\alpha(0,s) = 0$, and $G_\beta(0,s) = 0$ for all $s \in [0,1]$, we get $T_n^1(x,y), T_n^2(x,y) \in C^2[0,1]$, $T_n^1(x,y)(0) = T_n^2(x,y)(0) = (T_n^1(x,y))'(0) = (T_n^2(x,y))'(0) = 0$, $T_n^1(x,y) \geq 0$, $T_n^2(x,y) \geq 0$, $(T_n^1(x,y))' \geq 0$, $(T_n^2(x,y))' \geq 0$, $(T_n^1(x,y))'' \geq 0$, and $(T_n^2(x,y))'' \geq 0$ on $[0,1]$. Thus, T_n maps P into P. Suppose that $\{(x_k, y_k)\}_{k\geq 1}$ is a convergent sequence in P and $\lim_{k\to\infty}(x_k, y_k) = (x,y)$. Then, $\lim_{k\to\infty} x_k^{(j)}(t) = x^{(j)}(t)$ and $\lim_{k\to\infty} y_k^{(j)}(t) = y^{(j)}(t)$ uniformly on $[0,1]$ for $j = 0,1,2$. Since for $p, q \in (0,1)$, $\mu_i, \nu_i \in (1,2)$ for all $i = 1, 2, \ldots, m$,

$$\begin{aligned} &\left| D_{0^+}^{p} x_k(t) - D_{0^+}^{p} x(t) \right| \\ &\quad \leq \frac{\|x_k'' - x''\|}{\Gamma(2-p)} \int_0^t (t-s)^{1-p} ds \leq \frac{\|x_k'' - x''\|}{\Gamma(3-p)}, \\ &\left| D_{0^+}^{q} y_k(t) - D_{0^+}^{q} y(t) \right| \\ &\quad \leq \frac{\|y_k'' - y''\|}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} ds \leq \frac{\|y_k'' - y''\|}{\Gamma(3-q)}, \\ &\left| D_{0^+}^{\nu_i} x_k(t) - D_{0^+}^{\nu_i} x(t) \right| \\ &\quad \leq \frac{\|x_k'' - x''\|}{\Gamma(2-\nu_i)} \int_0^t (t-s)^{1-\nu_i} ds \leq \frac{\|x_k'' - x''\|}{\Gamma(3-\nu_i)}, \end{aligned}$$

$$\begin{aligned}
&\left|D_{0^+}^{\mu_i} y_k(t) - D_{0^+}^{\mu_i} y(t)\right| \\
&\quad \le \frac{\|y_k'' - y''\|}{\Gamma(2-\mu_i)} \int_0^t (t-s)^{1-\mu_i} ds \le \frac{\|y_k'' - y''\|}{\Gamma(3-\mu_i)}, \\
&\left|(\phi_j x_k)(t) - (\phi_j x)(t)\right| \\
&\quad \le \sup_{t\in[0,1]} \left|\int_0^t \gamma_j(t,s)\, ds\right| \|x_k - x\| = \gamma_0^j \|x_k - x\|, \\
&\left|(\psi_j y_k)(t) - (\psi_j y)(t)\right| \\
&\quad \le \sup_{t\in[0,1]} \left|\int_0^t \lambda_j(t,s)\, ds\right| \|y_k - y\| = \lambda_0^j \|y_k - y\|,
\end{aligned} \tag{21}$$

for $j = 1,2$ and $k \ge 1$, we have $\lim_{k\to\infty} D_{0^+}^{p} x_k(t) = D_{0^+}^{p} x(t)$, $\lim_{k\to\infty} D_{0^+}^{q} y_k(t) = D_{0^+}^{q} y(t)$, $\lim_{k\to\infty} D_{0^+}^{\nu_i} x_k(t) = D_{0^+}^{\nu_i} x(t)$, $\lim_{k\to\infty} D_{0^+}^{\mu_i} y_k(t) = D_{0^+}^{\mu_i} y(t)$ for $i = 1,2,\dots,m$, and $\lim_{k\to\infty}(\phi_j x_k)(t) = (\phi_j x)(t)$, $\lim_{k\to\infty}(\psi_j y_k)(t) = (\psi_j y)(t)$ uniformly on $[0,1]$ for $j = 1,2$. Also, we have

$$\begin{aligned}
\left\|D_{0^+}^{p} x_k\right\| &\le \frac{\|x_k''\|}{\Gamma(2-p)} \int_0^t (t-s)^{1-p} ds \le \frac{\|x_k''\|}{\Gamma(3-p)}, \\
\left\|D_{0^+}^{q} y_k\right\| &\le \frac{\|y_k''\|}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} ds \le \frac{\|y_k''\|}{\Gamma(3-q)}, \\
\left\|D_{0^+}^{\nu_i} x_k\right\| &\le \frac{\|x_k''\|}{\Gamma(2-\nu_i)} \int_0^t (t-s)^{1-\nu_i} ds \le \frac{\|x_k''\|}{\Gamma(3-\nu_i)}, \\
\left\|D_{0^+}^{\mu_i} y_k\right\| &\le \frac{\|y_k''\|}{\Gamma(2-\mu_i)} \int_0^t (t-s)^{1-\mu_i} ds \le \frac{\|y_k''\|}{\Gamma(3-\mu_i)},
\end{aligned} \tag{22}$$

and also $\|\phi_j x_k\| \le \gamma_0^j \|x_k\|$ and $\|\psi_j y_k\| \le \lambda_0^j \|y_k\|$ for $j = 1,2$ and $k \ge 1$. Now, put

$$\begin{aligned}
\rho_k^1(t) = f_n^1\big(&t, x_k(t), y_k(t), (\phi_1 x_k)(t), \\
&(\psi_1 y_k)(t), D_{0^+}^{p} x_k(t), D_{0^+}^{\mu_1} y_k(t), \\
&D_{0^+}^{\mu_2} y_k(t), \dots, D_{0^+}^{\mu_m} y_k(t)\big)
\end{aligned} \tag{23}$$

and $\rho_k^2(t) = f_n^2(t, x_k(t), y_k(t), (\phi_2 x_k)(t), (\psi_2 y_k)(t), D_{0^+}^{q} y_k(t), D_{0^+}^{\nu_1} x_k(t), D_{0^+}^{\nu_2} x_k(t), \dots, D_{0^+}^{\nu_m} x_k(t))$ for all $k \ge 1$. Then, $\lim_{k\to\infty}\rho_k^j(t) = \rho^j(t)$ for almost all $t \in [0,1]$ and $j = 1,2$. Since $f_n^j \in \mathrm{Car}([0,1]\times\mathbb{R}^{m+5})$ and $\{x_k\}_{k\ge1}$, $\{y_k\}_{k\ge1}$ are bounded sequences in $C^2[0,1]$ and also by using the above inequalities, $\{D_{0^+}^{p} x_k\}_{k\ge1}$, $\{D_{0^+}^{q} y_k\}_{k\ge1}$, $\{D_{0^+}^{\nu_i} x_k\}_{k\ge1}$, $\{D_{0^+}^{\mu_i} y_k\}_{k\ge1}$ for $i = 1,2,\dots,m$ and $\{\phi_j x_k\}_{k\ge1}$, $\{\psi_j x_k\}_{k\ge1}$ for $j = 1,2$ are bounded sequences in $C[0,1]$, there exist $g_1, g_2 \in L^1[0,1]$ such that $m_j \le \rho_k^j(t) \le g_j(t)$ for almost all $t \in [0,1]$ and $k \ge 1$. Thus, by using the Lebesgue dominated convergence theorem and the following relations:

$$\begin{aligned}
&\left|T_n^1(x_k,y_k)(t) - T_n^1(x,y)(t)\right| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^1 \left|\rho_k^1(s) - \rho^1(s)\right| ds, \\
&\left|\left(T_n^1(x_k,y_k)\right)'(t) - \left(T_n^1(x,y)\right)'(t)\right| \\
&\quad \le \frac{1}{\Gamma(\alpha-1)} \int_0^1 \left|\rho_k^1(s) - \rho^1(s)\right| ds, \\
&\left|\left(T_n^1(x_k,y_k)\right)''(t) - \left(T_n^1(x,y)\right)''(t)\right| \\
&\quad \le \frac{1}{\Gamma(\alpha-2)} \int_0^1 \left|\rho_k^1(s) - \rho^1(s)\right| ds, \\
&\left|T_n^2(x_k,y_k)(t) - T_n^2(x,y)(t)\right| \\
&\quad \le \frac{1}{\Gamma(\beta)} \int_0^1 \left|\rho_k^2(s) - \rho^2(s)\right| ds, \\
&\left|\left(T_n^2(x_k,y_k)\right)'(t) - \left(T_n^2(x,y)\right)'(t)\right| \\
&\quad \le \frac{1}{\Gamma(\beta-1)} \int_0^1 \left|\rho_k^2(s) - \rho^2(s)\right| ds, \\
&\left|\left(T_n^2(x_k,y_k)\right)''(t) - \left(T_n^2(x,y)\right)''(t)\right| \\
&\quad \le \frac{1}{\Gamma(\beta-2)} \int_0^1 \left|\rho_k^2(s) - \rho^2(s)\right| ds,
\end{aligned} \tag{24}$$

we get $\lim_{k\to\infty}(T_n^1(x_k,y_k))^{(j)}(t) = (T_n^1(x,y))^{(j)}(t)$ and $\lim_{k\to\infty}(T_n^2(x_k,y_k))^{(j)}(t) = (T_n^2(x,y))^{(j)}(t)$ uniformly on $[0,1]$ for $j = 0,1,2$. This implies that T_n^1 and T_n^2 are continuous operators and so T_n is a continuous operator. Let $\{(x_k,y_k)\}_{k\ge1}$ be a bounded sequence in P. We show that the sequence $\{T_n(x_k,y_k)\}_{k\ge1}$ is relatively compact in $X\times X$. By using the Arzela-Ascoli theorem, it is sufficient to prove that $\{T_n(x_k,y_k)\}_{k\ge1}$ is bounded in $X\times X$ and $\{(T_n(x_k,y_k))''\}$ is equicontinuous on $[0,1]$. Choose a positive number S such that $\|x_k\| \le S$, $\|y_k\| \le S$, $\|x_k'\| \le S$, $\|y_k'\| \le S$, $\|x_k''\| \le S$, $\|y_k''\| \le S$ for all $k \ge 1$. By using the above inequalities, one can see that $\|D_{0^+}^{p} x_k\| \le S/\Gamma(3-p)$, $\|D_{0^+}^{q} y_k\| \le S/\Gamma(3-q)$, $\|D_{0^+}^{\nu_i} x_k\| \le S/\Gamma(3-\nu_i)$, and $\|D_{0^+}^{\mu_i} y_k\| \le S/\Gamma(3-\mu_i)$ for $i = 1,2,\dots,m$ and $\|\phi_j x_k\| \le \gamma_0^j S$ and $\|\psi_j x_k\| \le \lambda_0^j S$ for $j = 1,2$. Since

$$\begin{aligned}
0 \le T_n^1(x_k,y_k)(t) &= \int_0^1 G_\alpha(t,s)\,\rho_k^1(s)\, ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^1 g_1(s)\, ds = \frac{\|g_1\|_1}{\Gamma(\alpha)},
\end{aligned}$$

$$0 \le \left(T_n^1(x_k, y_k)\right)'(t) = \int_0^1 \frac{\partial}{\partial t} G_\alpha(t,s)\, \rho_k^1(s)\, ds$$
$$\le \frac{1}{\Gamma(\alpha-1)} \int_0^1 g_1(s)\, ds = \frac{\|g_1\|_1}{\Gamma(\alpha-1)},$$
$$0 \le \left(T_n^1(x_k, y_k)\right)''(t) = \int_0^1 \frac{\partial^2}{\partial t^2} G_\alpha(t,s)\, \rho_k^1(s)\, ds$$
$$\le \frac{1}{\Gamma(\alpha-2)} \int_0^1 g_1(s)\, ds = \frac{\|g_1\|_1}{\Gamma(\alpha-2)}, \tag{25}$$

$\|T_n^1(x_k, y_k)\|_* \le \|g_1\|_1/\Gamma(\alpha-2)$. Similarly, we can verify that $\|T_n^2(x_k, y_k)\|_* \le \|g_2\|_1/\Gamma(\beta-2)$. Thus,

$$\|T_n(x_k, y_k)\|_{**} \le \max\left\{\frac{\|g_1\|_1}{\Gamma(\alpha-2)}, \frac{\|g_2\|_1}{\Gamma(\beta-2)}\right\} \tag{26}$$

for all k. Hence, $\{T_n(x_k, y_k)\}_{k\ge1}$ is bounded in $X\times X$. Also, for $0 \le t_1 < t_2 \le 1$ we have

$$\left|\left(T_n^1(x_k, y_k)\right)''(t_2) - \left(T_n^1(x_k, y_k)\right)''(t_1)\right|$$
$$\le \frac{t_2^{\alpha-3} - t_1^{\alpha-3}}{\Gamma(\alpha-2)} \int_0^1 (1-s)^{\alpha-\delta_1-1} \rho_k^1(s)\, ds + \frac{1}{\Gamma(\alpha-2)}$$
$$\times \left|\int_0^{t_2} (t_2-s)^{\alpha-3} \rho_k^1(s)\, ds - \int_0^{t_1} (t_1-s)^{\alpha-3} \rho_k^1(s)\, ds\right|$$
$$\le \frac{\|\rho_k^1\|_1}{\Gamma(\alpha-2)} \left(t_2^{\alpha-3} - t_1^{\alpha-3}\right) + \frac{1}{\Gamma(\alpha-2)}$$
$$\times \left[\int_{t_1}^{t_2} (t_2-s)^{\alpha-3} \rho_k^1(s)\, ds + \int_0^{t_1} \left((t_2-s)^{\alpha-3} - (t_1-s)^{\alpha-3}\right) \rho_k^1(s)\, ds\right]$$
$$\le \frac{\|g_1\|_1}{\Gamma(\alpha-2)} \left(t_2^{\alpha-3} - t_1^{\alpha-3}\right) + \frac{1}{\Gamma(\alpha-2)}$$
$$\times \left[(t_2-t_1)^{\alpha-3} \|g_1\|_1 + \int_0^{t_1} \left((t_2-s)^{\alpha-3} - (t_1-s)^{\alpha-3}\right) g_1(s)\, ds\right]. \tag{27}$$

Similarly, we have

$$\left|\left(T_n^2(x_k, y_k)\right)''(t_2) - \left(T_n^2(x_k, y_k)\right)''(t_1)\right|$$
$$\le \frac{\|g_2\|_1}{\Gamma(\beta-2)} \left(t_2^{\beta-3} - t_1^{\beta-3}\right) + \frac{1}{\Gamma(\beta-2)}$$
$$\times \left[(t_2-t_1)^{\beta-3} \|g_2\|_1 + \int_0^{t_1} \left((t_2-s)^{\beta-3} - (t_1-s)^{\beta-3}\right) g_2(s)\, ds\right]. \tag{28}$$

Let $\epsilon > 0$ be given. Since the functions $t^{\alpha-3}$ and $t^{\beta-3}$ are uniformly continuous on $[0,1]$ and the functions $|t-s|^{\alpha-3}$ and $|t-s|^{\beta-3}$ are uniformly continuous on $[0,1]\times[0,1]$, there exists $\delta > 0$ such that $0 < t_2^{\alpha-3} - t_1^{\alpha-3} < \epsilon$, $0 < t_2^{\beta-3} - t_1^{\beta-3} < \epsilon$, $0 < (t_2-s)^{\alpha-3} - (t_1-s)^{\alpha-3} < \epsilon$, and $0 < (t_2-s)^{\beta-3} - (t_1-s)^{\beta-3} < \epsilon$ for all $0 \le t_1 < t_2 \le 1$, $t_2 - t_1 < \delta$ and $0 \le s \le t_1$. If $0 \le t_1 < t_2 \le 1$ and $t_2 - t_1 < \min\{\delta, \sqrt[\alpha-3]{\epsilon}, \sqrt[\beta-3]{\epsilon}\}$, then

$$\left|\left(T_n^1(x_k, y_k)\right)''(t_2) - \left(T_n^1(x_k, y_k)\right)''(t_1)\right| \le \frac{3\epsilon}{\Gamma(\alpha-2)} \|g_1\|_1 \tag{29}$$

and $|(T_n^2(x_k, y_k))''(t_2) - (T_n^2(x_k, y_k))''(t_1)| \le (3\epsilon/\Gamma(\beta-2))\|g_2\|_1$. Therefore, $\{(T_n(x_k, y_k))''\}$ is equicontinuous on $[0,1]$ and so T_n is a completely continuous operator on P. □

We need the following result (see [10]).

Lemma 3. *Let Y be a Banach space, P a cone in Y, and Ω_1 and Ω_2 bounded open balls in Y centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that $\|Tx\| \ge \|x\|$ for all $x \in P \cap \partial\Omega_1$ and $\|Tx\| \le \|x\|$ for all $x \in P \cap \partial\Omega_2$. Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

Theorem 4. *For each natural number n, the system* (15), (2), *and* (3) *has a solution $(u_n, v_n) \in P$ such that $u_n \ge M_1 t^{\alpha-1}$ and $v_n(t) \ge M_2 t^{\beta-1}$ for all $t \in [0,1]$.*

Proof. Let $(x, y) \in P$ and $n \ge 1$ be given. Then, $T_n^1(x,y)(t) \ge m_1 \int_0^1 G_\alpha(t,s)ds \ge M_1 t^{\alpha-1}$ and $T_n^2(x,y)(t) \ge m_2 \int_0^1 G_\beta(t,s)ds \ge M_2 t^{\beta-1}$ for all $t \in [0,1]$. Hence, $\|T_n^1(x,y)\| \ge M_1$ and $\|T_n^2(x,y)\| \ge M_2$ and so $\|T_n^1(x,y)\|_* \ge M_1$ and $\|T_n^2(x,y)\|_* \ge M_2$. Thus, $\|T_n(x,y)\|_{**} \ge \max\{M_1, M_2\} = M$. Now, put

$$\Omega_1 = \left\{(x,y) \in X\times X : \|(x,y)\|_{**} < M\right\}. \tag{30}$$

Then, $\|T_n(x,y)\|_{**} \ge \|(x,y)\|_{**}$ for all $(x,y) \in P \cap \partial\Omega_1$. Let $W_n^j = P_j(1/n, 1/n, \ldots, 1/n)$ for $j = 1, 2$. Then, we have

$$0 \le \left(T_n^1(x,y)\right)''(t)$$
$$\le \frac{1}{\Gamma(\alpha-2)} \times \int_0^1 \Big(W_n^1 + k_1(s)\, h_1\big(1 + x(s), 1 + y(s), 1 + (\phi_1 x)(s), 1 + (\psi_1 y)(s), 1 + D_{0^+}^{p} x(s), 1 + D_{0^+}^{\mu_1} y(s), 1 + D_{0^+}^{\mu_2} y(s), \ldots, 1 + D_{0^+}^{\mu_m} y(s)\big)\Big)\, ds$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha-2)} \\
&\quad\times\Big(W_n^1 + h_1\Big(1+\|x\|, 1+\|y\|, \\
&\qquad 1+\|\phi_1 x\|, 1+\|\psi_1 y\|, \\
&\qquad 1+\left\|D_{0^+}^{p} x\right\|, 1+\left\|D_{0^+}^{\mu_1} y\right\|, \\
&\qquad 1+\left\|D_{0^+}^{\mu_2} y\right\|, \ldots, \\
&\qquad 1+\left\|D_{0^+}^{\mu_m} y\right\|\Big)\|k_1\|_1\Big), \\
0 &\leq \left(T_n^1(x,y)\right)'(t) = \int_0^t \left(T_n^1(x,y)\right)''(s)\,ds \\
&\leq \frac{1}{\Gamma(\alpha-2)} \\
&\quad\times\Big(W_n^1 + h_1\Big(1+\|x\|, \\
&\qquad 1+\|y\|, 1+\|\phi_1 x\|, 1+\|\psi_1 y\|, \\
&\qquad 1+\left\|D_{0^+}^{p} x\right\|, 1+\left\|D_{0^+}^{\mu_1} y\right\|, \\
&\qquad 1+\left\|D_{0^+}^{\mu_2} y\right\|, \ldots, 1+\left\|D_{0^+}^{\mu_m} y\right\|\Big)\|k_1\|_1\Big), \\
0 &\leq T_n^1(x,y)(t) = \int_0^t \left(T_n^1(x,y)\right)'(s)\,ds \\
&\leq \frac{1}{\Gamma(\alpha-2)}\Big(W_n^1 + h_1\Big(1+\|x\|, 1+\|y\|, \\
&\qquad 1+\|\phi_1 x\|, 1+\|\psi_1 y\|, \\
&\qquad 1+\left\|D_{0^+}^{p} x\right\|, 1+\left\|D_{0^+}^{\mu_1} y\right\|, \\
&\qquad 1+\left\|D_{0^+}^{\mu_2} y\right\|, \ldots, 1+\left\|D_{0^+}^{\mu_m} y\right\|\Big)\|k_1\|_1\Big)
\end{aligned} \tag{31}$$

for all $(x, y) \in P$ and $t \in [0, 1]$. Hence,

$$\begin{aligned}
&\left\|T_n^1(x,y)\right\|_* \\
&\leq \frac{1}{\Gamma(\alpha-2)}\Bigg(W_n^1 + h_1\Bigg(1+\|x\|_*, \\
&\qquad 1+\|y\|_*, 1+\gamma_0^1\|x\|_*, \\
&\qquad 1+\lambda_0^1\|y\|_*, 1+\frac{\|x\|_*}{\Gamma(3-p)}, \\
&\qquad 1+\frac{\|y\|_*}{\Gamma(3-\mu_1)}, 1+\frac{\|y\|_*}{\Gamma(3-\mu_2)}, \ldots, \\
&\qquad 1+\frac{\|y\|_*}{\Gamma(3-\mu_m)}\Bigg)\|k_1\|_1\Bigg) \\
&\leq \frac{1}{\Gamma(\alpha-2)}\Bigg(W_n^1 + h_1\Bigg(1+\|(x,y)\|_{**}, \\
&\qquad 1+\|(x,y)\|_{**}, 1+\gamma_0^1\|(x,y)\|_{**}, \\
&\qquad 1+\lambda_0^1\|(x,y)\|_{**}, 1+\frac{\|(x,y)\|_{**}}{\Gamma(3-p)}, \\
&\qquad 1+\frac{\|(x,y)\|_{**}}{\Gamma(3-\mu_1)}, \\
&\qquad 1+\frac{\|(x,y)\|_{**}}{\Gamma(3-\mu_2)}, \ldots, \\
&\qquad 1+\frac{\|(x,y)\|_{**}}{\Gamma(3-\mu_m)}\Bigg)\|k_1\|_1\Bigg).
\end{aligned} \tag{32}$$

Similarly, we have

$$\begin{aligned}
&\left\|T_n^2(x,y)\right\|_* \\
&\leq \frac{1}{\Gamma(\beta-2)}\Bigg(W_n^2 + h_2\Bigg(1+\|(x,y)\|_{**}, \\
&\qquad 1+\|(x,y)\|_{**}, 1+\gamma_0^2\|(x,y)\|_{**}, \\
&\qquad 1+\lambda_0^2\|(x,y)\|_{**}, 1+\frac{\|(x,y)\|_{**}}{\Gamma(3-q)}, \\
&\qquad 1+\frac{\|(x,y)\|_{**}}{\Gamma(3-\nu_1)}, 1+\frac{\|(x,y)\|_{**}}{\Gamma(3-\nu_2)}, \ldots, \\
&\qquad 1+\frac{\|(x,y)\|_{**}}{\Gamma(3-\nu_m)}\Bigg)\|k_2\|_1\Bigg)
\end{aligned} \tag{33}$$

for all $(x, y) \in P$. Since $\lim_{x\to\infty} h_j(x, x, \ldots, x)/x = 0$ for $j = 1, 2$, there exists $S > M > 0$ such that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha-2)}\Bigg(W_n^1 + h_1\Bigg(1+S, \\
&\qquad 1+S, 1+\gamma_0^1 S, 1+\lambda_0^1 S, \\
&\qquad 1+\frac{S}{\Gamma(3-p)}, 1+\frac{S}{\Gamma(3-\mu_1)}, \\
&\qquad 1+\frac{S}{\Gamma(3-\mu_2)}, \ldots, \\
&\qquad 1+\frac{S}{\Gamma(3-\mu_m)}\Bigg)\|k_1\|_1\Bigg) \leq S,
\end{aligned}$$

$$\frac{1}{\Gamma(\beta-2)}\Bigg(W_n^2+h_2\Bigg(1+S,\; 1+S, 1+\gamma_0^2 S, 1+\lambda_0^2 S,\; 1+\frac{S}{\Gamma(3-q)}, 1+\frac{S}{\Gamma(3-\nu_1)},\; 1+\frac{S}{\Gamma(3-\nu_2)},\dots,\; 1+\frac{S}{\Gamma(3-\nu_m)}\Bigg)\|k_2\|_1\Bigg)\leq S. \tag{34}$$

Put $\Omega_2 = \{(x,y) \in X\times X \text{ s.t } \|(x,y)\|_{**} < S\}$. Then, the above inequalities imply that $\|T_n(x,y)\|_{**} \leq \|(x,y)\|_{**}$ for all $(x,y)\in P\cap\partial\Omega_2$. Now by using Lemma 3, we conclude that the operator T_n has a fixed point (u_n, v_n) in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$. It is easy to see that (u_n,v_n) is a desired solution of the system, that is, $u_n \geq M_1 t^{\alpha-1}$ and $v_n(t)\geq M_2 t^{\beta-1}$ for all $t\in[0,1]$. □

We need the following lemma.

Lemma 5. *The set of solutions $\{(u_n,v_n)\}_{n\geq 1}$ of the system* (15), (2), *and* (3) *is a relatively compact subset of $X\times X$.*

Proof. It is easy to check that

$$\begin{aligned}
u_n(t) &= \int_0^1 G_\alpha(t,s)\, f_n^1\Big(s, u_n(s), v_n(s), (\phi_1 u_n)(s), (\psi_1 v_n)(s), D_{0^+}^{p} u_n(s), D_{0^+}^{\mu_1} v_n(s), D_{0^+}^{\mu_2} v_n(s),\dots, D_{0^+}^{\mu_m} v_n(s)\Big)\,ds,\\
v_n(t) &= \int_0^1 G_\beta(t,s)\, f_n^2\Big(s, u_n(s), v_n(s), (\phi_2 u_n)(s), (\psi_2 v_n)(s), D_{0^+}^{q} v_n(s), D_{0^+}^{\nu_1} u_n(s), D_{0^+}^{\nu_2} u_n(s),\dots, D_{0^+}^{\nu_m} u_n(s)\Big)\,ds
\end{aligned} \tag{35}$$

for $t\in[0,1]$ and $n\geq 1$, satisfy in Theorem 4. Also, $u_n'(t)\geq m_1\int_0^1(\partial/\partial t)G_\alpha(t,s)ds \geq m_1 t^{\alpha-2}/\Gamma(\alpha)(\alpha-\delta_1)$, $u_n''(t)\geq m_1\int_0^1(\partial^2/\partial t^2)G_\alpha(t,s)ds\geq m_1(\delta_1-2)t^{\alpha-3}/\Gamma(\alpha-1)(\alpha-\delta_1)$, $v_n'(t)\geq m_2\int_0^1(\partial/\partial t)G_\beta(t,s)ds\geq m_2 t^{\beta-2}/\Gamma(\beta)(\beta-\delta_2)$, and $v_n''(t)\geq m_2\int_0^1(\partial^2/\partial t^2)G_\beta(t,s)ds\geq m_2(\delta_2-2)t^{\beta-3}/\Gamma(\beta-1)(\beta-\delta_2)$ for $t\in[0,1]$ and $n\geq 1$. Moreover,

$$\begin{aligned}
D_{0^+}^{p} u_n(t) &= \frac{1}{\Gamma(2-p)}\int_0^t (t-s)^{1-p} u_n''(s)\,ds \geq \frac{m_1(\delta_1-2)}{\Gamma(2-p)\Gamma(\alpha-1)(\alpha-\delta_1)} \times\int_0^t (t-s)^{1-p}s^{\alpha-3}ds,\\
D_{0^+}^{q} v_n(t) &= \frac{1}{\Gamma(2-q)}\int_0^t (t-s)^{1-q} v_n''(s)\,ds \geq \frac{m_2(\delta_2-2)}{\Gamma(2-q)\Gamma(\beta-1)(\beta-\delta_2)} \times\int_0^t (t-s)^{1-q}s^{\beta-3}ds,\\
D_{0^+}^{\nu_i} u_n(t) &= \frac{1}{\Gamma(2-\nu_i)}\int_0^t (t-s)^{1-\nu_i} u_n''(s)\,ds \geq \frac{m_1(\delta_1-2)}{\Gamma(2-\nu_i)\Gamma(\alpha-1)(\alpha-\delta_1)} \times\int_0^t (t-s)^{1-\nu_i}s^{\alpha-3}ds,\\
D_{0^+}^{\mu_i} v_n(t) &= \frac{1}{\Gamma(2-\mu_i)}\int_0^t (t-s)^{1-\mu_i} v_n''(s)\,ds \geq \frac{m_2(\delta_2-2)}{\Gamma(2-\mu_i)\Gamma(\beta-1)(\beta-\delta_2)} \times\int_0^t (t-s)^{1-\mu_i}s^{\beta-3}ds
\end{aligned} \tag{36}$$

for $i=1,2,\dots,m$. Since

$$\begin{aligned}
\int_0^t (t-s)^{1-p}s^{\alpha-3}ds &= t^{\alpha-p-1}\times\int_0^1(1-\xi)^{1-p}\xi^{\alpha-3}d\xi\\
&= \frac{\Gamma(2-p)\Gamma(\alpha-2)}{\Gamma(\alpha-p)}t^{\alpha-p-1}\\
&\geq \frac{\Gamma(2-p)\Gamma(\alpha-2)}{\Gamma(\alpha-p)}t^{\alpha-p},
\end{aligned}$$

$$\int_0^t (t-s)^{1-q} s^{\beta-3} ds \geq \frac{\Gamma(2-q)\Gamma(\beta-2)}{\Gamma(\beta-q)} t^{\beta-q},$$
$$\int_0^t (t-s)^{1-\nu_i} s^{\alpha-3} ds \geq \frac{\Gamma(2-\nu_i)\Gamma(\alpha-2)}{\Gamma(\alpha-\nu_i)} t^{\alpha-\nu_i},$$
$$\int_0^t (t-s)^{1-\mu_i} s^{\beta-3} ds \geq \frac{\Gamma(2-\mu_i)\Gamma(\beta-2)}{\Gamma(\beta-\mu_i)} t^{\beta-\mu_i} \tag{37}$$

for $i = 1, 2, \ldots, m$, we get $D^p_{0^+} u_n(t) \geq (m_1(\delta_1 - 2)/(\alpha-2)(\alpha-\delta_1)\Gamma(\alpha-p))t^{\alpha-p}$, $D^q_{0^+} v_n(t) \geq (m_2(\delta_2-2)/(\beta-2)(\beta-\delta_2)\Gamma(\beta-q))t^{\beta-q}$, $D^{\nu_i}_{0^+} u_n(t) \geq (m_1(\delta_1-2)/(\alpha-2)(\alpha-\delta_1)\Gamma(\alpha-\nu_i))t^{\alpha-\nu_i}$, and $D^{\mu_i}_{0^+} v_n(t) \geq (m_2(\delta_2-2)/(\beta-2)(\beta-\delta_2)\Gamma(\beta-\mu_i))t^{\beta-\mu_i}$ for all $t \in [0,1]$ and $n \geq 1$. Since

$$m_1 \cdot \min\left\{\frac{1}{(\alpha-\delta_1)\Gamma(\alpha+1)}, \frac{1}{(\alpha-2)(\alpha-\delta_1)}\right\} = M_1 \tag{38}$$

and $m_2 \cdot \min\{1/(\beta-\delta_2)\Gamma(\beta+1), 1/(\beta-2)(\beta-\delta_2)\} = M_2$, for each $t \in [0,1]$ and $n \geq 1$, we get $u_n(t) \geq M_1 t^{\alpha-1}$, $v_n(t) \geq M_2 t^{\beta-1}$, $D^p_{0^+} u_n(t) \geq ((\delta_1-2)M_1/\Gamma(\alpha-p))t^{\alpha-p}$, $D^q_{0^+} v_n(t) \geq ((\delta_2-2)M_2/\Gamma(\beta-q))t^{\beta-q}$, $D^{\nu_i}_{0^+} u_n(t) \geq ((\delta_1-2)M_1/\Gamma(\alpha-\nu_i))t^{\alpha-\nu_i}$, and $D^{\mu_i}_{0^+} v_n(t) \geq ((\delta_2-2)M_2/\Gamma(\beta-\mu_i))t^{\beta-\mu_i}$ for $i = 1, 2, \ldots, m$ and $(\phi_j u_n)(t) \geq M_1 \int_0^t \gamma_j(t,s) s^{\alpha-1} ds$ and $(\psi_j v_n)(t) \geq M_2 \int_0^t \lambda_j(t,s) s^{\beta-1} ds$ for $j = 1, 2$. Thus,

$$\begin{aligned}
&P_1\left(u_n(t), v_n(t), (\phi_1 u_n)(t), (\psi_1 v_n)(t),\right.\\
&\quad\left. D^p_{0^+} u_n(t), D^{\mu_1}_{0^+} v_n(t), D^{\mu_2}_{0^+} v_n(t), \ldots, D^{\mu_m}_{0^+} v_n(t)\right)\\
&\leq P_1\left(M_1 t^{\alpha-1}, M_2 t^{\beta-1}, M_1 \int_0^t \gamma_1(t,s) s^{\alpha-1} ds,\right.\\
&\qquad M_2 \int_0^t \lambda_1(t,s) s^{\beta-1} ds, \frac{(\delta_1-2)M_1}{\Gamma(\alpha-p)} t^{\alpha-p},\\
&\qquad \frac{(\delta_2-2)M_2}{\Gamma(\beta-\mu_1)} t^{\beta-\mu_1}, \frac{(\delta_2-2)M_2}{\Gamma(\beta-\mu_2)} t^{\beta-\mu_2}, \ldots,\\
&\qquad \left.\frac{(\delta_2-2)M_2}{\Gamma(\beta-\mu_m)} t^{\beta-\mu_m}\right),
\end{aligned}$$

$$\begin{aligned}
&P_2\left(u_n(t), v_n(t), (\phi_2 u_n)(t), (\psi_2 v_n)(t), D^q_{0^+} v_n(t),\right.\\
&\quad\left. D^{\nu_1}_{0^+} u_n(t), D^{\nu_2}_{0^+} u_n(t), \ldots, D^{\nu_m}_{0^+} u_n(t)\right)\\
&\leq P_2\left(M_1 t^{\alpha-1}, M_2 t^{\beta-1}, M_1 \int_0^t \gamma_2(t,s) s^{\alpha-1} ds,\right.\\
&\qquad M_2 \int_0^t \lambda_2(t,s) s^{\beta-1} ds, \frac{(\delta_2-2)M_2}{\Gamma(\beta-q)} t^{\beta-q},\\
&\qquad \frac{(\delta_1-2)M_1}{\Gamma(\alpha-\nu_1)} t^{\alpha-\nu_1}, \frac{(\delta_1-2)M_1}{\Gamma(\alpha-\nu_2)} t^{\alpha-\nu_2}, \ldots,\\
&\qquad \left.\frac{(\delta_1-2)M_1}{\Gamma(\alpha-\nu_m)} t^{\alpha-\nu_m}\right)
\end{aligned} \tag{39}$$

for all $t \in [0,1]$ and $n \geq 1$ and so

$$\begin{aligned}
0 &\leq u''_n(t)\\
&= \int_0^1 \frac{\partial^2}{\partial t^2} G_\alpha(t,s) f^1_n\left(s, u_n(s), v_n(s),\right.\\
&\qquad (\phi_1 u_n)(s), (\psi_1 v_n)(s),\\
&\qquad D^p_{0^+} u_n(s), D^{\mu_1}_{0^+} v_n(s),\\
&\qquad \left. D^{\mu_2}_{0^+} v_n(s), \ldots, D^{\mu_m}_{0^+} v_n(s)\right) ds\\
&\leq \frac{1}{\Gamma(\alpha-2)} \left[\left[\int_0^1 P_1\left(M_1 s^{\alpha-1}, M_2 s^{\beta-1},\right.\right.\right.\\
&\qquad M_1 \int_0^s \gamma_1(s,\tau) \tau^{\alpha-1} d\tau,\\
&\qquad M_2 \int_0^s \lambda_1(s,\tau) \tau^{\beta-1} d\tau,\\
&\qquad \frac{(\delta_1-2)M_1}{\Gamma(\alpha-p)} s^{\alpha-p},\\
&\qquad \frac{(\delta_2-2)M_2}{\Gamma(\beta-\mu_1)} s^{\beta-\mu_1},\\
&\qquad \frac{(\delta_2-2)M_2}{\Gamma(\beta-\mu_2)} s^{\beta-\mu_2}, \ldots,\\
&\qquad \left.\frac{(\delta_2-2)M_2}{\Gamma(\beta-\mu_m)} s^{\beta-\mu_m}\right) ds\\
&\quad + h_1\left(1 + \|u_n\|_*, 1 + \|v_n\|_*,\right.\\
&\qquad 1 + \gamma^1_0 \|u_n\|_*, 1 + \lambda^1_0 \|v_n\|_*,
\end{aligned}$$

$$1+\frac{\|u_n\|_*}{\Gamma(3-p)},1+\frac{\|v_n\|_*}{\Gamma(3-\mu_1)},$$
$$1+\frac{\|v_n\|_*}{\Gamma(3-\mu_2)},\dots,1+\frac{\|v_n\|_*}{\Gamma(3-\mu_m)}\Bigg)$$
$$\times\int_0^1 k_1(s)\,ds\Bigg]$$
$$=\frac{1}{\Gamma(\alpha-2)}$$
$$\times\Bigg(\Lambda_1+h_1\Bigg(1+\|u_n\|_*,1+\|v_n\|_*,$$
$$1+\gamma_0^1\|u_n\|_*,1+\lambda_0^1\|v_n\|_*,$$
$$1+\frac{\|u_n\|_*}{\Gamma(3-p)},1+\frac{\|v_n\|_*}{\Gamma(3-\mu_1)},$$
$$1+\frac{\|v_n\|_*}{\Gamma(3-\mu_2)},\dots,$$
$$1+\frac{\|v_n\|_*}{\Gamma(3-\mu_m)}\Bigg)\|k_1\|_1\Bigg),$$
$$0\le u_n'(t)=\int_0^t u_n''(s)\,ds$$
$$\le\frac{1}{\Gamma(\alpha-2)}$$
$$\times\Bigg(\Lambda_1+h_1\Bigg(1+\|u_n\|_*,$$
$$1+\|v_n\|_*,1+\gamma_0^1\|u_n\|_*,$$
$$1+\lambda_0^1\|v_n\|_*,$$
$$1+\frac{\|u_n\|_*}{\Gamma(3-p)},1+\frac{\|v_n\|_*}{\Gamma(3-\mu_1)},$$
$$1+\frac{\|v_n\|_*}{\Gamma(3-\mu_2)},\dots,$$
$$1+\frac{\|v_n\|_*}{\Gamma(3-\mu_m)}\Bigg)\|k_1\|_1\Bigg),$$
$$0\le u_n(t)=\int_0^t u_n'(s)\,ds$$
$$\le\frac{1}{\Gamma(\alpha-2)}$$
$$\times\Bigg(\Lambda_1+h_1\Bigg(1+\|u_n\|_*,$$
$$1+\|v_n\|_*,1+\gamma_0^1\|u_n\|_*,$$
$$1+\lambda_0^1\|v_n\|_*,$$
$$1+\frac{\|u_n\|_*}{\Gamma(3-p)},1+\frac{\|v_n\|_*}{\Gamma(3-\mu_1)},$$
$$1+\frac{\|v_n\|_*}{\Gamma(3-\mu_2)},\dots,$$
$$1+\frac{\|v_n\|_*}{\Gamma(3-\mu_m)}\Bigg)\|k_1\|_1\Bigg) \tag{40}$$

for all $t\in[0,1]$ and $n\ge 1$ and also $\Lambda_1<\infty$, where

$$\Lambda_1=\int_0^1 P_1\Bigg(M_1 s^{\alpha-1},M_2 s^{\beta-1},$$
$$M_1\int_0^s \gamma_1(s,\tau)\,\tau^{\alpha-1}d\tau,$$
$$M_2\int_0^s \lambda_1(s,\tau)\,\tau^{\beta-1}d\tau,$$
$$\frac{(\delta_1-2)M_1}{\Gamma(\alpha-p)}s^{\alpha-p},\frac{(\delta_2-2)M_2}{\Gamma(\beta-\mu_1)}s^{\beta-\mu_1},$$
$$\frac{(\delta_2-2)M_2}{\Gamma(\beta-\mu_2)}s^{\beta-\mu_2},\dots,$$
$$\frac{(\delta_2-2)M_2}{\Gamma(\beta-\mu_m)}s^{\beta-\mu_m}\Bigg)ds. \tag{41}$$

Hence,

$$\|u_n\|_*\le\frac{1}{\Gamma(\alpha-2)}\Bigg(\Lambda_1+h_1\Bigg(1+\|u_n\|_*,$$
$$1+\|v_n\|_*,1+\gamma_0^1\|u_n\|_*,$$
$$1+\lambda_0^1\|v_n\|_*,$$
$$1+\frac{\|u_n\|_*}{\Gamma(3-p)},1+\frac{\|v_n\|_*}{\Gamma(3-\mu_1)},$$
$$1+\frac{\|v_n\|_*}{\Gamma(3-\mu_2)},\dots,$$
$$1+\frac{\|v_n\|_*}{\Gamma(3-\mu_m)}\Bigg)\|k_1\|_1\Bigg)$$

$$\leq \frac{1}{\Gamma(\alpha-2)}\left(\Lambda_1 + h_1\left(1+\|(u_n,v_n)\|_{**},\right.\right.$$
$$1+\|(u_n,v_n)\|_{**}, 1+\gamma_0^1\|(u_n,v_n)\|_{**},$$
$$1+\lambda_0^1\|(u_n,v_n)\|_{**},$$
$$1+\frac{\|(u_n,v_n)\|_{**}}{\Gamma(3-p)}, 1+\frac{\|(u_n,v_n)\|_{**}}{\Gamma(3-\mu_1)},$$
$$1+\frac{\|(u_n,v_n)\|_{**}}{\Gamma(3-\mu_2)},\ldots,$$
$$\left.\left.1+\frac{\|(u_n,v_n)\|_{**}}{\Gamma(3-\mu_m)}\right)\|k_1\|_1\right) = G_n^1 \tag{42}$$

for all $n \geq 1$. Similarly, we have

$$\|v_n\|_* \leq \frac{1}{\Gamma(\beta-2)}$$
$$\times\left(\Lambda_2 + h_2\left(1+\|(u_n,v_n)\|_{**}, 1+\|(u_n,v_n)\|_{**},\right.\right.$$
$$1+\gamma_0^2\|(u_n,v_n)\|_{**},$$
$$1+\lambda_0^2\|(u_n,v_n)\|_{**},$$
$$1+\frac{\|(u_n,v_n)\|_{**}}{\Gamma(3-q)}, 1+\frac{\|(u_n,v_n)\|_{**}}{\Gamma(3-\nu_1)},$$
$$1+\frac{\|(u_n,v_n)\|_{**}}{\Gamma(3-\nu_2)},\ldots,$$
$$\left.\left.1+\frac{\|(u_n,v_n)\|_{**}}{\Gamma(3-\nu_m)}\right)\|k_2\|_1\right) = G_n^2 \tag{43}$$

for all $n \geq 1$, where

$$\Lambda_2 = \int_0^1 P_2\left(M_1 s^{\alpha-1}, M_2 s^{\beta-1},\right.$$
$$M_1\int_0^s \gamma_2(s,\tau)\tau^{\alpha-1}d\tau,$$
$$M_2\int_0^s \lambda_2(s,\tau)\tau^{\beta-1}d\tau,$$
$$\frac{(\delta_2-2)M_2}{\Gamma(\beta-q)}s^{\beta-q},$$
$$\frac{(\delta_1-2)M_1}{\Gamma(\alpha-\nu_1)}s^{\alpha-\nu_1},$$
$$\frac{(\delta_1-2)M_1}{\Gamma(\alpha-\nu_2)}s^{\alpha-\nu_2},\ldots,$$
$$\left.\frac{(\delta_1-2)M_1}{\Gamma(\alpha-\nu_m)}s^{\alpha-\nu_m}\right)ds. \tag{44}$$

Thus, $\|(u_n,v_n)\|_{**} \leq \max\{G_n^1, G_n^2\}$ for all $n \geq 1$. Since $\lim_{x\to\infty} h_j(x,x,\ldots,x)/x = 0$ for $j=1,2$, there exists $L>0$ such that

$$\frac{1}{\Gamma(\alpha-2)}\left(\Lambda_1 + h_1\left(1+\upsilon, 1+\upsilon, 1+\gamma_0^1\upsilon,\right.\right.$$
$$1+\lambda_0^1\upsilon, 1+\frac{\upsilon}{\Gamma(3-p)},$$
$$1+\frac{\upsilon}{\Gamma(3-\mu_1)},$$
$$1+\frac{\upsilon}{\Gamma(3-\mu_2)},\ldots,$$
$$\left.\left.1+\frac{\upsilon}{\Gamma(3-\mu_m)}\right)\|k_1\|_1\right) < \upsilon,$$
$$\frac{1}{\Gamma(\beta-2)}\left(\Lambda_2 + h_2\left(1+\upsilon, 1+\upsilon, 1+\gamma_0^2\upsilon,\right.\right.$$
$$1+\lambda_0^2\upsilon, 1+\frac{\upsilon}{\Gamma(3-q)},$$
$$1+\frac{\upsilon}{\Gamma(3-\nu_1)},$$
$$1+\frac{\upsilon}{\Gamma(3-\nu_2)},\ldots,$$
$$\left.\left.1+\frac{\upsilon}{\Gamma(3-\nu_m)}\right)\|k_2\|_1\right) < \upsilon \tag{45}$$

for all $\upsilon \geq L$. Consequently, $\|(u_n,v_n)\|_{**} < L$ for all $n \geq 1$ and so $\{(u_n,v_n)\}_{n\geq1}$ is a bounded sequence in $X\times X$. It remains to prove that $\{(u_n,v_n)''\}$ is equicontinuous on $[0,1]$. Put

$$V_1 = h_1\left(1+L, 1+L, 1+\gamma_0^1 L,\right.$$
$$1+\lambda_0^1 L, 1+\frac{L}{\Gamma(3-p)}, 1+\frac{L}{\Gamma(3-\mu_1)},$$
$$\left.1+\frac{L}{\Gamma(3-\mu_2)},\ldots, 1+\frac{L}{\Gamma(3-\mu_m)}\right),$$

$$V_2 = h_2\Bigg(1+L, 1+L, 1+\gamma_0^2 L, 1+\lambda_0^2 L, 1+\frac{L}{\Gamma(3-q)}, 1+\frac{L}{\Gamma(3-\nu_1)}, 1+\frac{L}{\Gamma(3-\nu_2)}, \dots, 1+\frac{L}{\Gamma(3-\nu_m)}\Bigg),$$

$$\Phi_1(t) = P_1\Bigg(M_1 t^{\alpha-1}, M_2 t^{\beta-1}, M_1\int_0^t \gamma_1(t,s) s^{\alpha-1} ds, M_2\int_0^t \lambda_1(t,s) s^{\beta-1} ds \frac{(\delta_1-2)M_1}{\Gamma(\alpha-p)} t^{\alpha-p}, \frac{(\delta_2-2)M_2}{\Gamma(\beta-\mu_1)} t^{\beta-\mu_1}, \frac{(\delta_2-2)M_2}{\Gamma(\beta-\mu_2)} t^{\beta-\mu_2}, \dots, \frac{(\delta_2-2)M_2}{\Gamma(\beta-\mu_m)} t^{\beta-\mu_m}\Bigg),$$

$$\Phi_2(t) = P_2\Bigg(M_1 t^{\alpha-1}, M_2 t^{\beta-1}, M_1\int_0^t \gamma_2(t,s) s^{\alpha-1} ds, M_2\int_0^t \lambda_2(t,s) s^{\beta-1} ds, \frac{(\delta_2-2)M_2}{\Gamma(\beta-q)} t^{\beta-q}, \frac{(\delta_1-2)M_1}{\Gamma(\alpha-\nu_1)} t^{\alpha-\nu_1}, \frac{(\delta_1-2)M_1}{\Gamma(\alpha-\nu_2)} t^{\alpha-\nu_2}, \dots, \frac{(\delta_1-2)M_1}{\Gamma(\alpha-\nu_m)} t^{\alpha-\nu_m}\Bigg) \tag{46}$$

for all $t \in (0,1]$. Note that, $\Lambda_j = \int_0^1 \Phi_j(t)dt$ for $j = 1, 2$ and

$$\begin{aligned} &f_n^1\big(t, u_n(t), v_n(t), (\phi_1 u_n)(t), (\psi_1 v_n)(t), D_{0^+}^{p} u_n(t), D_{0^+}^{\mu_1} v_n(t), D_{0^+}^{\mu_2} v_n(t), \dots, D_{0^+}^{\mu_m} v_n(t)\big) \\ &\quad \le \Phi_1(t) + V_1 k_1(t), \\ &f_n^2\big(t, u_n(t), v_n(t), (\phi_2 u_n)(t), (\psi_2 v_n)(t), D_{0^+}^{q} v_n(t), D_{0^+}^{\nu_1} u_n(t), D_{0^+}^{\nu_2} u_n(t), \dots, D_{0^+}^{\nu_m} u_n(t)\big) \\ &\quad \le \Phi_2(t) + V_2 k_2(t) \end{aligned} \tag{47}$$

for almost all $t \in [0,1]$ and $n \ge 1$. If $0 \le t_1 < t_2 \le 1$, then

$$\begin{aligned} &\left|u_n''(t_2) - u_n''(t_1)\right| \\ &= \Bigg|\int_0^1 \left(\frac{\partial^2}{\partial t^2} G_\alpha(t_2,s) - \frac{\partial^2}{\partial t^2} G_\alpha(t_1,s)\right) f_n^1\big(s, u_n(s), v_n(s), (\phi_1 u_n)(s), (\psi_1 v_n)(s), D_{0^+}^{p} u_n(s), D_{0^+}^{\mu_1} v_n(s), D_{0^+}^{\mu_2} v_n(s), \dots, D_{0^+}^{\mu_m} v_n(s)\big) ds\Bigg| \\ &\le \frac{1}{\Gamma(\alpha-2)}\Bigg[\left(t_2^{\alpha-3} - t_1^{\alpha-3}\right)\int_0^1 (\Phi_1(s) + V_1 k_1(s)) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-3} (\Phi_1(s) + V_1 k_1(s)) ds \\ &\quad + \int_0^{t_1} \left((t_2-s)^{\alpha-3} - (t_1-s)^{\alpha-3}\right) \times (\Phi_1(s) + V_1 k_1(s)) ds\Bigg] \\ &\le \frac{1}{\Gamma(\alpha-2)}\Big[\left(t_2^{\alpha-3} - t_1^{\alpha-3}\right)(\Lambda_1 + V_1\|k_1\|_1) + (t_2-t_1)^{\alpha-3}(\Lambda_1 + V_1\|k_1\|_1) \\ &\quad + \int_0^{t_1}\left((t_2-s)^{\alpha-3} - (t_1-s)^{\alpha-3}\right) \times (\Phi_1(s) + V_1 k_1(s)) ds\Big]. \end{aligned} \tag{48}$$

Similarly, we have

$$\begin{aligned} &\left|v_n''(t_2) - v_n''(t_1)\right| \\ &\le \frac{1}{\Gamma(\beta-2)}\Big[\left(t_2^{\beta-3} - t_1^{\beta-3}\right)(\Lambda_2 + V_2\|k_2\|_1) + (t_2-t_1)^{\beta-3}(\Lambda_2 + V_2\|k_2\|_1) \\ &\quad + \int_0^{t_1}\left((t_2-s)^{\beta-3} - (t_1-s)^{\beta-3}\right) \times (\Phi_2(s) + V_2 k_2(s)) ds\Big]. \end{aligned} \tag{49}$$

Let $\epsilon > 0$ be given. Choose $\delta_0 > 0$ such that $0 < t_2^{\alpha-3} - t_1^{\alpha-3} < \epsilon$, $0 < t_2^{\beta-3} - t_1^{\beta-3} < \epsilon$,

$$0 < (t_2-s)^{\alpha-3} - (t_1-s)^{\alpha-3} < \epsilon \tag{50}$$

and $0 < (t_2-s)^{\beta-3} - (t_1-s)^{\beta-3} < \epsilon$ for all $0 \le t_1 < t_2 \le 1$, $t_2 - t_1 < \delta_0$ and $0 \le s \le t_1$. Thus, for each $0 \le t_1 < t_2 \le 1$ and $t_2 - t_1 < \min\{\delta_0, \sqrt[\alpha-3]{\epsilon}, \sqrt[\beta-3]{\epsilon}\}$, we have $|u_n''(t_2) - u_n''(t_1)| \le (3\epsilon/\Gamma(\alpha-2))(\Lambda_1 + V_1\|k_1\|_1)$ and $|v_n''(t_2) - v_n''(t_1)| \le (3\epsilon/\Gamma(\beta-2))(\Lambda_2 + V_2\|k_2\|_1)$. Therefore, $\{(u_n, v_n)''\}_{n\ge1}$ is equicontinuous on $[0,1]$. This completes the proof. □

Now, we give our main result.

Theorem 6. *The system* (1), (2), *and* (3) *has a positive solution* (u, v) *such that* $u(t) \geq M_1 t^{\alpha-1}$, $v(t) \geq M_2 t^{\beta-1}$, $D^p_{0^+}u(t) \geq ((\delta_1-2)M_1/\Gamma(\alpha-p))t^{\alpha-p}$, $D^q_{0^+}v(t) \geq ((\delta_2-2)M_2/\Gamma(\beta-q))t^{\beta-q}$, $D^{\nu_i}_{0^+}u(t) \geq ((\delta_1-2)M_1/\Gamma(\alpha-\nu_i))t^{\alpha-\nu_i}$, *and* $D^{\mu_i}_{0^+}v(t) \geq ((\delta_2-2)M_2/\Gamma(\beta-\mu_i))t^{\beta-\mu_i}$ *for* $i = 1, 2, \ldots, m$ *and* $t \in [0, 1]$ *and* $(\phi_j u)(t) \geq M_1 \int_0^t \gamma_j(t, s)s^{\alpha-1}ds$ *and*

$$\left(\psi_j v\right)(t) \geq M_2 \int_0^t \lambda_j(t, s) s^{\beta-1} ds \tag{51}$$

for $j = 1, 2$ *and* $t \in [0, 1]$.

Proof. By using Theorem 4, for each natural number n, the system (15), (2), and (3) has a solution (u_n, v_n) in P. Also by using Lemma 5, the set $\{(u_n, v_n)\}_{n\geq1}$ is a relatively compact subset of $X\times X$. By using the Arzela-Ascoli theorem, without loss of generality we can assume that $\{(u_n, v_n)\}_{n\geq1}$ is convergent in $X \times X$ to some element (u, v) of P. It is easy to check that (u, v) satisfy the boundary conditions (2) and (3) and also $\lim_{n\to\infty} D^p_{0^+}u_n = D^p_{0^+}u$, $\lim_{n\to\infty} D^q_{0^+}v_n = D^q_{0^+}v$, $\lim_{n\to\infty} D^{\nu_i}_{0^+}u_n = D^{\nu_i}_{0^+}u$, and $\lim_{n\to\infty} D^{\mu_i}_{0^+}v_n = D^{\mu_i}_{0^+}v$ for $i = 1, 2, \ldots, m$ and $\lim_{n\to\infty}\phi_j u_n = \phi_j u$ and $\lim_{n\to\infty}\psi_j v_n = \psi_j v$ for $j = 1, 2$. Thus, it is easy to see that (u, v) satisfy the desired conditions. Also,

$$\begin{aligned}
&\lim_{n\to\infty} f^1_n \left(t, u_n(t), v_n(t), (\phi_1 u_n)(t), \right.\\
&\qquad \left(\psi_1 v_n\right)(t), D^p_{0^+}u_n(t), D^{\mu_1}_{0^+}v_n(t),\\
&\qquad \left. D^{\mu_2}_{0^+}v_n(t), \ldots, D^{\mu_m}_{0^+}v_n(t)\right)\\
&\quad = f_1 \left(t, u(t), v(t), (\phi_1 u)(t), (\psi_1 v)(t),\right.\\
&\qquad D^p_{0^+}u(t), D^{\mu_1}_{0^+}v(t),\\
&\qquad \left. D^{\mu_2}_{0^+}v(t), \ldots, D^{\mu_m}_{0^+}v(t)\right),\\
&\lim_{n\to\infty} f^2_n \left(t, u_n(t), v_n(t), (\phi_2 u_n)(t),\right.\\
&\qquad \left(\psi_2 v_n\right)(t), D^q_{0^+}v_n(t), D^{\nu_1}_{0^+}u_n(t),\\
&\qquad \left. D^{\nu_2}_{0^+}u_n(t), \ldots, D^{\nu_m}_{0^+}u_n(t)\right)\\
&\quad = f_2 \left(t, u(t), v(t), (\phi_2 u)(t),\right.\\
&\qquad \left(\psi_2 v\right)(t), D^q_{0^+}v(t), D^{\nu_1}_{0^+}u(t),\\
&\qquad \left. D^{\nu_2}_{0^+}u(t), \ldots, D^{\nu_m}_{0^+}u(t)\right)
\end{aligned} \tag{52}$$

for almost all $t \in [0, 1]$. If $\xi = \sup\{\|(u_n, v_n)\|_{**} : n \geq 1\}$, then $\|D^p_{0^+}u_n\| \leq \xi/\Gamma(3-p)$, $\|D^q_{0^+}v_n\| \leq \xi/\Gamma(3-q)$, $\|D^{\nu_i}_{0^+}u_n\| \leq \xi/\Gamma(3-\nu_i)$, and $\|D^{\mu_i}_{0^+}v_n\| \leq \xi/\Gamma(3-\mu_i)$ for $i = 1, 2, \ldots, m$ and $n \geq 1$ $\|\phi_j u_n\| \leq \xi\gamma^j_0$ and $\|\psi_j v_n\| \leq \xi\lambda^j_0$ for $j = 1, 2$ and $n \geq 1$. Thus,

$$\begin{aligned}
&0 \leq G_\alpha(t, s) f^1_n \left(s, u_n(s), v_n(s),\right.\\
&\qquad \left(\phi_1 u_n\right)(s), \left(\psi_1 v_n\right)(s),\\
&\qquad D^p_{0^+}u_n(s), D^{\mu_1}_{0^+}v_n(s),\\
&\qquad \left. D^{\mu_2}_{0^+}v_n(s), \ldots, D^{\mu_m}_{0^+}v_n(s)\right)\\
&\leq \frac{1}{\Gamma(\alpha)} \left(\Phi_1(s) + h_1 \left(1 + \xi,\right.\right.\\
&\qquad 1 + \xi, 1 + \gamma^1_0\xi, 1 + \lambda^1_0\xi,\\
&\qquad 1 + \frac{\xi}{\Gamma(3-p)}, 1 + \frac{\xi}{\Gamma(3-\mu_1)},\\
&\qquad 1 + \frac{\xi}{\Gamma(3-\mu_2)}, \ldots,\\
&\qquad \left.\left. 1 + \frac{\xi}{\Gamma(3-\mu_m)}\right) k_1(s)\right),\\
&0 \leq G_\beta(t, s) f^2_n \left(s, u_n(s), v_n(s), (\phi_2 u_n)(s),\right.\\
&\qquad \left(\psi_2 v_n\right)(s), D^q_{0^+}v_n(s), D^{\nu_1}_{0^+}u_n(s),\\
&\qquad \left. D^{\nu_2}_{0^+}u_n(s), \ldots, D^{\nu_m}_{0^+}u_n(s)\right)\\
&\leq \frac{1}{\Gamma(\beta)} \left(\Phi_2(s) + h_2 \left(1 + \xi,\right.\right.\\
&\qquad 1 + \xi, 1 + \gamma^2_0\xi, 1 + \lambda^2_0\xi,\\
&\qquad 1 + \frac{\xi}{\Gamma(3-q)}, 1 + \frac{\xi}{\Gamma(3-\nu_1)},\\
&\qquad 1 + \frac{\xi}{\Gamma(3-\nu_2)}, \ldots,\\
&\qquad \left.\left. 1 + \frac{\xi}{\Gamma(3-\nu_m)}\right) k_2(s)\right)
\end{aligned} \tag{53}$$

for almost all $(t, s) \in [0, 1] \times [0, 1]$ and $n \geq 1$. Hence, by using the Lebesgue dominated convergence theorem we get

$$\begin{aligned}
&u(t)\\
&\quad = \int_0^1 G_\alpha(t, s) f_1 \left(s, u(s), v(s), (\phi_1 u)(s), (\psi_1 v)(s),\right.\\
&\qquad D^p_{0^+}u(s), D^{\mu_1}_{0^+}v(s), D^{\mu_2}_{0^+}v(s), \ldots,\\
&\qquad \left. D^{\mu_m}_{0^+}v(s)\right) ds,
\end{aligned}$$

$$
\begin{aligned}
v(t) &= \int_0^1 G_\beta(t,s) f_2\Big(s, u(s), v(s), (\phi_2 u)(s),\\
&\qquad (\psi_2 v)(s), D_{0^+}^{q} v(s), D_{0^+}^{\nu_1} u(s),\\
&\qquad D_{0^+}^{\nu_2} u(s), \ldots, D_{0^+}^{\nu_m} u(s)\Big)\, ds
\end{aligned} \tag{54}
$$

for all $t \in [0,1]$. Therefore, (u,v) is a positive solution of the system (1), (2), and (3). □

4. Example

Here, we give an example to illustrate our last result.

Example 1. Let $m_1 > 0$, $m_2 > 0$, $n \geq 4$, $n-1 < \alpha, \beta < n$, $0 < p, q < 1$, $1 < \mu_i, \nu_i < 2$ for $i = 1, 2, \ldots, m$, $\rho_1, \rho_2, \sigma_1, \sigma_2 \in L^1[0,1]$, $\sigma_1(t) \geq m_1$ and $\sigma_2(t) \geq m_2$ for almost all $t \in [0,1]$. Suppose that $a_1, c_1 \in (0, 1/(\alpha-1))$, $a_2, c_2 \in (0, 1/(\beta-1))$, $a_5 \in (0, 1/(\alpha-p))$, $c_5 \in (0, 1/(\beta-q))$, $a_i' \in (0, 1/(\beta-\mu_i))$, and $c_i' \in (0, 1/(\alpha-\nu_i))$ for $i = 1, 2, \ldots, m$ and also $b_1, b_2, b_3, b_4, b_5, b_1', b_2', \ldots, b_m', d_1, d_2, d_3, d_4, d_5, d_1', d_2', \ldots, d_m' \in (0, 1)$. Assume that a_3, a_4, c_3 and c_4 are positive real numbers such that $\int_0^1 (\int_0^s \gamma_1(s,\tau)\tau^{\alpha-1} d\tau)^{-a_3} ds < \infty$, $\int_0^1 (\int_0^s \gamma_2(s,\tau)\tau^{\alpha-1} d\tau)^{-c_3} ds < \infty$, $\int_0^1 (\int_0^s \lambda_1(s,\tau)\tau^{\beta-1} d\tau)^{-a_4} ds < \infty$, and $\int_0^1 (\int_0^s \lambda_2(s,\tau)\tau^{\beta-1} d\tau)^{-c_4} ds < \infty$. Define the functions f_1 and f_2 on $[0,1] \times \mathfrak{D}$ by

$$
\begin{aligned}
&f_1(t, x, y, z, w, v, u_1, u_2, \ldots, u_m)\\
&\quad = \frac{1}{x^{a_1}} + \frac{1}{y^{a_2}} + \frac{1}{z^{a_3}} + \frac{1}{w^{a_4}}\\
&\qquad + \frac{1}{v^{a_5}} + \frac{1}{u_1^{a_1'}} + \frac{1}{u_2^{a_2'}} + \cdots + \frac{1}{u_m^{a_m'}}\\
&\qquad + |\rho_1(t)| \Big(x^{b_1} + y^{b_2} + z^{b_3}\\
&\qquad\qquad + w^{b_4} + v^{b_5} + u_1^{b_1'}\\
&\qquad\qquad + u_2^{b_2'} + \cdots + u_m^{b_m'}\Big) + \sigma_1(t),\\
&f_2(t, x, y, z, w, v, u_1, u_2, \ldots, u_m)\\
&\quad = \frac{1}{x^{c_1}} + \frac{1}{y^{c_2}} + \frac{1}{z^{c_3}} + \frac{1}{w^{c_4}}\\
&\qquad + \frac{1}{v^{c_5}} + \frac{1}{u_1^{c_1'}} + \frac{1}{u_2^{c_2'}} + \cdots + \frac{1}{u_m^{c_m'}}\\
&\qquad + |\rho_2(t)| \Big(x^{d_1} + y^{d_2} + z^{d_3}\\
&\qquad\qquad + w^{d_4} + v^{d_5} + u_1^{d_1'}\\
&\qquad\qquad + u_2^{d_2'} + \cdots + u_m^{d_m'}\Big) + \sigma_2(t).
\end{aligned} \tag{55}
$$

Note that the functions f_1 and f_2 satisfy the conditions (H_1) and (H_2), where

$$
\begin{aligned}
&P_1(x, y, z, w, v, u_1, u_2, \ldots, u_m)\\
&\quad = \frac{1}{x^{a_1}} + \frac{1}{y^{a_2}} + \frac{1}{z^{a_3}} + \frac{1}{w^{a_4}}\\
&\qquad + \frac{1}{v^{a_5}} + \frac{1}{u_1^{a_1'}} + \frac{1}{u_2^{a_2'}} + \cdots + \frac{1}{u_m^{a_m'}},\\
&P_2(x, y, z, w, v, u_1, u_2, \ldots, u_m)\\
&\quad = \frac{1}{x^{c_1}} + \frac{1}{y^{c_2}} + \frac{1}{z^{c_3}} + \frac{1}{w^{c_4}}\\
&\qquad + \frac{1}{v^{c_5}} + \frac{1}{u_1^{c_1'}} + \frac{1}{u_2^{c_2'}} + \cdots + \frac{1}{u_m^{c_m'}},\\
&h_1(x, y, z, w, v, u_1, u_2, \ldots, u_m)\\
&\quad = 1 + x^{b_1} + y^{b_2} + z^{b_3} + w^{b_4}\\
&\qquad + v^{b_5} + u_1^{b_1'} + u_2^{b_2'} + \cdots + u_m^{b_m'},\\
&h_2(x, y, z, w, v, u_1, u_2, \ldots, u_m)\\
&\quad = 1 + x^{d_1} + y^{d_2} + z^{d_3} + w^{d_4}\\
&\qquad + v^{d_5} + u_1^{d_1'} + u_2^{d_2'} + \cdots + u_m^{d_m'},
\end{aligned} \tag{56}
$$

$k_1(t) = |\rho_1(t)| + \sigma_1(t)$ and $k_2(t) = |\rho_2(t)| + \sigma_2(t)$. Theorem 6 guarantees that the system (1), (2), and (3) via these functions has a positive solution (u,v) satisfying the desired inequalities in Theorem 6 whenever $M_1 = m_1/(\alpha-\delta_1)\Gamma(\alpha+1)$ and $M_2 = m_2/(\beta-\delta_2)\Gamma(\beta+1)$.

5. Conclusions

One of the most interesting branches is obtaining solutions of singular fractional differential equations via boundary value problems. Having these thought in mind we discuss the existence of positive solutions for a coupled system of multiterm singular fractional integrodifferential boundary value problems. An illustrative example illustrates the applicability of the proposed method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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