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Research Article

Hybrid Bernstein Block-Pulse Functions Method for Second Kind Integral Equations with Convergence Analysis

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We introduce a new combination of Bernstein polynomials (BPs) and Block-Pulse functions (BPFs) on the interval [0, 1]. These functions are suitable for finding an approximate solution of the second kind integral equation. We call this method Hybrid Bernstein Block-Pulse Functions Method (HBBPFM). This method is very simple such that an integral equation is reduced to a system of linear equations. On the other hand, convergence analysis for this method is discussed. The method is computationally very simple and attractive so that numerical examples illustrate the efficiency and accuracy of this method.

1. Introduction

In recent years, many different basic functions have been used for solving integral equations, such as Block-Pulse functions [1, 2], Triangular functions [3], Haar functions [4], Hybrid Legendre and Block-Pulse functions [5], Hybrid Chebyshev and Block-Pulse functions [6, 7], Hybrid Taylor and Block-Pulse functions [8], and Hybrid Fourier and Block-Pulse functions [9].

Block-Pulse functions were introduced in electrical engineering by Harmuth. After that study, several researchers have discussed applications of Block-Pulse functions [10, 11].

Bernstein polynomials have been applied in various fields of mathematics. For example, some researchers applied the Bernstein polynomials for solving high order differential equations [12], some classes of integral equations [13], partial differential equations, and optimal control problems [14]. Also, we introduced new operational matrices of fractional derivative and integral operators by Bernstein polynomials and then used them for solving fractional differential equations [15–17], system of fractional differential equations [18], and fractional optimal control problems [19, 20].

In this work, we combine the Bernstein polynomials (BPs) and Block-Pulse functions (BPFs) on the interval $[0, 1]$. Then, we use these bases for finding an approximate solution of the second kind integral equation. We call this method Hybrid Bernstein Block-Pulse Functions Method (HBBPFM). In this method the integral equation is reduced to a system of linear equations. Also, we discuss the convergence analysis for this method. Furthermore, we compare the accuracy of obtained results of BPFs, BPs, and HBBPFM by some examples.

The rest of this paper is as follows. In Section 2, HBBPFs are introduced; therefore we approximate functions by using HBBPFs and also we discuss best approximation and convergence analysis in Section 3. Then we apply HBBPF method to find an approximate solution for the second kind integral equations and we survey error analysis for proposed method in Section 4. Also, we apply the proposed method on some examples. We observe that the accuracy and efficiency of this method are more than the near methods. Finally, Section 6 concludes our work in this paper.

2. Hybrid of Bernstein and Block-Pulse Functions

In this section, we recall some definitions and properties of Bernstein polynomials and Block-Pulse functions.

Lemma 1 (see [19]). *The Bernstein polynomials (BPs) of mth-degree are defined on the interval* $[0,1]$ *as follows:*

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0,1,\ldots,m, \tag{1}$$

where

$$\binom{m}{i} = \frac{m!}{i!\,(m-i)!}. \tag{2}$$

Then $\{B_{0,m}, B_{1,m}, \ldots, B_{m,m}\}$ *in Hilbert space* $L^2[0,1]$ *is a complete basis. Therefore, any polynomial of degree m can be expanded in terms of linear combination of* $B_{i,m}(x)$ $(i = 0,1,\ldots,m)$.

Lemma 2. *Let a set of Block-Pulse functions (BPFs)* $b_i(t)$, $i = 1,2,\ldots,N$ *be on the interval* $[0,1)$ *such that.*

$$b_i(t) = \begin{cases} 1, & \dfrac{i-1}{N} \le t < \dfrac{i}{N}, \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

Then, the following properties for these functions satisfy the following:

(i) *disjointness,*

(ii) *orthogonality,*

(iii) *completeness.*

Proof. The disjointness property can be clearly obtained from the definition of Block-Pulse functions as follows:

$$b_i(t)\,b_j(t) = \begin{cases} b_i(t), & i = j, \\ 0, & i \ne j, \end{cases} \tag{4}$$

where $i, j = 1,2,\ldots,N$.

The other property is orthogonality. It is clear that

$$\int_0^1 b_i(t)\,b_j(t)\,dt = \frac{1}{N}\delta_{ij}, \tag{5}$$

where $i, j = 1,2,\ldots,N$ and δ_{ij} is the Kroneker delta.

The third property is completeness. For every $f \in L^2([0,1))$, when m approaches the infinity, Parseval's identity holds:

$$\int_0^1 f^2(x)\,dx = \sum_{i=0}^{\infty} \left(f_i^2 \|b_i(t)\|^2 \right), \tag{6}$$

where $f_i = N\int_0^1 f(t) b_i(t)dt$. □

Definition 3 (Hybrid Bernstein Block-Pulse Functions (HBBPFs)). $H_{n,m}(t)$, $n = 1,2,\ldots,N$, $m = 0,1,\ldots,M$, have three arguments; n and m are the order of BPFs and BPs, respectively, and t is the normalized time. HBBPFs are defined on the interval $[0,1)$ as follows:

$$H_{n,m}(t) = \begin{cases} B_{m,M}(Nt-n+1), & \dfrac{n-1}{N} \le t \le \dfrac{n}{N}, \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

In the next section, we deal with the problem of approximation of these functions.

3. Approximation of Functions by Using HBBPFs and Convergence Analysis

Theorem 4. *Suppose that the function* $f : [0,1] \to R$ *is* $m+1$ *times continuously differentiable, and* $S = \text{Span}\{B_{0,m}, B_{1,m}, \ldots, B_{m,m}\}$. *Then* $c^T B = s_0 = \sum_{i=0}^{m} c_i B_{i,m} \in S$ *is the best approximation f out of* $S \subseteq L^2[0,1]$ *with the following inner product:*

$$\begin{aligned} \langle f, B\rangle &= \int_0^1 f(x)\,B(x)^T dx \\ &= \left[\langle f, B_{0,m}\rangle, \langle f, B_{1,m}\rangle, \ldots, \langle f, B_{m,m}\rangle\right], \end{aligned} \tag{8}$$

where $B^T = [B_{0,m}, B_{1,m}, \ldots, B_{m,m}]$ *and* $c^T = [c_1, c_2, \ldots, c_m]$. *Also, one can obtain the following inequality:*

$$\left\| f - c^T B \right\|_{L^2[0,1]} \le \frac{\widehat{K}}{(m+1)!\sqrt{2m+3}}, \tag{9}$$

where $\widehat{K} = \max_{x\in[0,1]} |f^{(m+1)}(x)|$.

Proof. We prove that $c^T B$ is the best approximation for f out of S. We can prove that S is a convex subset of a real inner product space $L^2[0,1]$ (see [21]). Therefore, for any $x \in L^2[0,1]$, $\hat{x} \in S$ is its best approximation in S if and only if it satisfies

$$\langle x - \hat{x}, z - \hat{x}\rangle \le 0 \quad \forall z \in S, \tag{10}$$

where the inner product is defined by $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$. Then for any $x \in L^2[0,1]$, its best approximation is unique. Also, we know that $S \subset L^2[0,1]$ is a convex and closed finite-dimensional subset of an inner product space $L^2[0,1]$. Then for any $x \in L^2[0,1]$, there is a unique element $\hat{x} \in S$ such that $\|x - \hat{x}\| = \inf_{z\in S}\|x - z\|$. Therefore, there exist the unique coefficients c_i, $i = 0,1,\ldots,m$ such that

$$f \cong s_0 = \sum_{i=0}^{m} c_i B_{i,m} = c^T B. \tag{11}$$

On the other hand, we can consider that $\{1, x, \ldots, x^n\}$ is a basis for polynomials space of degree m. Therefore we define $y_1(x) = f(0) + x f'(0) + (x^2/2!) f''(0) + \cdots + (x^m/m!) f^{(m)}(0)$. Hence, from Taylor expansion we have

$$\left| f(x) - y_1(x) \right| = \left| f^{(m+1)}(\xi_x) \frac{x^{m+1}}{(m+1)!} \right|, \tag{12}$$

Table 1: Absolute errors by using BPFs for $N = 4$, BPs for $M = 3$, and HBBPFM for $N = 4$, $M = 3$ in Example 1.

t	BPFs $N = 4$	BPs $M = 3$	HBBPFM $N = 4, M = 3$
0	0.159448	0.000252739	2.57612×10^{-7}
0.1	0.0596148	0.0000539886	5.73616×10^{-8}
0.2	0.0392211	0.000110834	1.23088×10^{-7}
0.3	0.118936	0.0000398714	3.42659×10^{-7}
0.4	0.0250381	0.0000566614	2.06685×10^{-7}
0.5	0.167325	0.000106028	1.3331×10^{-6}
0.6	0.0821085	0.0000743689	3.07359×10^{-7}
0.7	0.00253325	0.0000243121	5.58694×10^{-7}
0.8	0.125405	0.000119641	7.24512×10^{-7}
0.9	0.0594347	0.000076931	4.20127×10^{-7}

Table 2: Absolute errors by using BPFs for $N = 5$, BPs for $M = 4$, and HBBPFs for $N = 5$, $M = 4$ in Example 1.

t	BPFs $N = 5$	BPs $M = 4$	HBBPFM $N = 5, M = 4$
0	0.120718	0.0000294061	9.38506×10^{-9}
0.1	0.0208845	0.0000117512	4.43327×10^{-10}
0.2	0.124426	4.28074×10^{-6}	7.87196×10^{-9}
0.3	0.0275755	7.98903×10^{-6}	1.13157×10^{-9}
0.4	0.124293	8.7962×10^{-6}	6.2511×10^{-9}
0.5	0.034286	2.23581×10^{-7}	1.14373×10^{-9}
0.6	0.120604	8.9113×10^{-6}	1.22382×10^{-8}
0.7	0.0410285	7.57027×10^{-6}	1.62177×10^{-9}
0.8	0.230475	7.94076×10^{-6}	4.74483×10^{-10}
0.9	0.00358729	0.0000229948	9.2012×10^{-9}

Table 3: Absolute errors by using BPFs for $N = 4$, BPs for $M = 3$, and HBBPFM for $N = 4$, $M = 3$ in Example 2.

t	BPFs $N = 4$	BPs $M = 3$	HBBPFM $N = 4, M = 3$
0	0.134438	0.000939946	2.60043×10^{-6}
0.1	0.0292675	0.000210236	6.00397×10^{-7}
0.2	0.0869644	0.000396173	1.08124×10^{-6}
0.3	0.103935	0.000126329	1.37399×10^{-6}
0.4	0.0380311	0.000213179	7.99831×10^{-7}
0.5	0.216077	0.00037144	4.28735×10^{-6}
0.6	0.0426798	0.000246979	9.89894×10^{-7}
0.7	0.148954	0.0000965353	1.78268×10^{-6}
0.8	0.167943	0.000412916	2.26538×10^{-6}
0.9	0.0661188	0.000254268	1.31873×10^{-6}

Table 4: Absolute errors by using BPFs for $N = 5$, BPs for $M = 4$, and HBBPFs for $N = 5$, $M = 4$ in Example 2.

t	BPFs $N = 5$	BPs $M = 4$	HBBPFM $N = 5, M = 4$
0	0.106159	0.0000526416	1.44355×10^{-8}
0.1	0.000988576	0.0000210365	1.12472×10^{-9}
0.2	0.128144	8.80224×10^{-6}	1.65937×10^{-8}
0.3	0.000312002	0.0000141662	1.06195×10^{-9}
0.4	0.155374	0.0000171016	9.87722×10^{-9}
0.5	0.00152224	7.80052×10^{-7}	2.99127×10^{-9}
0.6	0.189012	0.0000166986	4.29645×10^{-8}
0.7	0.00262215	0.0000156271	7.92177×10^{-9}
0.8	0.230475	7.94076×10^{-6}	4.74483×10^{-10}
0.9	0.00358729	0.0000229948	9.2012×10^{-9}

where $\xi_x \in (0,1)$. Since $c^T B$ is the best approximation f out of S, and we assume that $y_1 \in S$, therefore, we have

$$\begin{aligned}\left\| f - c^T B \right\|^2_{L^2[0,1]} &\le \left\| f - y_1 \right\|^2_{L^2[0,1]} \\ &= \int_0^1 \left| f(x) - y_1(x) \right|^2 dx \\ &= \int_0^1 \left| f^{(m+1)}(\xi_x) \right|^2 \left(\frac{x^{m+1}}{(m+1)!} \right)^2 dx \\ &\le \frac{\widehat{K}^2}{(m+1)!^2} \int_0^1 x^{2m+2} dx \\ &= \frac{\widehat{K}^2}{(m+1)!^2 (2m+3)}.\end{aligned} \tag{13}$$

Then by taking square roots, the proof is complete. □

The previous theorem shows that the error vanishes as $m \to \infty$.

Corollary 5. *One can write $c^T \langle B, B \rangle \cong \langle f, B \rangle$, such that one defines $Q = \langle B, B \rangle$ that is a $(m+1) \times (m+1)$ matrix and is said dual matrix of B, and one can obtain*

$$\begin{aligned}Q_{i+1,j+1} &= \int_0^1 B_{i,m}(x) B_{j,m}(x) dx \\ &= \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}}, \quad i, j = 0, 1, \ldots, m.\end{aligned} \tag{14}$$

Proof. We know

$$f \cong s_0 = \sum_{i=0}^{m} c_i B_{i,m} = c^T B; \tag{15}$$

therefore, the proof is complete. □

Corollary 6. *A function $f(t) \in L^2([0,1])$ may be expanded as follows:*

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} p_{n,m} H_{n,m}(t). \tag{16}$$

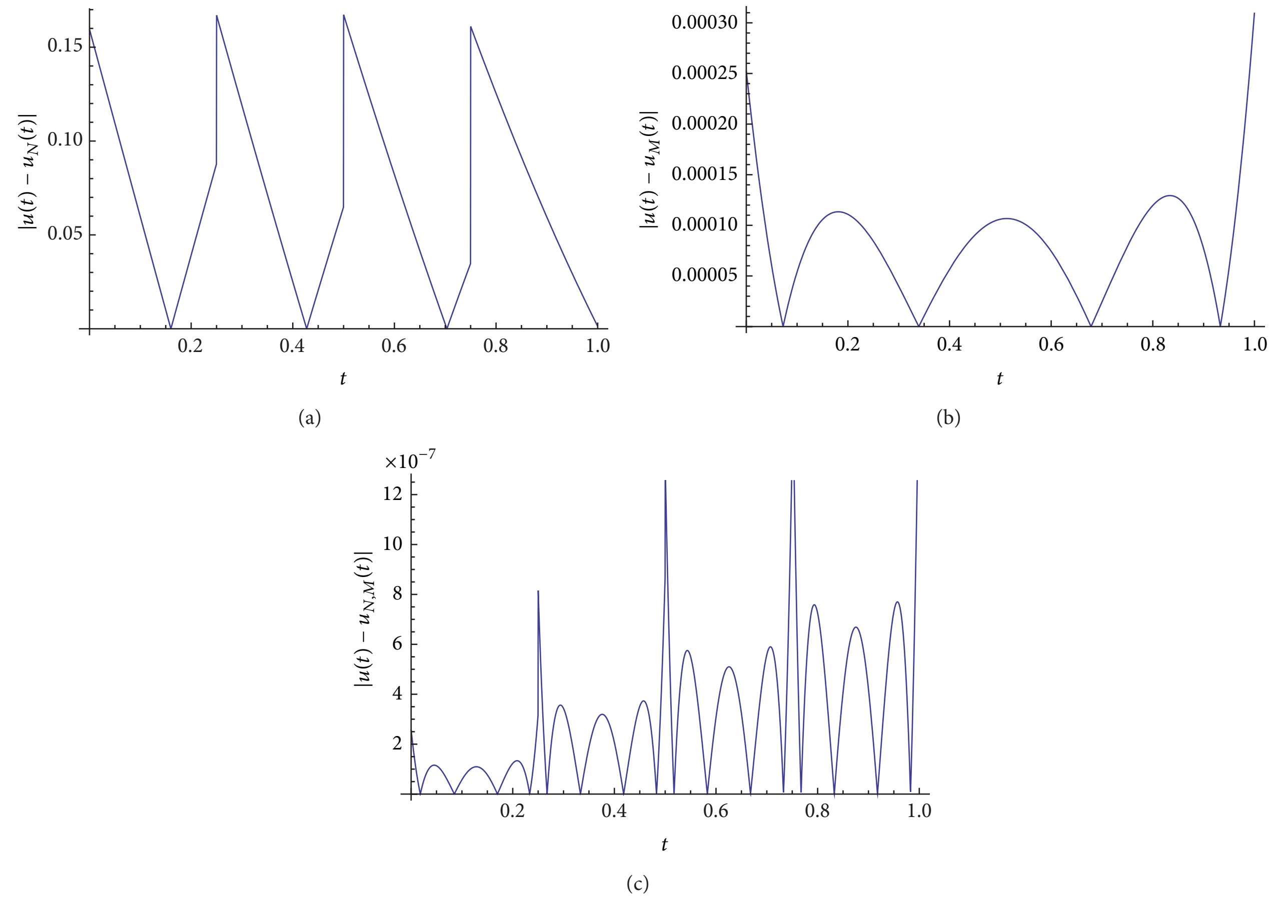

Figure 1: Plot of error functions by using BPFs for $N = 4$ (a), BPs for $M = 3$ (b), and HBBPFM for $N = 4$, $M = 3$ (c) in Example 1.

If the infinite series in (16) *is truncated, then we have*

$$f(t) \approx \sum_{n=1}^{N} \sum_{m=0}^{M} p_{n,m} H_{n,m}(t) = P^T H(t), \tag{17}$$

where

$$H(t) = [H_{1,0}(t), H_{1,1}(t), \ldots, H_{1,M}(t), H_{2,0}(t), H_{2,1}(t), \ldots, H_{N,M}(t)]^T, \tag{18}$$

$$P = [p_{1,0}, p_{1,1}, \ldots, p_{1,M}, p_{2,0}, p_{2,1}, \ldots, p_{N,M}]^T. \tag{19}$$

Therefore we can get

$$P^T \langle H(t), H(t) \rangle = \langle f(t), H(t) \rangle. \tag{20}$$

Then

$$P = D^{-1} \langle f(t), H(t) \rangle, \tag{21}$$

where

$$D = \langle H(t), H(t) \rangle = \int_0^1 H(t) H^T(t)\, dt = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_N \end{bmatrix}, \tag{22}$$

where by using (7)*,* D_n $(n = 1, 2, \ldots, N)$ *is defined as follows:*

$$\begin{aligned}(D_n)_{i+1,j+1} &= \int_{(n-1)/N}^{n/N} B_{i,M}(Nt - n + 1) B_{j,M}(Nt - n + 1)\, dt \\ &= \frac{1}{N} \int_0^1 B_{i,M}(t) B_{j,M}(t)\, dt \\ &= \frac{\binom{M}{i}\binom{M}{j}}{N(2M+1)\binom{2M}{i+j}}, \quad i, j = 0, 1, \ldots, M.\end{aligned} \tag{23}$$

We can also approximate the function $k(t, s) \in L^2([0, 1] \times [0, 1])$ *as follows:*

$$k(t, s) \approx H^T(t) K H(s), \tag{24}$$

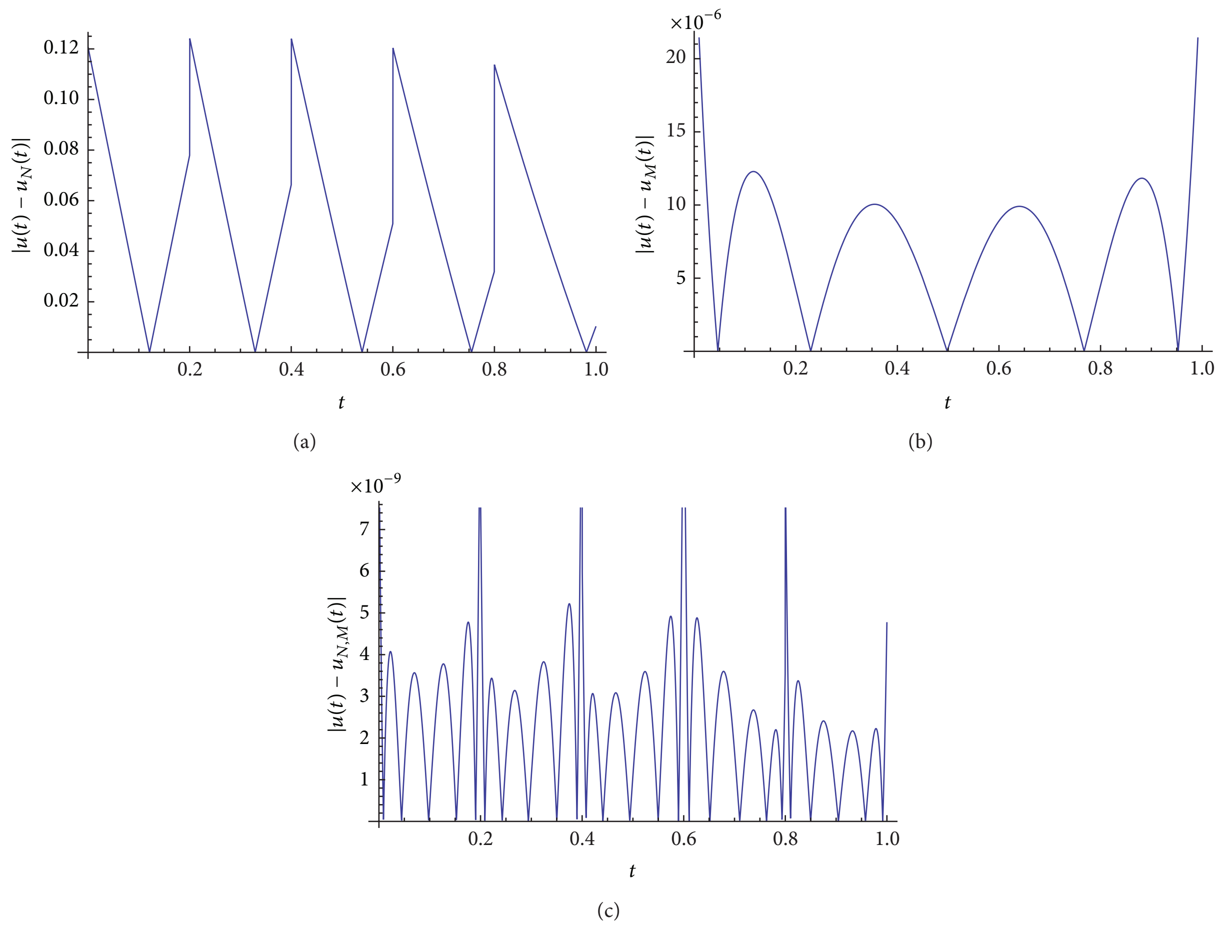

Figure 2: Plot of error functions by using BPFs for $N = 5$ (a), BPs for $M = 4$ (b), and HBBPFM for $N = 5$, $M = 4$ (c) in Example 1.

where K is an $N(M+1)\times N(M+1)$ matrix that we can obtain as follows:

$$K = D^{-1}\left\langle H(t), \left\langle k(t,s), H(s)\right\rangle\right\rangle D^{-1}. \tag{25}$$

Theorem 7. *Let the function $f : [0,1] \to R$ be $M+1$ times continuously differentiable; then we have*

$$\left\| f - P^T H \right\|_{L^2[0,1]} \le \frac{\widetilde{K}}{N^{M+1}(M+1)!\sqrt{2M+3}}, \tag{26}$$

where $\widetilde{K} = \max_{t\in[0,1]}|f^{(M+1)}(t)|$.

Proof. By using Theorem 4 we get

$$\begin{aligned}
&\left\| f - P^T H \right\|^2_{L^2[0,1]} \\
&= \int_0^1 \left| f(t) - P^T H(t) \right|^2 dx \\
&= \sum_{n=1}^{N} \left(\int_{(n-1)/N}^{n/N} \left| f(t) - \sum_{m=0}^{M} p_{n,m} B_{m,M}(Nt - n + 1) \right|^2 \right) dt \\
&= \frac{1}{N} \sum_{n=1}^{N} \int_0^1 \left| f\left(\frac{t+n-1}{N}\right) - \sum_{m=0}^{M} p_{n,m} B_{n,m}(t) \right|^2 dt \\
&\le \frac{1}{N^{2M+3}} \sum_{n=1}^{N} \int_0^1 \left| f^{(M+1)}(\xi_n) \right|^2 \frac{t^{2M+2}}{(M+1)!^2} dt \\
&\le \frac{1}{N^{2M+3}} \sum_{n=1}^{N} \frac{\widehat{K}_n^2}{(M+1)!^2 (2M+3)} \\
&\le \frac{\widetilde{K}^2}{N^{2M+2}(M+1)!^2(2M+3)},
\end{aligned} \tag{27}$$

where $\xi_n \in ((n-1)/N, n/N)$ and $\widehat{K}_n = \max_{t\in[(n-1)/N, n/N]}|f^{(M+1)}(t)|$. Therefore by taking square roots, the proof is complete. □

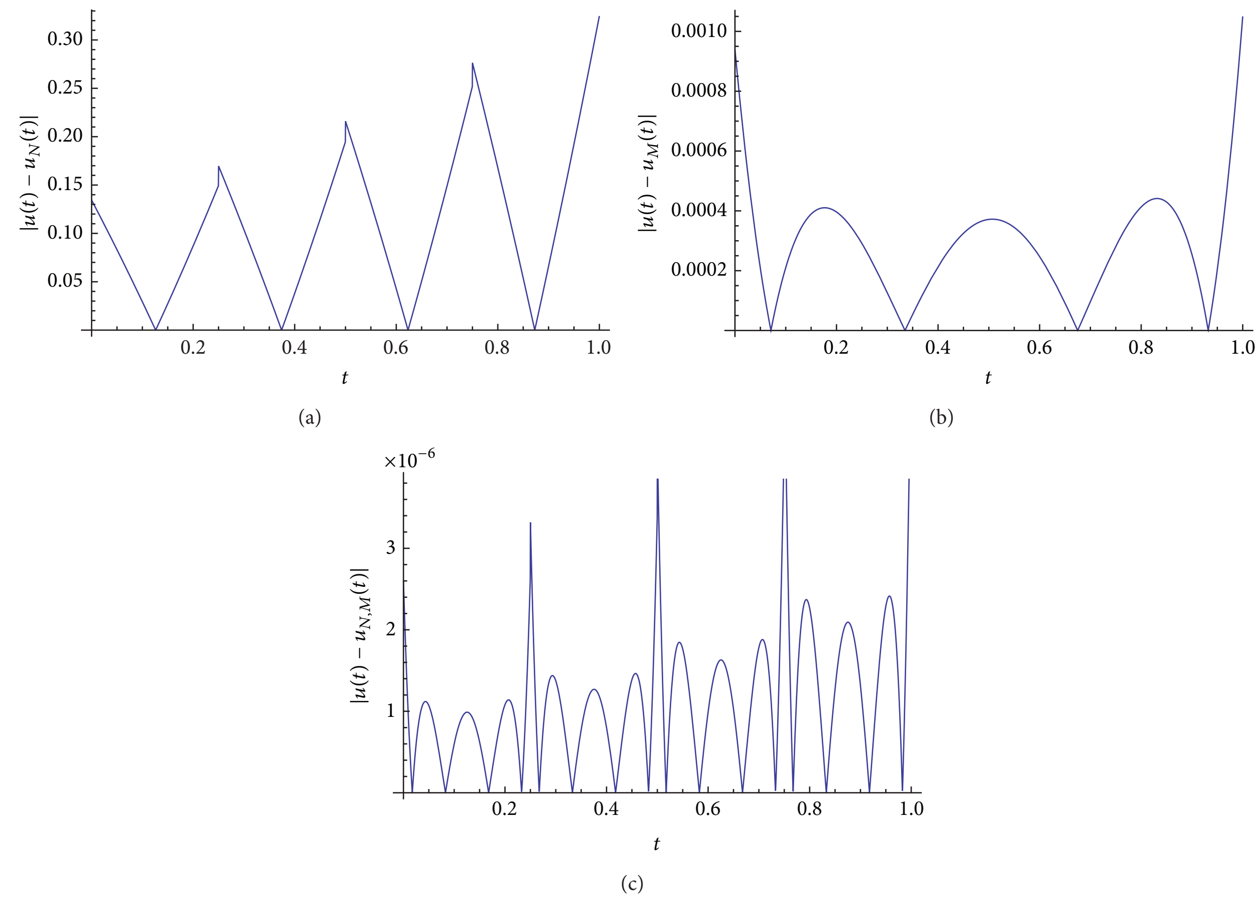

Figure 3: Plot of error functions by using BPFs for $N = 4$ (a), BPs for $M = 3$ (b), and HBBPFM for $N = 4$, $M = 3$ (c) in Example 2.

The above theorem shows that the approximation error vanishes as $M, N \to \infty$.

4. HBBPFs for the Second Kind Integral Equations and Error Analysis

In this section, we are dealing with the following Fredholm equations of the second kind:

$$u(t) = \int_0^1 k(t,s)\,u(s)\,ds + f(t), \tag{28}$$

where $u, f \in L^2([0,1])$, $k \in L^2([0,1] \times [0,1])$, and $u(t)$ is an unknown function.

Let us approximate u, f, and k by (18) and (25) as follows:

$$u(t) \approx U^T H(t), \qquad f(t) \approx F^T H(t),$$
$$k(t,s) \approx H^T(t)\,KH(t). \tag{29}$$

By substituting (29) in (28) we obtain

$$\begin{aligned} H^T(t)\,U &= \int_0^1 H^T(t)\,KH(s)\,H^T(s)\,U\,ds + H^T(t)\,F \\ &= H^T(t)\,K\left(\int_0^1 H(s)\,H^T(s)\,ds\right)U + H^T(t)\,F \\ &= H^T(t)\,KDU + H^T(t)\,F = H^T(t)\,(KDU + F). \end{aligned} \tag{30}$$

Therefore we have the following linear system:

$$(I - KD)\,U = F, \tag{31}$$

that by solving this linear system we can obtain the vector U.

Theorem 8. *Suppose that $u(t)$ is exact solution of (28) and $u_{N,M}(t)$ is approximate solution by HBBPFs for $u(t)$ and $E_{N,M}(t)$ is perturbation function that depends only on $u_{N,M}(t)$ (i.e., $u_{N,M}(t) = \int_0^1 k(t,s)u_{N,M}(s)ds + f(t) + E_{N,M}(t)$). Let $R = \max_{0\le t,s\le 1}|k(s,t)| < \infty$. Then $E_{N,M}(t) \to 0$ as $M, N \to \infty$.*

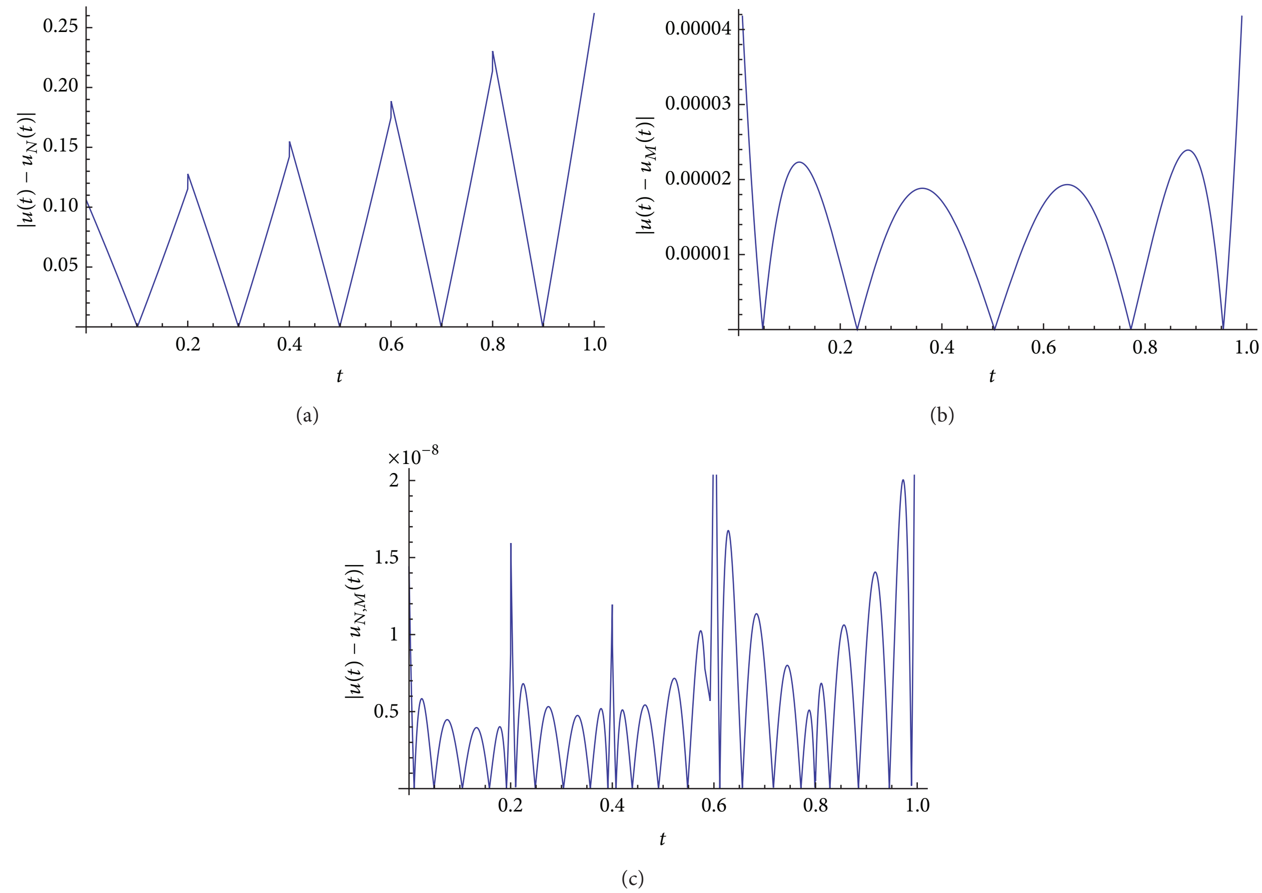

FIGURE 4: Plot of error functions by using BPFs for $N = 5$ (a), BPs for $M = 4$ (b), and HBBPFM for $N = 5$, $M = 4$ (c) in Example 2.

Proof. Suppose $e_{N,M}(t) = u(t) - u_{N,M}(t)$ is the error function of approximate solution $u_{N,M}(t)$ to the exact solution $u(t)$. Therefore we get

$$e_{N,M}(t) = \int_0^1 k(t,s)\, e_{N,M}(s)\, ds - E_{N,M}(t). \quad (32)$$

By taking absolute value and using Holder inequality we get

$$\begin{aligned} |E_{N,M}(t)| &\leq \int_0^1 |k(t,s)|\, |e_{N,M}(s)|\, ds + |e_{N,M}(t)| \\ &\leq \left(\int_0^1 |k(t,s)|^2 ds\right)^{1/2} \left(\int_0^1 |e_{N,M}(t)|^2 ds\right)^{1/2} \\ &\quad + |e_{N,M}(t)| \\ &\leq R \|e_{N,M}(t)\|_{L^2[0,1]} + |e_{N,M}(t)|. \end{aligned} \quad (33)$$

Now, by taking norm $L^2([0,1])$ we obtain

$$\|E_{N,M}(t)\|_{L^2[0,1]} \leq (R+1)\, \|e_{N,M}(t)\|_{L^2[0,1]}. \quad (34)$$

Finally, from Theorem 7 we can write

$$\|E_{N,M}(t)\|_{L^2[0,1]} \leq \frac{(R+1)\,\overline{K}}{N^{M+1}\,(M+1)!\,\sqrt{2M+3}}, \quad (35)$$

where $\overline{K} = \max_{t\in[0,1]} |u^{(M+1)}(t)|$.

Therefore, we can show that $E_{N,M}(t) \to 0$ as $M, N \to \infty$. □

5. Numerical Examples

In this section we discuss the implementation of the new method and investigate its accuracy by applying it to different examples. In the following examples, we suppose that $u_N(t)$, $u_M(t)$, and $u_{M,N}(t)$ are approximate solutions by BPFs, BPs, and HBBPFM for the exact solution $u(t)$, respectively.

Example 1. Consider the following integral equation:

$$\begin{aligned} u(t) = \int_0^1 (t+s)\, u(s)\, ds + \sin(t) \\ - t + (t+1)\cos(1) - \sin(1). \end{aligned} \quad (36)$$

We know that the exact solution is $u(t) = \sin(t)$. The obtained results of BPFs, BPs, and HBBPFs are reported in Tables 1 and 2 and are plotted in Figures 1 and 2. We compare the obtained results and observe that HBBPFM is very effective and accuracy of approximate solutions in this method is more than methods of BPFs and BPs.

Example 2. Consider the following integral equation:

$$u(t) = \int_0^1 tsu(s)\,ds + e^t - t, \quad (37)$$

with exact solution $u(t) = e^t$. We obtain the computational by BPFs, BPs, and HBBPFM with $N = 4$, $M = 3$, and $N = 5$, $M = 4$; then we compare them together. The results are reported in Tables 3 and 4 and are plotted in Figures 3 and 4. Similar to the previous example, we see that the method HBBPFM is very effective and accuracy of solution in this method is more than methods of BPFs and BPs.

6. Conclusion

In this paper, HBBPFs are used to solve second kind integral equations we call this method with HBBPFM. This method converts second kind integral equations to systems of linear equations whose answers are coefficient of HBBPFs expansion of the solution of second kind integral equations. Also, by using several lemmas and theorems, we have discussed convergence analysis of the proposed method. Numerical examples show the efficiency and accuracy of the method. Moreover we see that accuracy of solutions in HBBPFM is more satisfactory than the methods of BPFs and BPs.

Conflict of Interests

The authors declare that there is no conflict of interests in this paper.

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