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Investigation of the fractional diffusion equation based on generalized integral quadrature technique

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ABSTRACT

Nowadays, the conventional Euclidean models are mostly used to describe the behavior of fluid flow through porous media. These models assume the homogeneity of the reservoir, and in naturally fractured reservoir, the fractures are distributed uniformly and use the interconnected fractures assumption. However, several cases such as core, log, outcrop data, production behavior of reservoirs, and the dynamic behavior of reservoirs indicate that the reservoirs have a different behavior other than these assumptions in most cases. According to the fractal theory and the concept of fractional derivative, a generalized diffusion equation is presented to analyze the transport in fractal reservoirs. Three outer boundary conditions are investigated. Using exact analytical or semi-analytical solutions for generalized diffusion equation with fractional order differential equation and a fractal physical form, under the usual assumptions, requires large amounts of computation time and may produce inaccurate and fake results for some combinations of parameters. Because of fractionality, fractal shape, and therefore the existence of infinite series, large computation times occur, which is sometimes slowly convergent. This paper provides a computationally efficient and accurate method via differential quadrature (DQ) and generalized integral quadrature (GIQ) analyses of diffusion equation to overcome these difficulties. The presented method would overcome the imperfections in boundary conditions' implementations of second-order partial differential equation (PDE) encountered in such problems.

1. Introduction

The 60% of the remaining oil in the world is found in carbonate reservoirs. Carbonate reservoirs specifically include heterogeneous natural fractures on a wide range of spatial scales. These types of reservoirs that are heterogeneous cannot be easily described and the paths of flow are unpredictable. In spite of these complexities, the pressure transient models are developed under the assumption of standard geometry and homogeneity behavior, which are not realistic in most cases

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Nomenclature

$A_{ik}^{(m)}$	weighing coefficient of the mth-order derivative of p_D with respect to r_D
$\bar{A}_{jk}^{(n)}$	weighing coefficient of the nth-order derivative of p_D with respect to t_D
B	oil formation volume factor, RB/STB [L^3/L^3]
c_k^{ij}	weighing coefficient of the integral of p_D with respect to t_D
c_t	total isothermal compressibility factor, psi^{-1} [Lt^2/m]
d	fractal topological dimension
h	net formation thickness, ft [L]
k	reservoir rock permeability, md [L^2]
M	number of grids points in t_D direction
N	number of grids points in r_D direction
p_i	initial reservoir pressure, psi [$m/(Lt^2)$]
p_D	dimensionless pressure
p_{wf}	wellbore flowing pressure, psi [$m/(Lt^2)$]
q	flow rate, STB/D [L^3/t]
r	distance from the center of wellbore, ft [L]
r_D	dimensionless radius
r_{Di}	ith dimensionless grid point in r_D direction
r_{eD}	dimensionless external radius
r_w	wellbore radius, ft [L]
t_D	dimensionless time
t_{Di}	ith dimensionless grid point in t_D direction
v_r	radial velocity, ft/s [L/t]
θ	fractal dynamical index
μ	oil viscosity, cp [$m/(Lt)$]
ρ	oil density, lb_m/ft^3 (m/L^3)
ϕ	porosity of reservoir rock

[1–9]. One of the aspects of the geometrical complexity is that flow distribution is affected by the distribution of fractures most of the times. There may be some regions in the reservoir with a group of fractures whereas the reservoir does not contain fractures. The different scale of fractures shows an uncertainty relevant element in developing the mathematical model of reservoir. Thus, according to these high complexities, the weakness of the Euclidean models is clear in most cases. Alternatively, fractal geometry provides an appropriate approach to explain and model the complex fractured reservoirs [9,10].

Chang and Yortsos [11] were the first scholars who presented a mathematical model to describe the pressure behavior of transport in fractally fractured reservoirs. They also obtained the analytical results, which showed that a fractal reservoir can be identified by a log–log straight line with the slope equal to a function of the fractal topological dimension and the fractal dynamical index. Beier [12] and Aprilian [13] applied fractal reservoir model to analyze well test data for complex reservoirs which could not be matched by traditional model, and the results were consistent with field practice. Poon [14] extended the concept of fractal distribution to find out the effect of a composite reservoir. He et al. [15] established a fractal model for unsteady-state flow in double-porosity and permeability reservoirs based on Warren and Root [16] model, and solved it by the Correction Prediction method. Meanwhile, they analyzed pressure performances and their effect on different factors. On the basis of Warren and Root model, Zhang et al. [17], set up a model for deformed double-porosity fractal gas reservoirs by introducing fractal parameters and compressibility factors. They solved this model by finite element method with consideration of secondary boundary conditions, plotting the results into type curves. To consider threshold pressure gradient in low permeability reservoir and gradual pressure propagation in the formation, Hou and Tong [18] established the non-Darcy flow model for deformed double-porosity media [19]. The production of oil from fractally fractured systems indicates that the system has an anomalous behavior, which cannot be modeled by the classical diffusion equation. Since the history of flow is pivotal in all stages of production, the fractional derivative can be used as an appropriate approach to consider the history of flow in the mathematical model [20].

Since most analytical, semi-analytical, and numerical methods are slowly convergent and may produce inaccurate results to differential equations, Bellman et al. [21] introduced another numerical method that gives acceptable accurate results using small number of grid points [22–26] This alternative method was called differential quadrature method (DQM) and was further developed by Bellman and Roth [27]. The double-porosity reservoirs were discussed in Ref [9], and the response of these reservoirs assuming fractal structure was analyzed. A model was developed for analysis of double-porosity reservoirs where it was passed up the effect of fractal topological dimension (d), which was assumed to be equal to Euclidean dimension (D). The mathematical model was analyzed by using the Laplace transform.

This paper establishes the solution of generalized diffusion equation for single-porosity naturally fractured reservoirs based on a new application of DQM and generalized integral quadrature (GIQ). The method weighting coefficients are not exclusive and any methods that can be used in conventional DQM for evaluation of the weighting coefficients, such as

the generalized differential quadrature method (GDQM) which is claimed to be most computationally efficient and accurate [1], may be employed. Recall that three main types of reservoirs exist in petroleum engineering including infinite reservoir, bounded circular reservoir, and circular reservoir with constant pressure outer boundary, which are represented mathematically by three different outer boundary conditions. To prove the versatility of the presented approach, we investigated these three different outer boundary conditions.

The organization of this paper is as follows. Section 2 provides some preliminaries including the necessary definitions of fractional calculus and the necessary relations of DQ and GIQ methods. The fractal diffusion equation and generalized diffusion equation are discussed in Section 3. Section 4 presents the results of DQ and GIQ methods and their advantages in linear partial differential equations (PDEs). Our conclusions can be seen in Section 5.

2. Preliminaries

In this section, two preliminaries are presented, namely, fractional calculus and the differential quadrature method.

2.1. Basic definitions

There are various definitions for fractional order operator, and each of them has its unique prominent feature. For engineering applications, the Riemann–Liouville fractional order integral is the most applied operator for integration, which is defined as follows:

$$ {}_aD_{t_D}^{-\gamma} p_D(t_D) := \frac{1}{\Gamma(\gamma)} \int_a^{t_D} (t_D-\tau)^{\gamma-1} p_D(\tau) d\tau, \quad \gamma \in \mathbf{R}^+, \tag{1} $$

in which $\Gamma(\gamma) = \int_0^\infty e^{-z} z^{\gamma-1} dz$, $\gamma > 0$ is the Gamma function.

On the other hand, from an engineering view point, there are two operators for fractional order differentiations [28]:

1. Riemann–Liouville (RL) fractional order derivative for $m-1 < \gamma < m \in Z^+$:

$$ {}_a^{RL}D_{t_D}^{\gamma} p_D(t_D) := D^m {}_aD_{t_D}^{-(m-\gamma)} p_D(t_D) = \frac{1}{\Gamma(m-\gamma)} \frac{d^m}{dt^m} \int_a^{t_D} (t_D-\tau)^{m-\gamma-1} p_D(\tau) d\tau. \tag{2} $$

2. Caputo (C) fractional order derivative for $m-1 < \gamma < m \in Z^+$:

$$ {}_C D_{t_D}^{\gamma} p_D(t_D) := {}_a^{RL}D_{t_D}^{-(m-\gamma)} D^m p_D(t_D) = \frac{1}{\Gamma(m-\gamma)} \int_a^{t_D} (t_D-\tau)^{m-\gamma-1} p_D^{(m)}(\tau) d\tau. \tag{3} $$

2.2. The differential quadrature method

We recall that by using the DQM, the nth derivative of the function $p_D(r_D,t_D)$ at point r_{Di} and the mth derivative of the function $p_D(r_D,t_D)$ at point t_{Dj} are approximated as:

$$ p_{(Dr_D)}^{(n)}(r_{Di},t_{Dj}) = \sum_{k=1}^{N} A_{ik}^{(n)} p_D(r_{Dk},t_{Dj}), \tag{4} $$

$$ p_{(Dt_D)}^{(m)}(r_{Di},t_{Dj}) = \sum_{k=1}^{M} \bar{A}_{jk}^{(m)} p_D(r_{Di},t_{Dk}), \tag{5} $$

respectively, where $A_{ik}^{(n)}$ and $\bar{A}_{jk}^{(m)}$ are the weighting coefficients associated with the nth- and mth- order derivatives of p_D with respect to r_D and t_D, respectively. The number of grid points in the r_D direction and t_D direction are represented by N and M, respectively.

The weighting coefficients in our analysis are evaluated by GDQM. In this method, the weighting coefficients for evaluating derivatives of a function may be obtained directly, irrespective of the number and position of the grid points from an explicit formula [8]. These coefficients for the first-order derivatives are given by

$$ A_{ij}^{(1)} = \begin{cases} \frac{P^{(1)}(r_{Di})}{(r_{Di}-r_{Dj})P^{(1)}(r_{Dj})} & j \neq i \\ -\sum_{j=1, j\neq i}^{N} A_{ij}^{(1)} & \end{cases}, \tag{6} $$

for $i,j = 1,2,\ldots,N$

$$\bar{A}_{ij}^{(1)} = \begin{cases} \frac{Q^{(1)}(t_{Di})}{(t_{Di}-t_{Dj})Q^{(1)}(t_{Dj})} & j \neq i \\ -\sum_{j=1,j\neq i}^{M} \bar{A}_{ij}^{(1)} \end{cases}, \tag{7}$$

for $i,j = 1,2,\ldots,M$

where

$$P^{(1)}(r_{Di}) = \prod_{j=1,j\neq i}^{N} (r_{Di} - r_{Dj}), \tag{8}$$

and

$$Q^{(1)}(t_{Di}) = \prod_{j=1,j\neq i}^{M} (t_{Di} - t_{Dj}). \tag{9}$$

Now, the weighting coefficients of higher order derivatives can be obtained from the following recursive formulas:

$$A_{ij}^{(n)} = \begin{cases} n\left(A_{ij}^{(1)} A_{ii}^{(n-1)} - \frac{A_{ij}^{(n-1)}}{r_{Di}-r_{Dj}}\right) & j \neq i \\ -\sum_{j=1,j\neq i}^{N} A_{ij}^{(n)} \end{cases}, \tag{10}$$

for $n = 2,3,\ldots,N-1; i,j = 1,2,\ldots,N$

$$\bar{A}_{ij}^{(m)} = \begin{cases} m\left(\bar{A}_{ij}^{(1)} \bar{A}_{ii}^{(m-1)} - \frac{\bar{A}_{ij}^{(m-1)}}{t_{Di}-t_{Dj}}\right) & j \neq i \\ -\sum_{j=1,j\neq i}^{M} \bar{A}_{ij}^{(m)} \end{cases}, \tag{11}$$

for $m = 2,3,\ldots,M-1;\ i,j = 1,2,\ldots,M$

Equally spaced points as a convenient choice are often chosen for the grid points. Another kind of grid points are unequally spaced grid points which give more accurate results [8]. A well accepted and practical set of grid points is the cosine-type (or the Gauss–Lobatto–Chebyshev) points given by

$$r_{Di} = \frac{1}{2}\left[1 - \cos\frac{(i-1)\pi}{N-1}\right], \tag{12}$$

$$t_{Di} = \frac{1}{2}\left[1 - \cos\frac{(i-1)\pi}{M-1}\right], \tag{13}$$

in the r_D direction and t_D direction, respectively.

In this study, weighting coefficients are evaluated by GDQM in conjunction with unequally spaced grid points [29,30].

As a general case for GIQ, the integral of $p_D(r_D, t_D)$ over a part of the whole domain is approximated by a linear combination of all the functional values in the overall domain, namely, with the form

$$\int_{t_{Di}}^{t_{Dj}} p_D(r_{Di}, t_D)dt_D = \sum_{k=1}^{M} c_k^{ij} p_D(r_{Dl}, t_{Dk}), \tag{14}$$

where t_{Di} and t_{Dj} are the coordinates that can be altered. The c_k^{ij} coefficients are given by $c_k^{ij} = w_{jk} - w_{ik}$, where w_{jk} and w_{ik} are the elements of the matrix W. This matrix is obtained by reversing of another matrix A with the following elements:

$$a_{ij} = \frac{t_{Di} - c}{t_{Dj} - c} x_{ij}^{(1)}, \tag{15}$$

when $j \neq i$

$$a_{ii} = x_{ii}^{(1)} + \frac{1}{t_{Di} - c}, \tag{16}$$

where $x_{ij}^{(1)}$ is the weighting coefficient, which was used to calculate the first-order derivative in DQM. It should be noted that c must be considered a point other than the given grid points t_{Di} and t_{Dj} [31].

3. Problem statement and theory development

In spite of all the heterogeneous models, which are considered in the literature, the double-porosity reservoirs have been the most frequently discussed. They assume the existence of two distinct porous media regions within the formation. These two regions include the fissure system, with low storativity and high permeability, and the matrix blocks, with high storativity and low permeability. The main assumptions that are considered to develop a mathematical model are:

(1) The fissure system is assumed to be uniformly distributed throughout the reservoir.
(2) The matrix is not produced directly into the wellbore, but only into the fissures.
(3) Only the fissure system provides the total mobility, but the matrix blocks supply most of the storage capacity.

The first assumption may be violated in a real double-porosity system, and consequently the fluid flow can have a different behavior. Fractal geometry can be used as a useful and effective tool to analyze and model the effect of heterogeneities in disordered fractured media. For a naturally fractured reservoir (double porosity), when the matrix does not participate in the production (single porosity and single permeability), many approaches have been proposed to derive an appropriate diffusion equation for analyzing the behavior of these reservoirs. Basically, there are two equations to analyze these reservoirs.

3.1. Fractal diffusion equation: classical approaches

The main assumptions of the model include: (1) d-dimensional fractal flowing grids in two-dimensional Euclidean nonpermeable rock and full penetration of the wellbore into reservoir with thickness of h and under radial flow; (2) a single well with single-phase and slightly compressible reservoir fluid of density ρ; and (3) ignore the effect of gravity and threshold pressure gradient [19].

Before deriving differential equation for fluid flow in a porous medium, the following dimensionless variables should be introduced [9]:

$$p_D = \frac{2\pi k_w h(p_i - p)}{q\mu}, \quad r_D = \frac{r}{r_w}, \quad t_D = \frac{k_w t}{\phi_w \mu c_t r_w^2},$$

where p and p_i are reservoir pressure and initial reservoir pressure, respectively. k_w, h, q, and μ denote wellbore permeability, formation thickness, flow rate, and viscosity, respectively. r and r_w are radial distance and wellbore radius, respectively. t, ϕ_w, and c_t represent time, wellbore porosity, and total compressibility, respectively.

According to fractal theory [32], permeability and porosity of the system is

$$k(r) = k_w \left(\frac{r}{r_w}\right)^{d-\theta-2}, \tag{17}$$

$$\phi(r) = \phi_w \left(\frac{r}{r_w}\right)^{d-2}, \tag{18}$$

where d is the fractal topological dimension and θ is the dynamical fractal dimension that is related to the dynamic property of diffusion.

Continuity equation for fluid flow is

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r(r)) = -\frac{\partial}{\partial t}(\phi(r)\rho). \tag{19}$$

Darcy's law becomes:

$$v_r(r) = -\frac{k(r)}{\mu}\frac{\partial p}{\partial r}. \tag{20}$$

Combining Eqs. (17)–(20) yields the diffusion equation for flow through fractal porous media

$$\frac{\partial^2 p}{\partial r^2} + \frac{d-\theta-1}{r}\frac{\partial p}{\partial r} = \left(\frac{r}{r_w}\right)^{\theta}\frac{\mu\phi_w c_t}{k_w}\frac{\partial p}{\partial t}, \tag{21}$$

or

$$\frac{1}{r^{d-\theta-1}}\frac{\partial}{\partial r}\left(r^{d-\theta-1}\frac{\partial p}{\partial r}\right) = \left(\frac{r}{r_w}\right)^{\theta}\frac{\mu\phi_w c_t}{k_w}\frac{\partial p}{\partial t}, \tag{22}$$

where

$$c_l = \frac{1}{\rho}\frac{\partial \rho}{\partial p}, \quad c_\phi = \frac{1}{\phi}\frac{\partial \phi}{\partial p} \quad \text{and} \quad c_t = c_l + c_\phi.$$

The dimensionless forms of Eqs. (21) and (22) can be written as:

$$\frac{1}{r_D^{\theta}}\frac{\partial^2 p_D}{\partial r_D^2}+\frac{d-\theta-1}{r_D^{\theta+1}}\frac{\partial p_D}{\partial r_D}=\frac{\partial p_D}{\partial t_D}, \tag{23}$$

or

$$\frac{1}{r_D^{d-1}}\frac{\partial}{\partial r_D}\left(r_D^{d-\theta-1}\frac{\partial p_D}{\partial r_D}\right)=\frac{\partial p_D}{\partial t_D}, \tag{24}$$

respectively.

The initial pressure distribution is assumed uniform everywhere $p_D(r_D, 0) = 0$. The inner boundary condition without wellbore storage and skin effect is $(r_D^{(d-\theta-1)}\partial p_D/\partial r_D)_{r_D=1} = 1$ or equivalently $(\partial p_D/\partial r_D)_{r_D=1} = 1$. It should be noted that the inner boundary condition can be written as $(\partial p_D/\partial r_D)_{r_D=1} = -1$, if the positive direction inverses. The outer boundary conditions are $\lim_{r_D\to\infty} p_D(r_D, t_D) = 0$, $(\partial p_D/\partial r_D)_{r_D=r_{eD}} = 0$ and $p_D(r_{eD}, t_D) = 0$ for infinite reservoirs, bounded circular reservoirs, and circular reservoirs with constant pressure outer boundary, respectively.

Both the single porosity reservoir and the corresponding diffusion equation have been presented by O'Shaughnessy and Procaccia [33] and Chang and Yortsos [11], respectively. This expression should be abbreviated as the OP equation. For this case, gaskets, such as the Sierpinski gasket, are characterized in d dimensions by [33] $d=\frac{\ln(D+1)}{\ln 2}$ and $\theta = \ln(D + 3)/\ln 2 - 2$, where D is Euclidean dimension [8].

3.2. Generalized diffusion equation: Fractional calculus approach

As Velázquez et al. [8] stated, the fractal reservoirs cannot be fully described by the OP equation because the diffusion process of these reservoirs is history dependent. In fractally fractured reservoirs, the history of flow has an important role in all stages of production. Including a temporal fractional derivative, Metzler et al. [20] proposed a mathematical model that generalizes the OP equation:

$$\frac{1}{r_D^{\theta}}\frac{\partial^2 p_D}{\partial r_D^2}+\frac{d-\theta-1}{r_D^{\theta+1}}\frac{\partial p_D}{\partial r_D}=\frac{\partial^{\gamma} p_D}{\partial t_D^{\gamma}}, \tag{25}$$

where $\gamma = 2/(2 + \theta)$, and $\partial^{\gamma} p_D/\partial t_D^{\gamma}$ is given by the Caputo derivative. So, $0 < \gamma \leqslant 1$. This equation reduces to the OP equation (Barker 1988) when $\gamma = 1$ and to the traditional Euclidean diffusion equation when $\gamma = 1$, $\theta = 0$ and $d = 2$. This equation has been developed under the assumption that all fluid diffuses between backbone fractures and the fractal fracture loops only (i.e. the matrix is not participating). Eq. (25) may also be called the fractally fractional diffusion (FFD) equation [9]. It should be noted that Eq. (25) is given to describe the pressure behavior of single porosity reservoirs, while in Ref [9], two equations were developed together for the analysis of the pressure behavior in double-porosity reservoirs.

4. Results and discussion

There are several exterior boundary conditions to solve generalized diffusion equation numerically. As mentioned in Section 3.1, we provide the solution of three cases as given below.

4.1. Infinite reservoir case

For infinite reservoir, for purposes of convenience during numerical computations, r_D is normalized by

$$R_D = 1 - \exp(1 - r_D). \tag{26}$$

Using Eq. (26), Eq. (25) can be transformed to

$$\frac{(1-R_D)^2}{(1-\ln(1-R_D))^{\theta}}\frac{\partial^2 p_D}{\partial R_D^2}+\frac{(1-R_D)(d-\theta-2+\ln(1-R_D))}{(1-\ln(1-R_D))^{\theta+1}}\frac{\partial p_D}{\partial R_D}=\frac{\partial^{\gamma} p_D}{\partial t_D^{\gamma}}. \tag{27}$$

Using DQM, the left-hand side of Eq. (27) can be discretized as follow:

$$I=\frac{(1-R_{Di})^2}{(1-\ln(1-R_{Di}))^{\theta}}\sum_{k=1}^{N}A_{ik}^{(2)}p_D(R_{Dk},t_{Dj})+\frac{(1-R_{Di})(d-\theta-2+\ln(1-R_{Di}))}{(1-\ln(1-R_{Di}))^{\theta+1}}\sum_{k=1}^{N}A_{ik}^{(1)}p_D(R_{Dk},t_{Dj}), \tag{28}$$

that can be written as

$$I=\sum_{k=1}^{N}u_{ik}p_D(R_{Dk},t_{Dj}), \tag{29}$$

where

$$u_{ik} = \frac{(1-R_{Di})^2}{(1-\ln(1-R_{Di}))^{\theta}} A_{ik}^{(2)} + \frac{(1-R_{Di})(d-\theta-2+\ln(1-R_{Di}))}{(1-\ln(1-R_{Di}))^{\theta+1}} A_{ik}^{(1)}, \tag{30}$$

for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$

For infinite reservoir, the following boundary conditions are given

$$\sum_{k=1}^{N} A_{1k}^{(1)} p_D(R_{Dk}, t_{Dj}) = 1, \tag{31}$$

$$p_D(R_{DN}, t_{Dj}) = 0, \tag{32}$$

for $j = 1, 2, \ldots, M$

Solving Eqs. (31) and (32) simultaneously gives

$$p_D(R_{D1}, t_{Dj}) = \frac{1 - \sum_{j=2}^{N-1} A_{1k}^{(1)} p_D(R_{Dk}, t_{Dj})}{A_{11}^{(1)}}. \tag{33}$$

Substitution of Eq. (33) into Eq. (29) yields

$$I = \frac{u_{i1}}{A_{11}^{(1)}} + \sum_{k=2}^{N-1} \left(u_{ik} - \frac{A_{1k}^{(1)}}{A_{11}^{(1)}} u_{i1} \right) p_D(R_{Dk}, t_{Dj}). \tag{34}$$

By the Caputo definition of differentiation, the right-hand side of Eq. (27) may be written as:

$$J = \frac{1}{\Gamma(1-\gamma)} \int_0^{t_D} (t_D - \tau)^{-\gamma} p_D^{(1)}(R_D, \tau) d\tau. \tag{35}$$

To overcome discontinuity of integrand at t_D, it can be used integration by parts that gives

$$J = \frac{1}{(1-\gamma)\Gamma(1-\gamma)} \left(t_D^{1-\gamma} p_D^{(1)}(R_D, 0) + \int_0^{t_D} (t_D - \tau)^{1-\gamma} p_D^{(2)}(R_D, \tau) d\tau \right). \tag{36}$$

It should be noted that $0 < \gamma \leqslant 1$. And when $\gamma = 1$, the expression $(1-\gamma)\Gamma(1-\gamma)$ may be considered equal to 1.

Based on DQM, the discretization of Eq. (36) is as follow:

$$J = \frac{1}{(1-\gamma)\Gamma(1-\gamma)} \left(t_{Dj}^{1-\gamma} \sum_{l=2}^{M} \bar{A}_{jl}^{(1)} p_D(R_{Di}, t_{Dl}) + \sum_{l=2}^{M} g_{jl} p_D(R_{Di}, t_{Dl}) \right), \tag{37}$$

where

$$g_{jl} = \sum_{k=1}^{M} c_k^{1j} \bar{A}_{kl}^{(2)} (t_{Dj} - t_{Dk})^{1-\gamma}. \tag{38}$$

The initial condition of infinite reservoir in the DQM form is

$$p_D(R_D, t_{Dj1}) = 0, \tag{39}$$

which is applied in Eq. (37).

and finally, we have

$$J = \sum_{l=2}^{M} v_{jl} p_D(R_{Di}, t_{Dj}), \tag{40}$$

where

$$v_{jl} = \frac{1}{(1-\gamma)\Gamma(1-\gamma)} \left(t_{Dj}^{1-\gamma} \bar{A}_{jl}^{(1)} + g_{jl} \right). \tag{41}$$

Equating Eqs. (34) and (40), a linear algebraic system of equations will be obtained that can be solved by any standard method (direct and iterative methods) such as Gaussian elimination method (direct), *LU* decomposition (direct), Gauss–Seidel (iterative) method, or successive over-relaxation (SOR) iteration (iterative).

4.2. Bounded circular reservoir case

Using DQM, the left-hand side of Eq. (25) for finite reservoirs can be expressed as:

$$I = \sum_{k=1}^{N} u_{ik} p_D(r_{Dk}, t_{Dj}), \tag{42}$$

where

$$u_{ik} = \frac{A_{ik}^{(2)}}{r_i^{\theta}} + \frac{d-\theta-1}{r_i^{\theta+1}} A_{ik}^{(1)}, \tag{43}$$

for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$

Application of the DQ method to discretize the boundary condition of finite reservoir case gives

$$\sum_{k=1}^{N} A_{1k}^{(1)} p_D(r_{Dk}, t_{Dj}) = 1, \tag{44}$$

$$\sum_{k=1}^{N} A_{Nk}^{(1)} p_D(r_{Dk}, t_{Dj}) = 0, \tag{45}$$

for $j = 1, 2, \ldots, M$

From Eqs. (44) and (45), we have

$$p_D(r_{D1}, t_{Dj}) = \frac{A_{NN}^{(1)} + \sum_{k=2}^{N-1} (A_{1N}^{(1)} A_{Nk}^{(1)} - A_{NN}^{(1)} A_{1k}^{(1)}) p_D(r_{Dk}, t_{Dj})}{A_{11}^{(1)} A_{NN}^{(1)} - A_{1N}^{(1)} A_{N1}^{(1)}}, \tag{46}$$

$$p_D(r_{DN}, t_{Dj}) = \frac{-A_{N1}^{(1)} + \sum_{k=2}^{N-1} (A_{N1}^{(1)} A_{1k}^{(1)} - A_{11}^{(1)} A_{Nk}^{(1)}) p_D(r_{Dk}, t_{Dj})}{A_{11}^{(1)} A_{NN}^{(1)} - A_{1N}^{(1)} A_{N1}^{(1)}}. \tag{47}$$

Substitution of Eqs. (46) and (47) into Eq. (42) gives

$$I = \frac{u_{i1} A_{NN}^{(1)} - u_{iN} A_{N1}^{(1)}}{A_{11}^{(1)} A_{NN}^{(1)} - A_{1N}^{(1)} A_{N1}^{(1)}} + \sum_{k=2}^{N-1} \left(u_{ik} + u_{i1} \frac{A_{1N}^{(1)} A_{Nk}^{(1)} - A_{NN}^{(1)} A_{1k}^{(1)}}{A_{11}^{(1)} A_{NN}^{(1)} - A_{1N}^{(1)} A_{N1}^{(1)}} + u_{iN} \frac{A_{N1}^{(1)} A_{1k}^{(1)} - A_{11}^{(1)} A_{Nk}^{(1)}}{A_{11}^{(1)} A_{NN}^{(1)} - A_{1N}^{(1)} A_{N1}^{(1)}} \right) p_D(r_{Dk}, t_{Dj}). \tag{48}$$

The right-hand side of Eq. (25) can be written by Eq. (40). Again, equating Eqs. (48) and (40) yields a linear algebraic system of equations that can be solved by some standard methods. These standard methods were mentioned in the previous case.

4.3. Constant-pressure outer boundary case

The discretization of constant-pressure outer boundary reservoirs can be expressed by Eqs. (42) and (43). The boundary conditions of this case can be represented by Eq. (44) and the following equation:

$$p_D(r_{DN}, t_{Dj}) = 0, \tag{49}$$

for $j = 1, 2, \ldots, M$

Substitution of Eqs. (44) and (49) into Eq. (42) gives

$$I = \frac{u_{i1}}{A_{11}^{(1)}} + \sum_{k=2}^{N-1} \left(u_{ik} - \frac{A_{1k}^{(1)}}{A_{11}^{(1)}} u_{i1} \right) p_D(r_{Dk}, t_{Dj}). \tag{50}$$

The right-hand side of Eq. (25) can be written by Eq. (40). Similar to two previous cases, equating Eqs. (50) and (40) yields a linear algebraic system of equations that can be solved by some standard methods.

In the above DQ discretization, the grid points are the cosine type (or the Gauss–Lobatto–Chebyshev) points given by

$$r_{Di} = \frac{1}{2} \left(1 - \cos \left(\frac{i-1}{N-1} \right) \pi \right) (r_{eD} - 1) + 1, \tag{51}$$

for bounded and constant-pressure outer boundary reservoirs, and

$$R_{Di} = \frac{1}{2} \left(1 - \cos \left(\frac{i-1}{N-1} \right) \pi \right), \tag{52}$$

for infinite reservoirs.

Razminia et al. [22] analyzed diffusion equation using DQM and compared with the other methods. Their results show that DQM is an accurate and stable approach for diffusion equation.

The numerical results of this method are presented in Figs. 1–9, which are shown as type curves without wellbore storage and skin effects. Three different discretizations $N \times M = \{25 \times 25, 35 \times 25, 35 \times 35\}$ are considered for all three cases (infinite, bounded, and constant-pressure outer boundary). Figs. 1–3 show the analysis results for infinite reservoirs. The numerical results of DQ and GIQ for the Euclidean model are given in Fig. 1 ($d = 2$, $\theta = 0$ ($\gamma = 1$)). Figs. 2 and 3 show the results of the given approach for two different cases of FFD model $d = \{1.8, 1.6\}$ and $\theta = \{0.2, 0.4\}$ ($\gamma = \{0.9090, 0.8333\}$). The applicability of DQ and GIQ for closed-circle reservoir is presented in Figs. 4–6. Fig. 4 shows the numerical results of the Euclidean model,

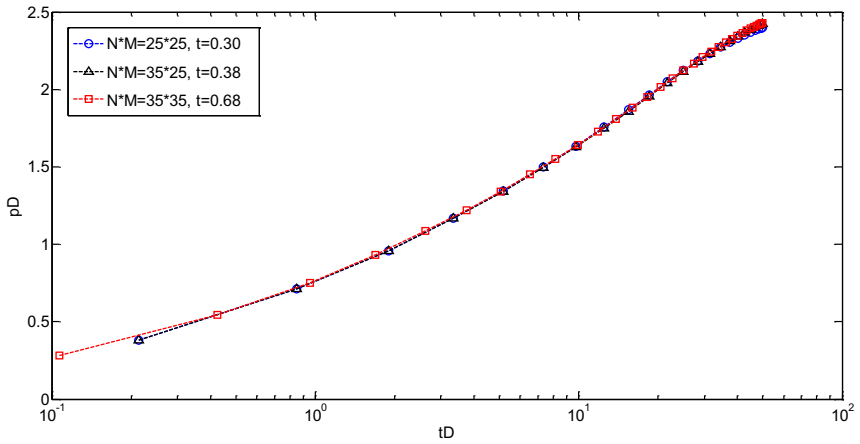

Fig. 1. Plot of p_D vs. t_D for infinite reservoir case with $d = 2$, $\theta = 0$ ($\gamma = 1$).

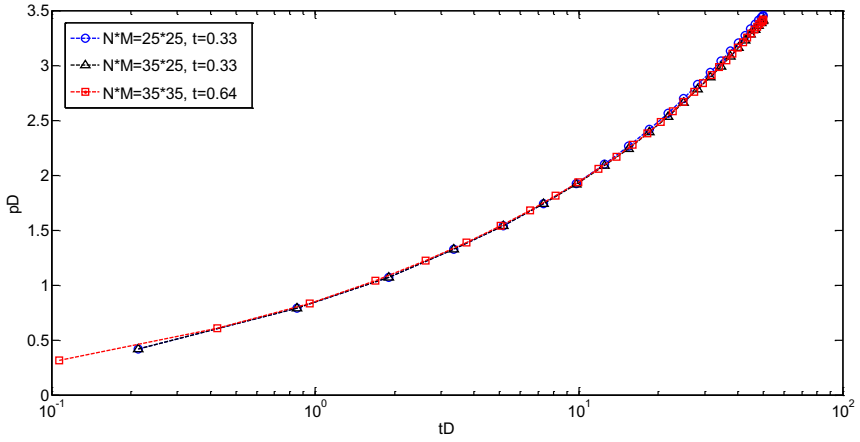

Fig. 2. Plot of p_D vs. t_D for infinite reservoir case with $d = 1.8$, $\theta = 0.2$ ($\gamma = 0.9090$).

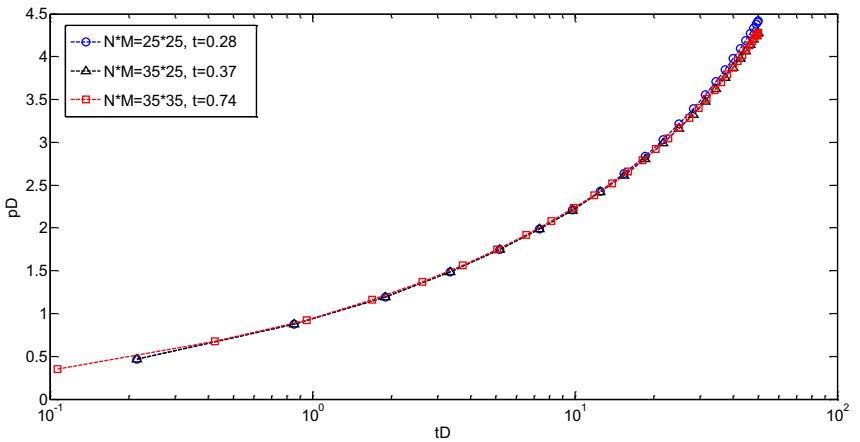

Fig. 3. Plot of p_D vs. t_D for infinite reservoir case with $d = 1.6$, $\theta = 0.4$ ($\gamma = 0.8333$).

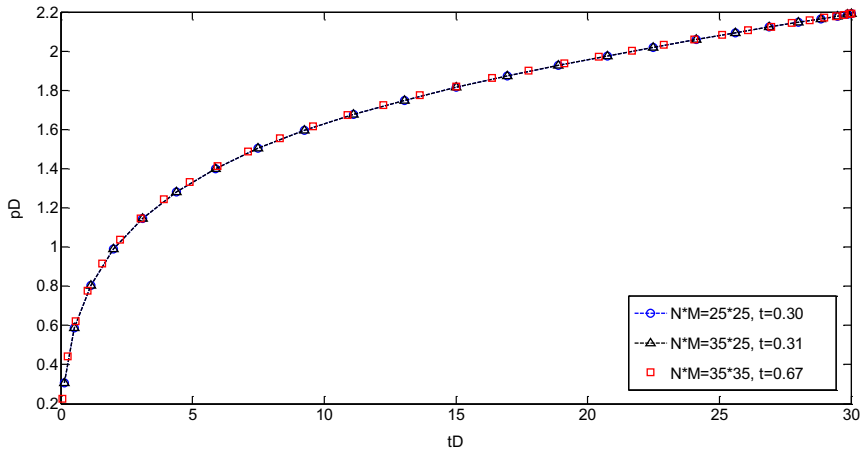

Fig. 4. Plot of p_D vs. t_D for bounded circular reservoir case ($r_{eD} = 10$) with $d = 2$, $\theta = 0$ ($\gamma = 1$).

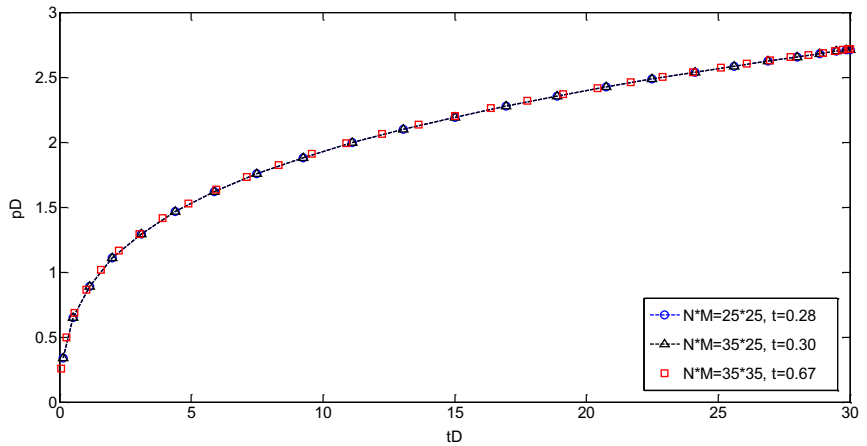

Fig. 5. Plot of p_D vs. t_D for bounded circular reservoir case ($r_{eD} = 10$) with $d = 1.8$, $\theta = 0.2$ ($\gamma = 0.9090$).

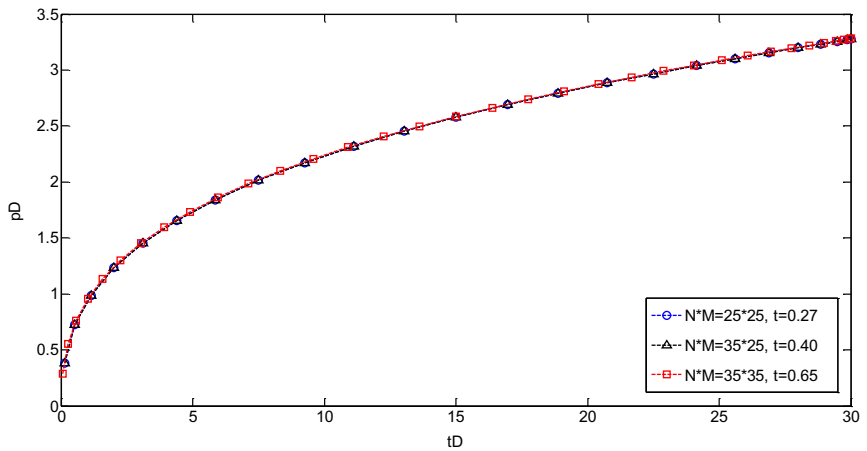

Fig. 6. Plot of p_D vs. t_D for bounded circular reservoir case ($r_{eD} = 10$) with $d = 1.6$, $\theta = 0.4$ ($\gamma = 0.8333$).

where ($d = 2$, $\theta = 0$ ($\gamma = 1$)). Figs. 5 and 6 present the results of FFD model for two different cases $d = \{1.8, 1.6\}$ and $\theta = \{0.2, 0.4\}$ ($\gamma = \{0.9090, 0.8333\}$). Finally, DQ and GIQ are applied to generalized diffusion equation for constant-pressure boundary reservoir, where the numerical results are shown in Figs. 7–9. The numerical analysis of the Euclidean model is shown in Fig. 7

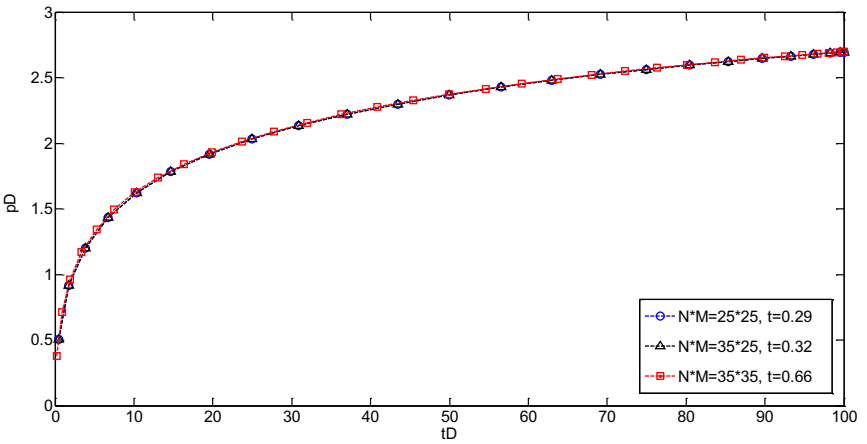

Fig. 7. Plot of p_D vs. t_D for constant-pressure outer boundary case ($r_{eD} = 20$) with $d = 2$, $\theta = 0$ ($\gamma = 1$).

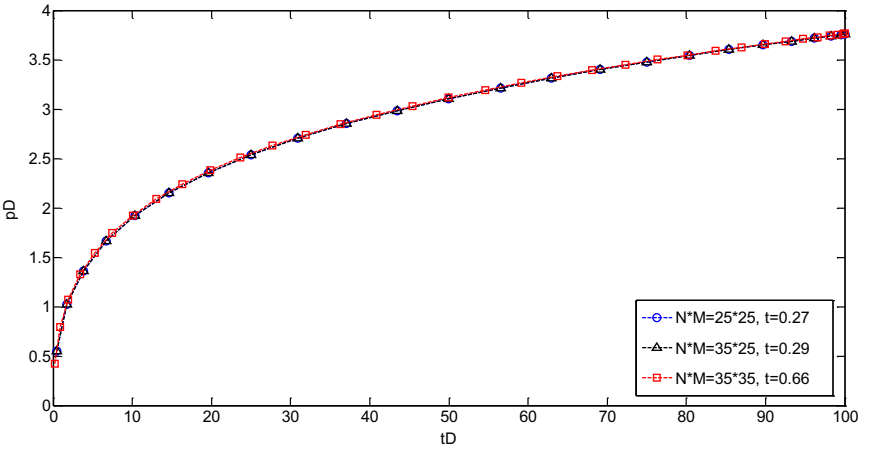

Fig. 8. Plot of p_D vs. t_D for constant-pressure outer boundary case ($r_{eD} = 20$) with $d = 1.8$, $\theta = 0.2$ ($\gamma = 0.9090$).

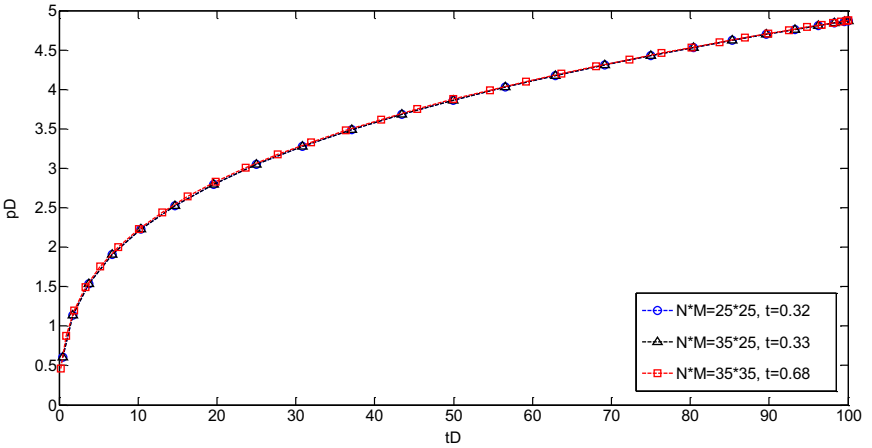

Fig. 9. Plot of p_D vs. t_D for constant-pressure outer boundary case ($r_{eD} = 20$) with $d = 1.6$, $\theta = 0.4$ ($\gamma = 0.8333$).

($d = 2$, $\theta = 0$ ($\gamma = 1$)). The numerical results of FFD model are depicted in Figs. 8 and 9 for two different cases $d = \{1.8, 1.6\}$ and $\theta = \{0.2, 0.4\}$ ($\gamma = \{0.9090, 0.8333\}$). The amounts of computation times are <1 s in all cases, which implies the great applicability of this method in solving such fractional PDEs. The presented results of all three cases indicate that as the fractal topological dimension (d) decreases and the fractal dynamical index (θ) increases, the dimensionless pressure increases. This indicates the faster diffusion in Euclidean reservoirs compared to fractal reservoirs.

As can be seen from Figs. 1–9, the grid numbers have no effect on the accuracy and stability of numerical results. This implies the accuracy of DQ and GIQ which can be used as practical methods to physical problems. The small amounts of computation time for DQ and GIQ demonstrates the fast rate of convergence. On the other hand, the stability and consistency of DQ and GIQ, which are necessary conditions of convergence, are independent of the number of grid points. The presented results, which are shown in Figs. 1–9, indicate that the different values of fractal topological dimension (d) and fractal dynamical index (θ) have no effect on the applicability of DQ and GIQ methods.

Based on the results obtained in Ref [22], the DQM can be used as a practical and powerful method to solve the linear PDEs because of two important reasons. First, the DQM has a fast rate of convergence in comparison with the other numerical methods (e.g. finite difference method) because the acceptable results can be obtained by small number of grid points. Second, the stability and consistency of the other numerical methods (e.g. finite difference method) depend on the number of grid points, whereas the stability and consistency of DQM are independent of the number of grid points.

5. Conclusions

This work introduces a numerical approach to show a new application of DQM to generalized diffusion equation with three geometry forms of boundaries. By generalized diffusion equation we mean the fractional order PDE of the equation. Examples demonstrate the fast rate of convergence with few numbers of grid points. The amounts of computation times are small in all cases, which is suitable and appropriate in solving such PDEs and can be considered as a good approach for practical problems. The accuracy of the results with a few number of grid points shows that DQM can be used as an unconditionally stable and efficient method for practical applications.

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