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Abundant new solutions of the transmission of nerve impulses of an excitable system

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Abstract This research investigates the dynamical behavior of the transmission of nerve impulses of a nervous system (the neuron) by studying the computational solutions of the FitzHugh–Nagumo equation that is used as a model of the transmission of nerve impulses. For achieving our goal, we employ two recent computational schemes (the extended simplest equation method and Sinh–Cosh expansion method) to evaluate some novel computational solutions of these models. Moreover, we study the stability property of the obtained solutions to show the applicability of them in life. For more explanation of this transmission, some sketches are given for the analytical obtained solutions. A comparison between our results and that obtained in previous work is also represented and discussed in detail to show the novelty for our solutions. The performance of the two used methods shows power, practical and their ability to apply to other nonlinear partial differential equations.

1 Introduction

Recently, the focus of many researchers is on biomathematics sciences. This branch of science represents many distinct data about the biological phenomena such as DNA, bacteria cell and its distribution, viruses, nerve system and the transmission of its impulses and so on. These vital issues have been formulated in mathematical structure based on collecting data from biological experiments or statistics to allow mathematical studying and investigations that are usually used in the construction of these biological phenomena in isolation by using modern experimental biology. The properties of these biological and impact factor of them are represented in the mathematical formula as functions and parameters. Solving these formulas gives accurate solutions that are used to improve the point of view for these models and also to control them by controlling the parameters.

According to the arising in the number of the mathematical and biological models and increasing of the attention for these models, many accurate schemes have been being formulated to study the computational and numerical solutions of them. These computational methods investigate the traveling wave solutions of these models that are considered as one of the most important motivations for development and derive new numerical schemes that

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are employed to evaluate the approximate solutions of these models. The examples of these schemes are the complex hyperbolic method, the generalized method of Riccati equation, Fan-expansion method, extended Fan-expansion method, the Jacobi elliptic-expansion method, Exp-expansion method, tanh-expansion method, Sinh–Cosh expansion method, tanh–sech expansion method, variational iteration method, the tanh-function method, homotopy disorder, Adomian analysis, Khater method, modified Khater method, the exp $(-\varphi(\Theta))$—expansion method, the modified simplest equation method, B-spline schemes and so on [1,3–5,10–13,18,19,21,23–29,32,33,35–37,39–44,46–48,52,54].

In this paper, we study one of the biological mathematical models that discusses a prototype of an excitable system. We study the transmission of the nerve impulses (neuron) in its mathematical formulate by employing the extended simplest equation method and Sinh–Cosh expansion method. This mathematical model is known with the FitzHugh–Nagumo (FN) equation. This model is also considered as an other version of the Hodgkin–Huxley model that is given by [20,45]

$$\begin{cases} \mathcal{H}_i = \varrho_i\,(\mathcal{L}_m - \mathcal{L}_i), \\ \mathcal{H} = \mathcal{M}_m\,\frac{d\,\mathcal{L}_m}{d\,t} + \varrho_K\,(\mathcal{L}_m - \mathcal{L}_K) + \varrho_{N_a} \\ \quad \times(\mathcal{L}_m - \mathcal{L}_{N_a}) + \varrho_i\,(\mathcal{L}_m - \mathcal{L}_{N_i}), \\ \mathcal{H}_c = \mathcal{M}_m\,\frac{d\,\mathcal{L}_m}{d\,t}, \end{cases} \tag{1}$$

where $[\varrho_i,\ \mathcal{L}_{N_a},\ \mathcal{L}_K,\ \mathcal{L}_i,\ \mathcal{L}_m,\ \varrho_l,\ n,\ \varrho_n,\ \mathcal{M}_m]$, respectively, represent the leak conductance per unit area, sodium reversal potentials, the potassium, membrane potential, ion pumps, leak channels, the specific ion channel, voltage-gated ion channels and the lipid bilayer. This system is used to describe the deactivation and activation dynamics of a neuron. This model is named with this name according to Richard FitzHugh (1922–2007). He created this system in 1961 with the help of J. Nagumo et al. who proved the equivalent circuit. This system is used to describe a prototype of an excitable system, and it is considered as a model of a relaxation oscillator. The original form of the model is given by [15–17]

$$\begin{cases} \mathcal{R}' = \mathcal{R} - \frac{\mathcal{R}^3}{3} - \mathcal{H} + \Gamma_{\mathcal{EXT}}, \\ \Theta\,\mathcal{H}' = \mathcal{R} + \Omega - \Upsilon\,\mathcal{H}, \end{cases} \tag{2}$$

where $\Gamma_{\mathcal{EXT}}, \mathcal{H},\ \Theta,\ \Omega,\ \Upsilon$ are arbitrary constants, while $\mathcal{H},\ \mathcal{R}$ receptively, represent the right and left branch of the cubic nullcline. It is also called as Bonhoeffer–van der Pol oscillator when $\Omega = \Upsilon = 0$.

The mathematical formula of FN equation that will be investigated in our paper is given by [14,34,38,50]

$$\mathcal{F}_{x\,x} - \mathcal{F}\,(1 - \mathcal{F})\,(\beta - \mathcal{F}) - \mathcal{F}_t = 0, \tag{3}$$

where β is arbitrary constant and $\mathcal{F} = \mathcal{F}(x, t)$ is function of x, t that describe the displacement and time. Equation (3) can be reduced to Newell–Whitehead ($\mathcal{NW}$) equation when $\beta = 0$.

The rest of the sections of this paper are ordered as follows. Section 2.1 applies the extended simplest equation method and Sinh–Cosh expansion method [6–9,22,30,31,49,51,53] to

FN equation to calculate the exact and solitary wave solutions. Moreover, some plots are represented to show more physical properties of the transmission of the nerve impulses. Section 3 studies the stability property of the obtained solutions and their applicability in different studies. Section 4 shows the novelty of our obtained solutions by representing a comparison between them and that obtained in previous research papers. Section 5 explains the conclusion of the whole paper.

2 Application

This section employs the extended simplest equation method and Sinh–Cosh expansion method to evaluate the solitary wave solutions of the FN equation. Using the next transformation $[\mathcal{F} = \mathcal{F}(x,t) = \mathcal{F}(\wp),\ \wp = k\,x + \omega\,t]$ on Eq. (3), leads to transform it to the following ordinary differential equation

$$k^2\,\mathcal{F}'' - \mathcal{F}\,(1-\mathcal{F})\,(\rho - \mathcal{F}) - \omega\,\mathcal{F}' = 0. \tag{4}$$

Calculating the balance value in Eq. (4) according to the nonlinear term and highest order derivative term, yields $N = 1$.

2.1 Extended simplest equation method

According to the balance value and the general solution that is suggested by the extended simplest equation method, the solution of Eq. (4) is given in the following formula:

$$\mathcal{F}(\wp) = \sum_{i=-n}^{n} a_i\, f(\wp)^i = \frac{a_{-1}}{f(\wp)} + a_0 + a_1\, f(\wp), \tag{5}$$

where $[a_i,\ (i = -1,\ 0,\ 1)]$. Additionally, $f(\wp)$ follows the following ODE:

$$f'(\wp) = \delta f(\wp)^2 + \rho f(\wp) + \chi,$$

where $\delta,\ \rho,\ \chi$ are arbitrary constants. Substituting Eq. (5) along its derivatives into Eq. (4), collecting all coefficients of the same power of $[f(\wp)^i,\ (i = -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3)]$ and equating them to zero, lead to a system of algebraic equations. Solving this system yields:

Family I:

$$\Bigg[a_0 \to \frac{\rho^2}{2\sqrt{\rho^4 - 4\delta\rho^2\chi}} + \frac{1}{2}, a_{-1} \to \frac{\rho\chi}{\sqrt{\rho^4 - 4\delta\rho^2\chi}}, a_1 \to 0, k \to -\frac{1}{\sqrt{2}\sqrt{\rho^2 - 4\delta\chi}}, \omega \to \frac{\rho - 2\beta\rho}{2\sqrt{\rho^4 - 4\delta\rho^2\chi}},$$
$$where \left(\rho^2 > 4\delta\chi,\ \rho \neq 0,\ \chi \neq 0,\ \rho \neq 2\beta\rho\right)\Bigg].$$

Thus, the soliton solutions of the nonlinear FN equation are evaluated in the following formulas:

For $[4\,\delta\,\chi\ <\ \rho^2]$

$$\mathcal{F}_1(x,t) = -\frac{2\delta\rho\chi}{\sqrt{\rho^4 - 4\delta\rho^2\chi}\left(\rho + \sqrt{\rho^2 - 4\delta\chi}\tanh\left(\frac{1}{2}\sqrt{\rho^2 - 4\delta\chi}\left(\frac{(1-2\beta)\rho t}{2\sqrt{\rho^4 - 4\delta\rho^2\chi}} - \frac{x}{\sqrt{2}\sqrt{\rho^2 - 4\delta\chi}} + \vartheta\right)\right)\right)}$$

$$+\frac{\rho^2}{2\sqrt{\rho^4-4\delta\rho^2\chi}}+\frac{1}{2}, \tag{6}$$

$$\mathcal{F}_2(x,t)=-\frac{2\delta\rho\chi}{\sqrt{\rho^4-4\delta\rho^2\chi}\left(\rho+\sqrt{\rho^2-4\delta\chi}\coth\left(\frac{1}{2}\sqrt{\rho^2-4\delta\chi}\left(\frac{(1-2\beta)\rho t}{2\sqrt{\rho^4-4\delta\rho^2\chi}}-\frac{x}{\sqrt{2}\sqrt{\rho^2-4\delta\chi}}+\vartheta\right)\right)\right)}$$
$$+\frac{\rho^2}{2\sqrt{\rho^4-4\delta\rho^2\chi}}+\frac{1}{2}. \tag{7}$$

Family II:

$$\left[a_0\to\frac{\rho^2}{2\sqrt{\rho^4-4\delta\rho^2\chi}}+\frac{1}{2},a_{-1}\to 0,a_1\to\frac{\delta\rho}{\sqrt{\rho^4-4\delta\rho^2\chi}},k\to\right.$$
$$-\frac{1}{\sqrt{2}\sqrt{\rho^2-4\delta\chi}},\omega\to\frac{(2\beta-1)\rho}{2\sqrt{\rho^4-4\delta\rho^2\chi}},$$
$$\left.\mathit{where}\ \left(\rho^2>4\delta\chi,\ \rho\neq 0,\ \rho\neq 2\beta\rho\right)\right].$$

Thus, the soliton solutions of the nonlinear FN equation are evaluated in the following formulas:

Case I $\left(\chi=0\right)$ For $\left[\rho>0\right]$:

$$\mathcal{F}_3(x,t)=\frac{1}{2}\left(\left(-\frac{2}{\delta\exp\left(\rho\left(\frac{(2\beta-1)\rho t}{2\sqrt{\rho^4}}-\frac{x}{\sqrt{2}\sqrt{\rho^2}}+\vartheta\right)\right)-1}-1\right)+1\right), \tag{8}$$

$$\mathcal{F}_4(x,t)=\frac{1}{2}\left(\frac{1}{\rho}\left(2\delta\left(\frac{1}{\delta\exp\left(\rho\left(\frac{(2\beta-1)\rho t}{2\sqrt{\rho^4}}-\frac{x}{\sqrt{2}\sqrt{\rho^2}}+\vartheta\right)\right)+1}-1\right)+\rho\right)+1\right) \tag{9}$$

Case II

For $[4\delta\chi<\rho^2]$

$$\mathcal{F}_5(x,t)=\frac{1}{2}-\frac{\sqrt{\rho^4-4\delta\rho^2\chi}\tanh\left(\frac{1}{2}\sqrt{\rho^2-4\delta\chi}\left(\frac{(2\beta-1)\rho t}{2\sqrt{\rho^4-4\delta\rho^2\chi}}-\frac{x}{\sqrt{2}\sqrt{\rho^2-4\delta\chi}}+\vartheta\right)\right)}{2\rho\sqrt{\rho^2-4\delta\chi}}, \tag{10}$$

$$\mathcal{F}_6(x,t)=\frac{1}{2}-\frac{\sqrt{\rho^4-4\delta\rho^2\chi}\coth\left(\frac{1}{2}\sqrt{\rho^2-4\delta\chi}\left(\frac{(2\beta-1)\rho t}{2\sqrt{\rho^4-4\delta\rho^2\chi}}-\frac{x}{\sqrt{2}\sqrt{\rho^2-4\delta\chi}}+\vartheta\right)\right)}{2\rho\sqrt{\rho^2-4\delta\chi}}. \tag{11}$$

Family III:

$$\left[a_0\to\frac{\rho-\sqrt{\rho^2-4\delta\chi}}{2\rho},a_{-1}\to-\frac{\chi}{\rho},a_1\to 0,k\to-\frac{1}{\sqrt{2}\rho},\right.$$
$$\omega\to\frac{\sqrt{\rho^2-4\delta\chi}-2\rho}{2\rho^2},\beta\to 1-\frac{\sqrt{\rho^2-4\delta\chi}}{\rho},$$

$$\text{where } \left(\rho^2 > 4\delta\chi,\ \rho \neq 0,\ \chi \neq 0,\ \rho \neq \sqrt{\rho^2 - 4\delta\chi}\right)\Big].$$

Thus, the soliton solutions of the nonlinear FN equation are evaluated in the following formulas:

For $[4\delta\chi < \rho^2]$

$$\mathcal{F}_7(x,t) = -\frac{\sqrt{\rho^2-4\delta\chi}}{2\rho} + \frac{2\delta\chi}{\rho^2 + \rho\sqrt{\rho^2-4\delta\chi}\tanh\left(\frac{\sqrt{\rho^2-4\delta\chi}\left(t\left(\sqrt{\rho^2-4\delta\chi}-2\rho\right)+\rho\left(2\rho\vartheta-\sqrt{2}x\right)\right)}{4\rho^2}\right)} + \frac{1}{2}, \tag{12}$$

$$\mathcal{F}_8(x,t) = -\frac{\sqrt{\rho^2-4\delta\chi}}{2\rho} + \frac{2\delta\chi}{\rho^2 + \rho\sqrt{\rho^2-4\delta\chi}\coth\left(\frac{\sqrt{\rho^2-4\delta\chi}\left(t\left(\sqrt{\rho^2-4\delta\chi}-2\rho\right)+\rho\left(2\rho\vartheta-\sqrt{2}x\right)\right)}{4\rho^2}\right)} + \frac{1}{2}. \tag{13}$$

Family IV:

$$\Big[a_0 \to \frac{\rho-\sqrt{\rho^2-4\delta\chi}}{2\rho}, a_{-1} \to 0, a_1 \to \frac{\delta}{\rho}, k \to -\frac{1}{\sqrt{2}\rho},$$
$$\omega \to \frac{\sqrt{\rho^2-4\delta\chi}+2\rho}{2\rho^2}, \beta \to -\frac{\sqrt{\rho^2-4\delta\chi}}{\rho},$$
$$\text{where } \left(\rho^2 > 4\delta\chi,\ \rho \neq 0,\ \rho \neq \sqrt{\rho^2-4\delta\chi}\right)\Big].$$

Thus, the soliton solutions of the nonlinear FN equation are evaluated in the following formulas:

Case I ($\chi = 0$)

For $[\rho > 0]$:

$$\mathcal{F}_9(x,t) = -\frac{\rho}{2\sqrt{\rho^2}} + \frac{\delta}{e^{-\frac{\rho t}{2\sqrt{\rho^2}}-t+\frac{x}{\sqrt{2}}-\rho\vartheta} - \delta} + \frac{1}{2}, \tag{14}$$

$$\mathcal{F}_{10}(x,t) = \frac{1}{2} - \frac{1}{2\rho}\left[\sqrt{\rho^2} + \frac{2\delta^2}{\delta + e^{-\frac{\rho t}{2\sqrt{\rho^2}}-t+\frac{x}{\sqrt{2}}-\rho\vartheta}}\right]. \tag{15}$$

Case II

For $[4\delta\chi < \rho^2]$

$$\mathcal{F}_{11}(x,t) = -\frac{\sqrt{\rho^2-4\delta\chi}}{2\rho}\left(\tanh\left(\frac{\sqrt{\rho^2-4\delta\chi}\left(t\left(\sqrt{\rho^2-4\delta\chi}+2\rho\right)+\rho\left(2\rho\vartheta-\sqrt{2}x\right)\right)}{4\rho^2}\right)+1\right), \tag{16}$$

$$\mathcal{F}_{12}(x,t) = -\frac{\sqrt{\rho^2-4\delta\chi}}{2\rho}\left(\coth\left(\frac{\sqrt{\rho^2-4\delta\chi}\left(t\left(\sqrt{\rho^2-4\delta\chi}+2\rho\right)+\rho\left(2\rho\vartheta-\sqrt{2}x\right)\right)}{4\rho^2}\right)+1\right). \tag{17}$$

where ϑ is arbitrary constant.

2.2 Solitary wave solutions via Sinh–Cosh expansion method

According to the balance value and the general solution that is suggested by the extended simplest equation method, the solution of Eq. (4) is given in the following formula:

$$\mathcal{F}(\wp)=\sum_{i=1}^{n}\sinh^{i-1}(\wp)\,(a_i\sinh(\wp)+b_i\cosh(\wp))+a_0=a_1\sinh(\wp)+a_0+b_1\cosh(\wp), \tag{18}$$

where $[a_0,\ a_1,\ b_1]$ are arbitrary constants. Substituting Eq. (18) along its derivatives into Eq. (4), collecting all coefficients of the same power of $[\sinh(\wp),\ \sinh^2(\wp),\ \sinh^3(\wp),\ \cosh(\wp),\ \sinh(\wp)\cosh(\wp),\ \sinh^2(\wp)\cosh(\wp)]$, and equating them to zero, lead to a system of algebraic equations. Solving this system yields:

Family I:

$$\left[a_0\to 0,\ a_1\to\frac{ib_1}{\sqrt{3}},\ k\to\frac{\sqrt{3b_1^2-2}}{\sqrt{2}},\ \omega\to-\frac{1}{2}i\sqrt{3}b_1^2,\ \beta\to-1,\ where\ \left(b_1<\sqrt{\frac{4}{6}},\ b_1\neq 0\right)\right].$$

Thus, the soliton solutions of the nonlinear FN equation are evaluated in the following formulas:

$$\mathcal{F}_{13}(x,t)=\frac{1}{3}\left(\sqrt{3}b_1\sin\left(\frac{1}{2}\sqrt{3}b_1^2t+\frac{1}{2}i\sqrt{6b_1^2-4x}\right)+3b_1\cos\left(\frac{1}{2}\sqrt{3}b_1^2t+\frac{1}{2}i\sqrt{6b_1^2-4x}\right)\right). \tag{19}$$

Family II:

$$\left[a_0\to\frac{1}{2},\ a_1\to\frac{ib_1}{\sqrt{3}},\ k\to\frac{1}{2}\sqrt{6b_1^2-1},\ \omega\to-\frac{1}{2}i\sqrt{3}b_1^2,\ \beta\to\frac{1}{2},\ where\ \left(b_1<\sqrt{\frac{1}{6}},\ b_1\neq 0\right)\right].$$

Thus, the soliton solutions of the nonlinear FN equation are evaluated in the following formulas:

$$\begin{aligned}\mathcal{F}_{14}(x,t)=\frac{1}{6}\Bigg(&2\sqrt{3}b_1\sin\left(\frac{1}{2}\sqrt{3}b_1^2t+\frac{1}{2}i\sqrt{6b_1^2-1x}\right)\\&+6b_1\cos\left(\frac{1}{2}\sqrt{3}b_1^2t+\frac{1}{2}i\sqrt{6b_1^2-1x}\right)+3\Bigg).\end{aligned} \tag{20}$$

Family III:

$$\left[a_0\to 1,\ a_1\to\frac{ib_1}{\sqrt{3}},\ k\to\frac{\sqrt{3b_1^2-2}}{\sqrt{2}},\ \omega\to-\frac{1}{2}i\sqrt{3}b_1^2,\ \beta\to 2,\ \textbf{where}\ \left(b_1<\sqrt{\frac{4}{6}},\ b_1\neq 0\right)\right].$$

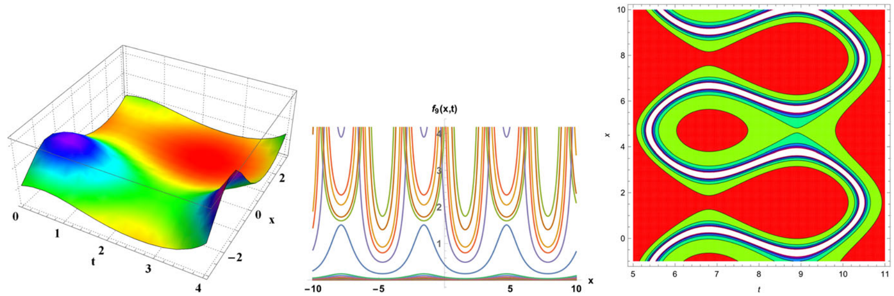

Fig. 1 Breath-soliton wave in three different forms of Eq. (14) for $[\delta = 1,\ \rho = 4,\ \vartheta = -1]$

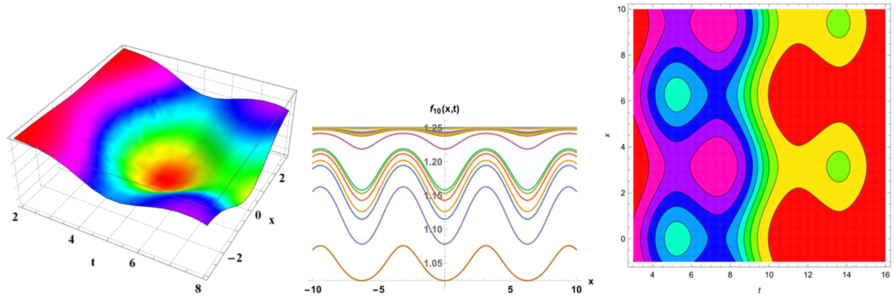

Fig. 2 Dark-soliton wave in three different forms of Eq. (15) for $[\delta = 1,\ \rho = -4,\ \vartheta = -1]$

Thus, the soliton solutions of the nonlinear FN equation are evaluated in the following formulas (Fig. 1):

$$\mathcal{F}_{15}(x,t) = \frac{1}{3}\left(\sqrt{3}b_1 \sin\left(\frac{1}{2}\sqrt{3}b_1^2 t + \frac{1}{2}i\sqrt{6b_1^2 - 4x}\right) + 3b_1 \cos\left(\frac{1}{2}\sqrt{3}b_1^2 t + \frac{1}{2}i\sqrt{6b_1^2 - 4x}\right) + 3\right). \tag{21}$$

3 Stability property

The stability property of some results is investigated in this section depending on the properties of the Hamiltonian system that gives the momentum $\varXi$ in the following formula (Figs. 2 and 3).

$$\varXi = \frac{1}{2}\int_{-\nu}^{\nu} \mathcal{F}^2(\wp)\, d\wp, \tag{22}$$

where $\mathcal{F}(\wp)$ is the solution of the FN equation. Thus, the necessary condition to make this solution stable one is derived in the next form:

$$\frac{\partial \varXi}{\partial \omega} > 0, \tag{23}$$

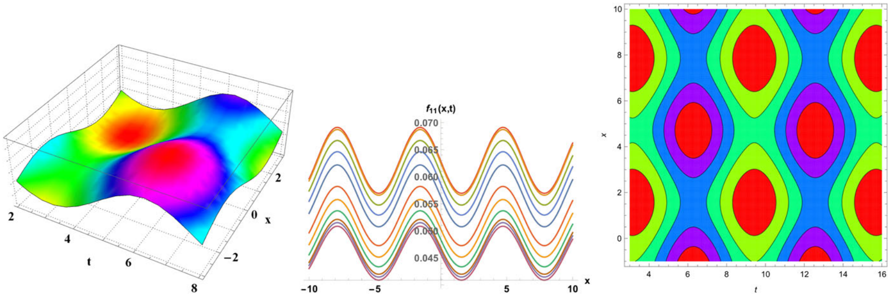

Fig. 3 Periodic-solitary wave in three different forms of Eq. (16) for $[\delta = 1,\ \rho = 5,\ \chi = 6,\ \vartheta = -1]$

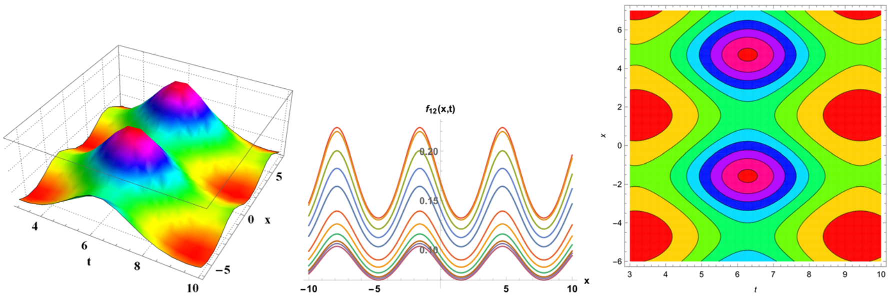

Fig. 4 Cone-soliton wave in three different forms of Eq. (17) for $[\delta = 1,\ \rho = 5,\ \chi = 6,\ \vartheta = -1]$

where ω is the wave velocity, so that the studying of the stability property of the FN equation takes the following steps (Figs. 4 and 5)

$$\begin{aligned}\varXi =&\frac{1}{10\sqrt{2}\omega}\Bigg[2\,\mathrm{Li}_2\left(-e^{5\omega+\frac{1}{\sqrt{2}}}\right)-2\,\mathrm{Li}_2\left(-e^{5\omega-\frac{1}{\sqrt{2}}}\right)-2\,\mathrm{Li}_2\left(-e^{\frac{1}{\sqrt{2}}-5\omega}\right)\\&+2\,\mathrm{Li}_2\left(-e^{-5\,\omega-\frac{1}{\sqrt{2}}}\right)+\left(10\omega+\sqrt{2}\right)\log\left(e^{5\omega+\frac{1}{\sqrt{2}}}+1\right)\\&-10\,\omega\,\log\left(e^{5\omega-\frac{1}{\sqrt{2}}}+1\right)-\left(\sqrt{2}-10\omega\right)\log\left(e^{\frac{1}{\sqrt{2}}-5\omega}+1\right)-\left(10\,\omega+\sqrt{2}\right)\\&\times\log\left(e^{-5\,\omega-\frac{1}{\sqrt{2}}}+1\right)+\sqrt{2}\,\log\left(e^{5\omega}+e^{\frac{1}{\sqrt{2}}}\right)-4\,\log\left(\cosh\left(\frac{1}{4}\left(10\omega+\sqrt{2}\right)\right)\right)\\&+4\log\left(\cosh\left(\frac{1}{4}\left(\sqrt{2}-10\omega\right)\right)\right)-1\Bigg]\end{aligned} \tag{24}$$

and thus

$$\frac{\partial \varXi}{\partial\omega}|_{\omega=\frac{11}{50}} = 0.3212360331 > 0.$$

Consequently, this solution is stable. Thus, using the same steps on the other obtained solutions yields a good investigation of the stability property of each of them.

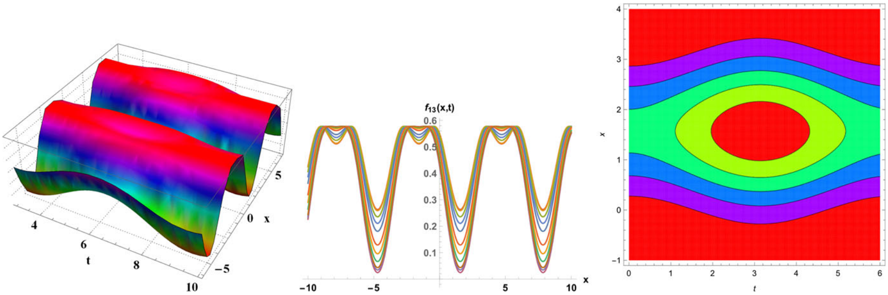

Fig. 5 Bright and dark-soliton wave in three different forms of Eq. (19) for $[b_1 = \frac{1}{2}]$

4 Results and discussion

This section gives a comparison between our two used method and that used in previous paper. Also, it gives a comparison between our results and that obtained by using these different methods.

1. **Comparison between the methods**
   In this part, we show the comparison between our two used methods and that employed in [2]
   - **The extended simplest equation method Vs the exp $-\varphi(\xi)$—expansion method:** Both methods are equal when $[e^{\varphi(\xi)} = f(\wp),\ \chi = 1,\ \rho = \lambda,\ \delta = \mu]$.
   - Sinh–Cosh expansion method Vs the exp $-\varphi(\xi)$—expansion method: Both methods are different.
2. **Comparison between the results**
   - Equation (12) is equal to Eq. (20) in [2] for $[\rho^2 - 4\,\delta\,\chi = \left(\frac{\lambda\,\rho\,\chi\,\delta}{\mu}\right)^2,\ \lambda = -\rho^2,\ C_1 = 0]$.
   - Equation (14) is equal to Eq. (21) in [2] when $[a_1\,\lambda = -1,\ \delta = -1,\ \lambda = 1,\ C_1 = 0]$.
   - All our other obtained solutions are new and different from that obtained in [2].

5 Conclusion

Two analytical methods were successfully employed to find exact and solitary wave solutions of the FN equation. These methods were the extended simplest equation method and Sinh–Cosh expansion method. Many solutions were obtained in different types such as dark, dark and bright, periodic, cone and breath solutions. These solutions were tested to investigate the stability property of them by using the characteristics of the Hamiltonian system. Moreover, a comparison between our solutions and that obtained in a previous research paper was investigated in detail. The performance of the used methods shows the effective power of them and their ability to apply other nonlinear evolution equations.

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