FREDHOLM TYPE INTEGRAL EQUATIONS

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ABSTRACT

FREDHOLM TYPE INTEGRAL EQUATIONS

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The aim of this thesis is to provide a comprehensive study on Fredholm Integral Equations and the methods to find exact solutions. We also seek to present some effective methods to find the exact solutions for linear and nonlinear Fredholm Integral Equations. Moreover, in order to study the convergence between the numerical solution and the exact solution, we applied the Newton-Kantorovich method as a model to find a numerical solution for a special type of Fredholm Integral Equation and compared the results with the exact solution. The results showed the accuracy of the numerical result and proximity of the exact results, thereby proving the effectiveness and simplicity of the Newton-Kantorovich method.

Keywords: Fredholm Integral Equations, First Kind, Second Kind, Special Kind, Analytical Methods of Solution, Newton-Kantorovich Method for Numerical Solution.

ÖZ

FREDHOLM TİPİ İNTEGRAL DENKLEMLER

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Bu tezin amacı, Fredholm Tümlevsel Denklemleri ve kesin çözümlerini bulma yöntemleri hakkında geniş kapsamlı bir çalışma yapmaktır. Bunun yanı sıra diğer bir amacımızda doğrusal olan ve olmayan Fredholm Tümlevsel Denklemlerinin kesin sonuçlarını bulmaya yarar bazı etkili yöntemlerde sunmaktır. Bundan başka, rakamsal çözüm ile kesin çözüm arasındaki yakınsaklığı irdelemek, özel türden olan bir Fredholm Tümlevsel Denkleminin rakamsal çözümünü bulmak için model teşkil etmek üzere Newton-Kantorovich yöntemini uyguladık ve sonuçları kesin çözüm ile karşılaştırdık. Sonuçlar rakamsal sonucun doğruluğunu ve kesin sonuçların yakınlığını gösterdi; böylece Newton-Kantorovich yönteminin etkililiği ve kolaylığı kanıtlanmış oldu.

Anahtar Kelimeler: FredholmTümlevsel Denklemler, Birinci Tür, İkinci Tür,Özel Tür, Çözümün Analitik Yöntemleri, Rakamsal Çözümün Newton-Kantorovich Yöntemi.

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TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

LIST OF TABLES

TABLES

CHAPTER 1

INTRODUCTION

1.1 Background

There is no doubt that integral equations have great importance in scientific and practical fields. The first integral equation was produced in 1825 by an Italian mathematician named Abel, who produced an integral equation relating to the problem of tautochrone [1]. In fact, the emergence of these equations was the result of the presence of a number of problems and issues in fields such as physics, heat and mass transfer, chemical engineering, as well as in economics, medicine, etc. [1,2,5].

Most integral equations closely connected with differential equations are Fredholm Integral Equations [4]. Therefore, Fredholm Integral Equations are derived from boundary value problems for differential equations and then solved by many simplified methods. A strong motivation existed to discover this kind of equation by Fredholm. The equations were renamed Fredholm Integral Equations after being discovered by Fredholm. It was the point of departure to resolve important obstacles which were impeding the development of mathematics [3]. Finally, we mention that Fredholm Integral Equations are also found in linear and non-linear forms, including the homogeneous and non-homogeneous variety [1, 2].

1.2 Organization of the Thesis

We can summarize what has been organized in this thesis as follows:

In Chapter 2, the study is about definitions and basic concepts that would show the way to enter into the world of integral equations of the Fredholm type in addition to the method which can derive the Fredholm Integral Equation from boundary value problems. In Chapter 3, the researcher provides some theorems which prove the existence and uniqueness of solutions to Fredholm Integral Equations.

Chapter 4 is dedicated to the study of the methods of finding solutions to the Fredholm Integral Equation and the system of Fredholm Integral Equations according to the classification of the Fredholm Integral Equation. It is worth mentioning that we will often use the Fredholm Integral Equation of degenerate or separable kernels.

Chapter 5 will present a study of numerical solutions to special nonlinear Fredholm Integral Equations using the Newton-Kantorovich method, after which the numerical results are compared with the analytical results.

Finally, in Chapter 6 expounds the conclusion that has been reached, according to the idea of thesis.

CHAPTER 2

DEFINITIONS

In this chapter, we will review some definitions and mathematical concepts for integral equations and we will classify and identify the types of integral equations. In particular, we will present the Fredholm Integral Equation and their types. Finally, we will discuss how to derive the Fredholm Integral Equation from boundary value problems with examples.

2.1 Basic Definition

Definition 1. [1-4] An equation which includes the integral of an unknown function $u(x)$ appearing inside the integral sign is called an integral equation. The general formula of integral equations is as follows:

$$u(x) = h(x) + \mu \int_{\varphi(x)}^{\delta(x)} F(x, y) u(y) dy , \tag{2.1}$$

where $F(x, y)$ represents the kernel of the integral equation, the limits of integration are $\phi(x)$ and $\delta(x)$ and μ is a fixed parameter [1,2]. The functions $h(x)$ and $f(x, y)$ are given. The limits of integration $\phi(x)$ and $\delta(x)$ might together be variables, constants or a combination of both, and they may be in one, two or more dimensions.

Definition 2. [1-4] In integral equations, when the unknown function inside the integral sign has an exponent of one, this integral equation is referred to as a linear equation for example:

$$u(x) = \frac{5}{2} x - \frac{7}{3} + \mu \int_0^1 (x - y) u(y) dy . \tag{2.2}$$

Definition 3. [1-4] In integral equations, when the unknown function inside the integral sign has an exponent not equaling one or includes nonlinear functions, such as *cos u*, e^u , *ln*(7 + *u*), this integral equation is called a nonlinear integral equation [3]; for example:

$$u(x) = 5 + \int_0^1 (1 + x - y)u^4(y)dy \,. \tag{2.3}$$

2.2 Classification of Integral Equations [1,2,4]

We can classify certain types of integral equations based on the kernel of the equation and the limits of integration. Therefore, in this chapter we will present the following four major types of integral equations [2,4]:

1. Fredholm Integral Equations
2. Volterra integral equations
3. Singular integral equations
4. Integro-differential equations

Because the subject of this thesis is Fredholm type Integral Equations, we will focus on these integral equations.

2.2.1 Fredholm integral equations

The integral equation (2.1) is called a Fredholm Integral Equation when $\phi(x) = a, \delta(x) = b$; *a*, *b* are constant; in other words, the limits of integration are fixed in the following form:

$$k(x)u(x) = h(x) + \mu\int_a^b f(x, y)u(y)dy \,. \tag{2.4}$$

It should be noted that there are three types of Fredholm Integral Equation according to the following:

2.2.1.1 Fredholm integral equations of the first kind

The integral equations (2.1) is called a Fredholm Integral Equation of the first kind when *k*(*x*)=0 as follows:

$$h(x) + \mu\int_a^b k(x, y)u(y)dy = 0 \quad . \tag{2.5}$$

2.2.1.2 Fredholm integral equations of the second kind

The integral equations (2.1) is called a Fredholm Integral Equation of the second kinds when $k(x)$=1 as follows

$$u(x) = h(x) + \mu\int_a^b k(x, y)u(y)dy \tag{2.6}$$

2.2.1.3 Fredholm integral equations of the third kind

If $k(x)$ is neither 0 nor 1, then (1.2) is referred to as a Fredholm Integral Equation of the third kind [2].

2.3 Homogeneity of Integral Equations [1,2,4]

Proceeding from the principle of homogeneity of integral equations, integral equations of the second kind can be classified as either homogeneous or non-homogeneous.

2.3.1 Homogeneous integral equations

Suppose that we have the Fredholm Integral Equation of the second kind as follows:

$$u(x) = h(x) + \mu\int_a^b k(x, y)u(y)dy\,. \tag{2.7}$$

The equation (2.7) is called homogeneous if the given function $h(x)$ is identically zero. The following is an example of such an equation:

$$u(x) = \int_0^1 (x - y)^2 u(y)dy\,. \tag{2.8}$$

In this equation $h(x)$=0; therefore, it is homogeneous.

2.3.2 Non- homogeneous integral equation

Fredholm Integral Equations of the second kind (2.7) are called non-homogeneous Fredholm Integral Equations if the function $h(x)$ is nonzero; for example

$$u(x) = \sin x + \int_0^1 xyu(y)dy\,. \tag{2.9}$$

In this equation, $h(x)$= $\sin x$. Therefore, it is non-homogeneous.

2.4 Converting Boundary Value Problem to Fredholm Integral Equations [2,4]

There is a connection between integral equations and boundary value problem where one can convert the boundary value problems to the Fredholm Integral Equation. In order to prove that, we will consider the following example

Example 2.1

Suppose that we have the following boundary value problem

$$y''(x) + g(x)y(x) = h(x), \quad 0 < x < 1, \tag{2.10}$$

with the following conditions:

$$y(0) = \alpha \, , \, y(1) = \beta \, . \tag{2.11}$$

We will show how we can reduce the boundary value problem to the Fredholm Integral Equation:

At first, we put

$$y''(x) = u(x) \, . \tag{2.12}$$

By integrating both sides of (2.12) from 0 to *x*, we get

$$\int_0^x y''(t)dt = \int_0^x u(t)dt \, . \tag{2.13}$$

The result is

$$y'(x) = y'(0) + \int_0^x u(t)dt \, . \tag{2.14}$$

It is noted that in a boundary value problem, the initial condition $y'(0)$ is not given. Therefore, by using the boundary condition at *x*=1, this condition will be determined later. Now, we will integrate both sides of (2.14) from 0 to *x* to get

$$y(x) = y(0) + xy'(0) + \int_0^x \int_0^x u(t)dt. \tag{2.15}$$

By substituting $y(0)=\alpha$, we obtain

$$y(x) = \alpha + xy'(0) + \int_0^x (x-t)u(t)dt. \tag{2.16}$$

As mentioned above, to determine $y'(0)$, we substitute *x*=1 into both sides of equation(2.16) and using the boundary condition at $y(1)=\beta$, we find

$$y(1) = \alpha + y'(0) + \int_0^1 (1-t)u(t)dt. \tag{2.17}$$

By substituting $y(1)=\beta$, we obtain

$$\beta = \alpha + y'(0) + \int_0^1 (1-t)u(t)dt, \tag{2.18}$$

which gives

$$y'(0) = (\beta-\alpha) - \int_0^1 (1-t)u(t)dt. \tag{2.19}$$

By substituting Equation (2.19) into (2.16), we get

$$y(x) = \alpha + (\beta-\alpha)x - \int_0^1 x(1-t)utdt + \int_0^x (x-t)u(t)dt. \tag{2.20}$$

Also, by substituting equation (2.12)and (2.20) into (2.10), we get

$$u(x) + \alpha g(x) + (\beta-\alpha)xg(x) - \int_0^1 xg(x)(1-t)u(t)dt$$

$$+\int_0^x g(x)(x-t)u(t)dt = f(x). \tag{2.21}$$

we can use the formula

$$\int_0^1 (.) = \int_0^x (.) + \int_x^1 (.), \tag{2.22}$$

to carry equation (2.21) to

$$\begin{aligned} u(x) &= h(x) - \alpha g(x) - (\beta-\alpha)xg(x) - g(x)\int_0^x (x-t)u(t)dt \\ &+ xg(x)\left[\int_0^x (1-t)utdt + \int_x^1 (1-t)u(t)dt\right], \end{aligned} \tag{2.23}$$

that gives

$$u(x) = f(x) + \int_0^x t(1-x)g(x)u(t)dt + \int_x^1 x(1-t)g(x)u(t)dt. \tag{2.24}$$

This in turn leads to the Fredholm Integral Equation:

$$u(x) = h(x) + \int_0^1 f(x,t)u(t)dt, \tag{2.25}$$

where

$$h(x) = f(x) - \alpha g(x) - (\beta-\alpha)xg(x), \tag{2.26}$$

and the kernel $f(x,t)$ is represented by

$$f(x,t) = \begin{cases} t(1-x)g(x), & 0 \le t \le x, \\ x(1-t)g(x), & x \le t \le 1. \end{cases} \tag{2.27}$$

From the above, it can be concluded that it is special case of (2.10) where $y(0)=y(1)=0$ or $\alpha=\beta=0$. Since the two boundaries of a moving string are fixed, we see clearly that $f(x)=h(x)$ in this case, thereby implying that the homogeneity concept to the Fredholm equation in (1.25) depends on the case of the boundary value problem in (2.10); that is, if the boundary value problem is homogeneous, then the Fredholm equation is also homogeneous, and if the boundary value problem is non-homogeneous, then the Fredholm equation is non-homogeneous when $\alpha = \beta = 0$.

CHAPTER 3

THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

In a mathematical concept, there are several integral equations which may have one solution, multiple solutions, infinite solutions or even no solution. In this chapter, we will present some theorems which prove the existence and uniqueness of the solution of the Fredholm Integral Equation.

3.1 Basic Theorems

Theorem 1. (Fredholm Alternative Theorem) [2,5,21]

This theorem states that if the homogeneous Fredholm Integral Equation as in the following formula has only the trivial solution i.e. $u(x)=0$

$$u(x) = \mu \int_a^b f(x,y)u(y)dy, \tag{3.1}$$

then the corresponding non-homogeneous Fredholm equation

$$u(x) = h(x) + \mu \int_a^b f(x,y)u(y)dy \tag{3.2}$$

always has a unique solution.

Theorem 2. (Uniqueness of the Solution) [2,5,21]

Suppose we have the following Fredholm Integral Equation

$$u(x) = h(x) + \mu \int_a^b f(x,y)u(y)dy\,. \tag{3.3}$$

It follows that:

1. the kernel $f(x,y)$ in equation (3.3) is continuous in the square

$$\prod = \{(x,y);\ a \le x \le b,\ a \le y \le b\},$$

2. $f(x)$ is a continuous real valued function,

Then the following condition

$$|\mu| M(b-a) < 1, \tag{3.4}$$

where

$$|f(x,y)| \le M \in R. \tag{3.5}$$

should be available for the existence of a unique solution for equation (3.3). In the case of the contrary, if the necessary condition (3.4) is not achieved, then a continuous solution may exist for the Fredholm Integral Equation.
To illustrate this theorem, we will discuss the following example

Example 3.1

Assume that we have the Fredholm Integral Equation

$$u(x) = 2 - 3x + \int_0^1 (3x+t)u(t)dt. \tag{3.6}$$

We can understand that $\mu = 1, |f(x,t)| \le 4$ and $(b-a)$ =1. This means

$$|\mu| M(b-a) = 4 > 1, \tag{3.7}$$

Consequently, the value of the exact solution to equation (3.6) is

$$u(x) = 6x. \tag{3.8}$$

3.2 Basic Definition: Complete Metric Spaces

We should be fully aware of the following topics where this topic represents the fundamentals necessary for the concept of a fixed point theorem.

3.2.1 Metric space [4,22]

A metric space, which is specified as (M,d), is a set M and a distance function d $(d : M \times M \to R)$ such that this distance (or metric) satisfies the following three conditions:

a) For every $x, y \in M$, $0 \le d(x,y)$ and $d(x,y) = 0$ if and only if x=y. (3.9)

This means that the distance between any two elements is always nonnegative, and the distance is identical.

b) For all $x, y \in M$ $d(x, y) = d(y, x)$. (3.10)

That is, the distance is symmetric in $x$and y

c) $\forall x, y, z \in M$ $d(x, z) \le d(x, y) + d(y, z)$. (3.11)

This means that the present general definition of distance still satisfies an inequality that parallels the usual triangle inequality

$$|x - z| \le |x - y| + |y - z|. \quad (3.12)$$

The triangle inequality (3.11) will be used very often in proofs of the basic theorems.

3.2.2 Fixed point of a mapping [4,22]

Suppose that we have a metric space (M, d) and S a subset of M with the mapping

$$T : S \to M,\ T(u) = v.$$

When $T(u_0) = u_0$, it leads an element $u_0 \in S$, which is a fixed point of the mapping *T*. For all elements $u \in S$, in particular elements u_0 are not affected under the transformation *T*, that is

$$u_0 = T(u_0) \quad (3.13)$$

3.2.3 Cauchy sequence [4,22]

A sequence is called a Cauchy sequence if all the elements become arbitrarily close to each other as the sequence progresses more precisely,(x_n) is said to be a Cauchy sequence if $\forall \varepsilon > 0, \exists N \ge 0$ such that in the case of real numbers $\forall m, n \ge N \to |x_m - x_n| < \varepsilon$.

3.2.4 Contractive mapping

The mapping T in ($u_0=T(u_0)$ is called contractive if there is a nonnegative real number α; $0 \le \alpha < 1$, such that $\forall\ u_1, u_2 \in S$ we have $d(T(u_1), T(u_2)) \le \alpha\, d(u_1, u_2)$

3.2.5 Lipschitz condition

For any real-valued function, this functions satisfies a Lipschitz condition on a set $Q(a, b)\times$R if the inequality $|f(x) - f(y)| \le M|x - y|$ holds $\forall\ x, y \in Q(a, b)\times$R, where M is a constant that is independent of x and y and R is the set of real numbers.

3.3 Fixed Point Theorem [4,6]

Suppose that we have a complete metric space (M, d), and we let the mapping $T : M \to M$, be a contraction, then T has exactly one fixed point.

Proof: We are to prove the following for the mapping $u = T(u)$

a) The uniqueness of the fixed point when it exists.

b) The existence of the fixed point, where we show first that the sequence of the successive approximations $u_{n+1} = T(u_n)$ is Cauchy convergent due to its being in a complete metric space and thus being convergent. We show that the limit point for this convergent sequence $u = \lim_{n\to\infty} u_n$ is indeed the fixed point of the actual problem $u = T(u)$.

a) To prove the uniqueness of the fixed point, we assume that there are two distinct points u and v ,$u \neq v$, implying $u=T(u)$ and $v=T(v)$, $u \neq v$. Since $u \neq v$, the distance between them is not zero: $d(u, v) \neq 0$. Because u and v are fixed points of T, we also have

$$d(T(u),T(v))=d(u, v) \neq 0.$$

However, since the mapping T is also contractive, we have according to

$$d(T(u_1),T(u_2)) \le \alpha\, d(u_1,u_2),$$

$$d(T(u),T(v)) \le \alpha\, d(u, v),\ 0 \le \alpha \le 1.$$

If we combine $d(T(u),T(v))=d(u, v) \neq 0$ *with* $d(T(u),T(v)) \le \alpha\, d(u, v)$, we see clearly that it is a contradiction of $d(u, v)= d(T(u),T(v)) \le \alpha\, d(u, v)$, $(1-\alpha)\, d(u, v) \le 0$.

since $d(u,v)>0$ by assumption $(1-\alpha) \leq 0$, $\alpha \geq 1$, which contradicts the assumption of contraction mapping whose α is strictly less than 1. Hence the distance $d(u\ ,v)>0$ must be identically zero, which is equivalent to u being equal to v. This proves the uniqueness of the fixed point when it exists.

b) To prove the existence of a limit point as a fixed point for $u=T(u)$, we will first prove that the sequence u_n of the iterative process $u_{n+1}=T\ (u_n)$ is a Cauchy sequence.

with the help of the contraction property, we will find the distance $d(u_n,u_{n+1})$ between two consecutive approximations in terms of the distance $d(u_1,u_2)$ between the first two approximations (input estimates) $u_1$and u_2.

The next step is to find the distance $d(u_{n,}\ u_{n+p})$, that we need to use in proving the Cauchy convergence($d(u_n, u_m) < \mathcal{E}$) for n, $m>N(\mathcal{E})$. From the following form

$$u_{n+1}(x) = \int F(x, y, u_n(y))dy,$$

we have

$$d(u_2,u_3)=d(T(u_1),T(u_2))\leq \alpha d(u_1,u_2).$$

By the same reasoning,

$$d(u_3,u_4)=d(T(u_2),T(u_3))\leq \alpha\ d(u_2,u_3).$$

And if we invoke on the right side the previous result for $d(u_2,u_3)$, we have

$$d(u_3,u_4) \leq \alpha\ d(u_2,u_3) \leq \alpha\ 2d(u_1,u_2).$$

If we continue this to u_n and u_{n+1},we have

$$d(u_n,u_{n+1}) \leq \alpha^{n-1}\mathrm{d}(u_1,u_2),$$

where clearly the higher-orderconsecutiveterms$u_n.u_{n+1}$are much closer together than the first ones, u_1 and u_2, due to the geometric factor α_{n-1}. Still, we need to show the Cauchy convergence, which will entail the use of the previous result and the triangle inequality of the metric $d(u_n\ ,u_{n+p})$. Observe that

$$d(u_n, u_{n+p}) \leq d(u_n,u_{n+1})+ d(u_{n+1},u_{n+2})+ d(u_{n+2},u_{n+3})+\ldots+ d(u_{n+p-1},u_{n+p})$$

after repeated use of the triangle inequality. Now, we use property

$$d(u_n,u_{n+1}) \leq \alpha_{n-1}\, d(u_1,u_2)$$

on each of the terms on the right side, which are distances for consecutive sequences. Therefore,

$$d(u_n, u_{n+p}) \leq d(u_n,u_{n+1})+ d(u_{n+1},u_{n+2})+ d(u_{n+2},u_{n+3})+\ldots+ d(u_{n+p-1},u_{n+p})$$

$$\leq \alpha^{n-1}d(u_1,u_2)+ \alpha^{n}d(u_1,u_2)+ \alpha^{n+1}d(u_1,u_2) +\ldots+ \alpha^{n+p-2}d(u_1,u_2)$$

$$=[\ \alpha^{n-1}+ \alpha^{n}+ \alpha^{n+1}+\ldots+ \alpha^{n+p-2}]d(u_1,u_2)$$

$=\alpha^{n-1}(1+\alpha+\alpha^2+\ldots+\alpha^{p-1})\, d(u_1,u_2)$

$$= \alpha^{n-1}\frac{1-\alpha^p}{1-\alpha}d(u_1,u_2)$$

After realizing that we have a geometric series in the parentheses above, and since $0 \le \alpha \le 1$,the right side would clearly approach zero as $n\to\infty$, which makes $d(u_n,u_{n+p})\to 0$ as $n\to\infty$ (i.e., the sequence converges in the Cauchy sequence). Since this sequence u_n is an element of a complete metric space, it will converge to a limit u in this space (i.e., $\lim_{n\to\infty} u_n = u$). What remains is to show that this limit point u is indeed the fixed point of our equation; that is, it must satisfy $u=T(u)$, in other words, $d(u\ ,T(u))=0$, hence we have $u_{n+1}=T(u_n)$ and from the proof of the existence of the limit point above, we can say that

$$\lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} u_n = u,$$

or

$$d(u,\ T(u_n))=d(u,u_{n+1})\to\infty,\ d(u,\ u_n)\to 0\ as\ n\to\infty.$$

with these results, we will use the triangular inequality to have

$$d(u\ ,T(u)) \le d(u,\ T(u_n))+\ d(T(u_n),T(u)) \le d(u,\ T(u_n))+\alpha d(u_n\ ,u)$$

$$\le d(u,u_{n+1})+\ \alpha\ d(u_n,\ u)$$

After using the contraction property of the operator T in the last term, as $n\to\infty$ each of the two terms on the right would approach zero, which makes $d(u\ ,T(u)) \le 0$. However, since the metric d is nonnegative by definition, we must have $d(u,\ T(u))=0$, which means that $u=T(u)$, the desired result of the fixed point theorem.

Next, we will illustrate the important Banach fixed point theorem to prove the existence of unique solutions to linear and nonlinear Fredholm Integral Equations that exhibit contraction.

3.3.1 Existence of the solution for fredholm integral equations [2,4,5]

3.3.1.1 Existence solution for linear fredholm integral equations

Suppose the linear Fredholm Integral Equation

$$u(x) = h(x) + \mu\int_a^b f(x,y)u(y)dy. \qquad (3.14)$$

First we should found *α*

We suppose that $h(x)$ is continuous on the interval $[a\ ,b]$ and $f(x, y)$ is continuous on the square $D=\{(x\ ,y): x \in [a\ ,b], y\ x \in [a\ ,b]\}$. For such functions we shall work with the complete metric space C[*a*, *b*] of continuous functions and its metric $d(x, y)$ as in the metric

$$d(f(x), g(x)) = \max_{x\in[a,b]}|f(x) - g(x)|;\ f\ , g\ \in C[a, b]. \qquad (3.15)$$

To find a sufficient condition for the mapping $T(u)$ of (3.15) to be contractive, we first indicate that the kernel $f(x\ ,y)$ here is bounded which means that $[f(x\ ,y) \leq M]$ since it is continuous on the bounded domain *D* (3.14) of the square in the following figure 3.1.

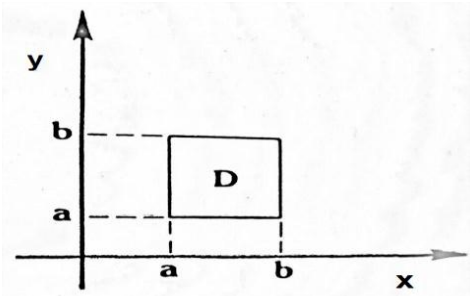

Figure 3.1 Domain D (3.14)

To show the contraction property of *T*, we use the metric of (3.15) on the images $T(\beta(x))$ and $T(\gamma(x))$ of the two continuous functions $\beta(x), \gamma(x)$ in C[*a* ,*b*],

$$d(T(\beta(x)), T(\gamma(x))) = \max_{x\in[a,b]}\Big|g(x) + \mu\int_a^b f(x,y)\beta(y)dy - [g(x)$$

$$+\mu\int_a^b f(x,y)\gamma(y)dy\Big|$$

$$= \max_{x\in[a,b]}\left|\mu\int_a^b f(x,y)[\beta(y) - \gamma(y)dy\right|$$

$$\leq \max_{x\in[a,b]}\int_a^b|\mu f(x,y)[\beta(y) - \gamma(y)]|dy$$

$$\leq |\mu| M \max_{x\in[a,b]}\int_a^b|[\beta(y) - \gamma(y)]|dy$$

$$\leq |\mu| M \max_{x\in[a,b]}|\beta(x) - \gamma(x)|\int_a^b dy$$

$$\leq |\mu| M(b-a) d(\beta(x), \gamma(x)) = \alpha d(\beta(x), \gamma(x)). \qquad (3.16)$$

After using the upper bound M for $f(x, y)$.Hence with

$$\alpha = |\mu M(b-a)| < 1 \quad . \tag{3.17}$$

The mapping of the linear Fredholm equation (3.14) becomes contractive after we found in (3.17) that $\alpha=\mu M(b\text{-}a)$,which gives a contractive mapping if we insist that $\alpha<1$ (*i.e.,* $\mu < \frac{1}{M(b-a)}$), where M is the upper bound of $f(x,y)$ on the square of Fig.(3.1). In this case the u_n estimate as an input would produce an output u_{n+1} that has a maximum error bounded.

$$\varepsilon = \max_{x\in[a,b]}|u-u_{n+1}| \le \frac{|\mu M(b-a)|^n}{1-|\mu M(b-a)|}\max_{x\in[a,b]}|u_2-u_1|, \tag{3.18}$$

where $|\mu M(b-a)| < 1$.

In terms of the maximum difference between the first two estimates, u_2 and u_1.

3.3.1.2 Existence solution for nonlinear fredholm integral equation [2]

Suppose the nonlinear Fredholm Integral Equation

$$u(x) = h(x) + \mu\int_a^b F(x,y,u(y))dy. \tag{3.19}$$

There are special conditions under which the solution exists for the nonlinear Fredholm Integral Equations as follows:

1. The function $h(x)$ is bounded, $|h(x)| < R$, *in* $a \le x \le b$.
2. The function $F(x,y,u(y))$ is integrable and bounded where

$$|F(x,y,u(y))| < M, \text{ in a} \le x, \text{ y} \le \text{b}.$$

3. The function $F(x,y,u(y))$ satisfies the Lipschitz condition

$$|\mathrm{F}(x, y, z_1) - \mathrm{F}(x, y, z_2)| < M|z_1 - z_2|.$$

Theorem 3.3 [21]

Suppose the nonlinear Fredholm Integral Equation

$$u(x) = h(x) + \mu\int_a^b F(x,y,u(y))dy. \tag{3.20}$$

And suppose that the function F(x,y,z) is defined and continuous on the set $Q(a\ ,b)\times R$ and satisfies a Lipschitz condition

$|\mathrm{F}(x,y,z_1)-\mathrm{F}(x,y,z_2)|<M|z_1-z_2|$.

Then the nonlinear Fredholm Integral Equation (3.19) has a unique solution on the interval [a ,b] whenever

$$|\mu|<1/(M(b-a)).$$

Proof: the proof can be found in [21].

CHAPTER 4

METHODS OF SOLUTIONS

In this chapter, we will review the solution methods for the Fredholm Integral Equation and the system of Fredholm Integral Equations according to the classification or type of the Fredholm Integral Equation in terms of being a linear F.I.E or non-linear F.I.E and the methods to solve system of Fredholm Integral Equation according to the type of equation. It is worth mentioning that we will often use the Fredholm Integral Equation of degenerate or separable kernels.

4.1 Methods to Solve Linear Fredholm Integral Equations

In this part, we will study linear Fredholm Integral Equations and method of their solutions. Also, we will study the system of nonlinear Fredholm Integral Equations and method of their solutions.

4.1.1 Methods to solve fredholm integral equations of the second kind

In this section, the study will be on Fredholm Integral Equation of the formula

$$u(x) = h(x) + \mu\int_a^b f(x,y)u(y)dy \quad , \tag{4.1}$$

where $u(x)$ is an unknown function, $f(x, y)$ and $h(x)$ are known function, x and y are real variable varying in the interval (a, b), and μ is a parameter.

In the following we shall present the various methods of solutions to the Fredholm Integral Equation (4.1).

4.1.1.1 The successive approximations method [1,2,5]

The method of successive approximations grants us a scheme that can be used for solving integral equations or initial value problems. Moreover, the technique of this method is finding successive approximations to the solution of issue by starting with an initial guess these are called the zeroth approximation. As a matter of fact, the zeroth approximation can be any real valued function $u_0(x)$ that will be used in a recurrence relation to determine the other approximations. Also, we can select any real-valued function for the zeroth approximation; the most frequently used values are *0, 1, or x*. To illustrate more, let consider Fredholm Integral Equation of the second kind (4.1). We have put $u_0(x)=h(x)$. It must be remembered that the zeroth approximation can be any selected real valued function $u_0(x)$, $a \le x \le b$. As a result, the first approximation $u_1(x)$ of the solution of $u(x)$ is defined by

$$u_1(x) = h(x) + \mu\int_a^b f(x,y)u_0(y)dy\,. \tag{4.2}$$

We can get the second approximation $u_2(x)$ of the solution $u(x)$ by replacing $u_0(x)$ in equation (4.2) by the previously obtained $u_1(x)$;thus, we find

$$u_2(x) = h(x) + \mu\int_a^b f(x,y)u_1(y)dy\quad . \tag{4.3}$$

These steps can be continued in the same technique above to get the n^{th} approximation. In other words, the successive approximations method can be summed up by $u_0(x)$ = any selective real valued function

$$u_{n+1}(x) = h(x) + \mu\int_a^b f(x,y)u_n(y)dy\,, n \ge 0. \tag{4.4}$$

we obtain the final solution by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x)\,. \tag{4.5}$$

In other words, according to this method, we get a solution if it exists by (4.5). To illustrate this method we will study the following example.

Example 4.1

Solve the following Fredholm Integral Equation

$$u_1(x) = x + \mu\int_{-1}^{1} xyu(y)dy \tag{4.6}$$

by using the successive approximation method.

To start with, for the zeroth approximation $u_0(x)$, we can select

$$u_0(x) = x, \tag{4.7}$$

and use the iteration formula to get

$$u_{n+1}(x) = x + \mu\int_{-1}^{1} xyu_n(y)dy, n \geq 0 \tag{4.8}$$

Substituting (4.7) into(4.8)we get

$$u_1(x) = x + \frac{2}{3}\mu x,$$

$$u_2(x) = x + \frac{2}{3}\mu x + \left(\frac{2}{3}\right)^2 \mu^2 x,$$

$$u_3(x) = x + \frac{2}{3}\mu x + \left(\frac{2}{3}\right)^2 \mu^2 x + \left(\frac{2}{3}\right)^3 \mu^3 x,$$

(4.9)

$\vdots$

$$u_{n+1}(x) = x + \frac{2}{3}\mu x + \left(\frac{2}{3}\right)^2 \mu^2 x + \left(\frac{2}{3}\right)^3 \mu^3 x + \ldots + \left(\frac{2}{3}\right)^{n+1} \mu^{n+1} x.$$

The solution $u(x)$ of (4.6) is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \frac{3x}{3-2\mu}, 0 < \mu < \frac{3}{2}. \tag{4.10}$$

4.1.1.2 The adomian decomposition method [1,2,7]

This method was recently introduced by George Adomian [7]. This method offers the solution in a series form, according to the decomposition series [1]:

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{4.11}$$

Which means that, this method is composed of decomposing the unknown function $u(x)$ into a sum of an infinite number of components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$,. . .[2]. The components $u_n(x)$, $n\geq 0$ will be determined recurrently. Beside, this method concerns itself with finding the components u_0, u_1, u_2,...individually. At the connected meaning, we can establish the recurrence relation by substitute (4.11) into the Fredholm Integral Equation (4.1) to get

$$\sum_{n=0}^{\infty} u_n(x) = h(x) + \mu\int_a^b f(x,y)\left\{\sum_{n=0}^{\infty} u_n(y)\right\}dy \quad . \tag{4.12}$$

Or equivalently

$$u_0(x)u_1(x)+u_2(x)+...=h(x)+\mu\int_a^b f(x,y)\{u_0(y)+u_1(y)+...\}dy. \tag{4.13}$$

The components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ... are completely determined by setting the recurrence relation

$$u_0(x)=h(x),\ u_{n+1}(x)=\mu\int_a^b f(x,y)u_n(y)dy, n\geq 0\,. \tag{4.14}$$

$$\begin{aligned} u_0(x) &= h(x), \\ u_1(x) &= \mu\int_a^b f(x,y)u_0(y)dy, \\ u_2(x) &= \mu\int_a^b f(x,y)u_1(y)dy, \\ u_3(x) &= \mu\int_a^b f(x,y)u_2(y)dy, \\ \vdots &= \vdots \\ u_n(x) &= \mu\int_a^b f(x,y)u_{n-1}(y)dy. \end{aligned} \tag{4.15}$$

Thus, for other components, the zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. It is clear to us, when the previous $u_0(x)$ is known, then it is easy to successively determine $u_1(x)$, $u_2(x)$, $u_3(x)$,. . . ., and so on for other components. According to equation (4.14) the components $u_j(x)$, $j\geq 0$ follow immediately. Once these components are determined, the solution $u(x)$ can be obtained using the series (4.11). It is very clear that the decomposition method converted the integral equation into an neat determination of computable components, when an exact solution exists for the problem, so the obtained series converges very rapidly to that exact solution. By studying the following example, this method will be clear.

Example 4.2

Use the Adomian decomposition method to solve the following Fredholm Integral Equation

$$u(x)=e^x-x+x\int_0^1 yu(y)dy. \tag{4.16}$$

According to the Adomian decomposition method, the solution $u(x)$ was assumed to be a series form given in (4.11). When substituting the decomposition series (4.11) into both sides of (4.16), we get

$$\sum_{n=0}^{\infty} u_n(x) = e^x - x + x\int_0^1 y \sum_{n=0}^{\infty} u_n(y)dy, \tag{4.17}$$

or equivalently

$$u_0(x)u_1(x) + u_2(x) + \ldots = e^x - x + x\int_a^b y\{u_0(y)u_1(y) + u_2(y) + \ldots\}dy. \tag{4.18}$$

Then we identify the zeroth component by all terms that are not included under the integral sign. For that reason, we obtain the following recurrence relation

$$u_0(x) = e^x - x, \quad u_{k+1}(x) = x\int_0^1 y u_k(y)dy, \; k \geq 0. \tag{4.19}$$

Consequently, we obtain

$$\begin{aligned}
&u_0(x) = e^x - x,\\
&u_1(x) = x\int_0^1 y u_0(y)dy = x\int_0^1 y(e^y - y)dy = \frac{2}{3}x,\\
&u_2(x) = x\int_0^1 y u_1(y)dy = x\int_0^1 \frac{2}{3}t^2 dy = \frac{2}{9}x,\\
&u_3(x) = x\int_0^1 y u_2(y)dt = x\int_0^1 \frac{2}{9}y^2 dy = \frac{2}{27}x,\\
&u_4(x) = x\int_0^1 y u_3(y)dy = x\int_0^1 \frac{2}{27}y^2 dy = \frac{2}{81}x,
\end{aligned} \tag{4.20}$$

and so on. using (4.11) gives the series solution

$$u(x) = e^x - x + \frac{2}{3}x\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots\right). \tag{4.21}$$

We note that the infinite geometric series at the right side has $a_1 = 1$, and the ratio $r = \frac{1}{3}$ .The sum of the infinite series will be on the following formula

$$S = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}. \tag{4.22}$$

The series solution (4.21) converges to the closed form solution

$$u(x) = e^x, \tag{4.23}$$

We obtained it by substituting(4.22) into (4.21).

4.1.1.3 The modified decomposition method [2,8]:

According to the above, the solutions from the Adomian decomposition method are an infinite series of components. Also, it is easy to compute the components $u_j, j \geq 0$ if the inhomogeneous term $h(x)$ in the Fredholm Integral Equation

$$u(x) = h(x) + \mu \int_a^b f(x, y) u(y) dy \tag{4.24}$$

consists of a polynomial of one or two terms. In addition, if the function $h(x)$ consists of a combination of two or more of polynomials, trigonometric functions, hyperbolic functions, and others, the evaluation of the components u_j, $j \geq 0$ requires more work. It is worth mentioning that there is a reliable modification of the Adomian decomposition method that was developed by Wazwaz [2]. Where the amendment as presented before, the modified decomposition method depends mainly on splitting the function $h(x)$ into two parts, and as a result, this method cannot be used if the $h(x)$ consists of only one term. To illustrate more of this method, we remember that the standard Adomian decomposition method uses the recurrence relation

$$u_0(x) = h(x) \; u_{k+1}(x) = \mu \int_a^b f(x, y) u_k(y) dy, k \geq 0 \quad , \tag{4.25}$$

where the expression of the solution $u(x)$ by an infinite sum of components is defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{4.26}$$

According to the (4.25), the components $u_n(x)$, $n \geq 0$ are easily obtained. The modified decomposition method offers a simple variation to the recurrence relation (4.25) to determine the components of $u(x)$ in the easiest and fastest style. For many cases, we can set

$$h(x) = h_1(x) + h_2(x), \tag{4.27}$$

where we put the function $h(x)$ as the sum of two partial functions, namely $h_1(x)$ and $h_2(x)$. By means of the modified decomposition method, the zeroth component $u_0(x)$ is delimited by one part of $h(x)$, namely $h_1(x)$ or $h_2(x)$. The other part of $h(x)$ can be added to the component $u_1(x)$ that exists in the standard recurrence relation. The modified decomposition method acknowledges the use of the modified recurrence relation:

$$u_0(x) = h_1(x),$$
$$u_1(x) = h_2(x) + \mu\int_a^b f(x,y)u_0(y)dy, \tag{4.28}$$
$$u_{k+1}(x) = \mu\int_a^b f(x,y)u_k(y)dy, k \ge 1.$$

In view of (4.25)and (4.28), we find that the difference between them rests only in the formation of the first two components $u_0(x)$ and $u_1(x)$. Moreover, we note that the remaining components u_j ,j ≥ 0 are still lingering in the two recurrence relations. In addition, we find that the slight difference between the formation of $u_0(x)$ and $u_1(x)$ is useful for accelerating the convergence of the solution and reducing the size of the calculations. Moreover, the effects of reducing the number of terms in $h_1(x)$ occur not only in the component $u_1(x)$ but also in the other components. As a result of the study of the modified decomposition method, we can make two remarks:

if we make the proper choice of functions $h_1(x)$ and $h_2(x)$, the exact solution $u(x)$ may be obtained by using very few iterations, and sometimes by evaluating only two components. This means that the success of this method depends mainly on the proper choice of $h_1(x)$ and $h_2(x)$ [10], and this we can find by trials only. The modified decomposition method cannot be used if the case of $h(x)$ consists of one term only.

Example 4.3

Solve the Fredholm Integral Equation

$$u(x) = \sec^2 x + x^2 + x - \int_0^{\frac{\pi}{4}} \left(\frac{4}{\pi} x^2 + xu(y) \right) dy. \tag{4.29}$$

By using the modified decomposition method.

It is clear that the function $h(x)$ consists of three terms. Therefore, we can split $h(x)$ thus:

$$h(x) = \sec^2 x + x^2 + x. \tag{4.30}$$

Now, we put the function $h(x)$ as the sum of two partial functions, namely $h_1(x)$ and $h_2(x)$:

$$h_1(x) = \sec^2 x,\ h_2(x) = x^2 + x. \tag{4.31}$$

Then, using the modified recurrence formula (4.28) gives

$$u_0(x) = h_1(x) = \sec^2 x,$$

$$u_1(x) = x^2 - x + \int_0^{\frac{\pi}{4}} \left(\frac{4}{\pi} x^2 + x u_0(y) \right) dy = 0, \tag{4.32}$$

$$u_{k+1}(x) = \mu \int_0^{\frac{\pi}{4}} p(x,y) u_k(y) dy = 0, k \geq 1.$$

As a result, the exact solution is given by

$$u(x) = \sec^2 x. \tag{4.33}$$

4.1.1.4 The noise terms phenomenon method [2]

Noise terms are defined as the identical terms, with opposite signs that may appear between components $u_0(x)$ and $u_1(x)$. Furthermore, they may appear between other components. These noise terms, if they appear, are very useful as they provide us with effective tools to find solutions quickly by using only two iterations. When the noise terms exist between the components $u_0(x)$ and $u_1(x)$, they provide the exact solution by using only the first two iterations. By means of canceling the noise terms between $u_0(x)$ and $u_1(x)$, even though $u_1(x)$ contains further terms, the remaining non-canceled terms of $u_0(x)$ may give the exact solution of the integral equation. Moreover, canceling the noise terms between $u_0(x)$ and $u_1(x)$ is not always sufficient to obtain the exact solution in spite of the appearance of these noise terms. Therefore, it is necessary to show that the non-canceled terms of $u_0(x)$ satisfy the given integral equation. The necessary condition for the appearance of the noise terms is required. In other words, the zeroth component $u_0(x)$ must contain the exact solution $u(x)$ among other terms. The following example clarifies this further:

Example 4.4

By using the noise terms phenomenon, we solve the Fredholm Integral Equation

$$u(x) = x \sin x - x + \int_0^{\frac{\pi}{2}} x u(y) dy. \tag{4.44}$$

According to the Adomian method, we have the recurrence relation

$$u_0(x) = x\sin x - x,$$
$$u_{k+1}(x) = \int_0^{\frac{\pi}{2}} x u_k(y)dy, k \geq 0. \tag{4.45}$$

This gives

$$u_0(x) = x\sin x - x,$$
$$u_1(x) = \int_0^{\frac{\pi}{2}} x u_0(y)dy = x - \frac{\pi^2}{8}x. \tag{4.46}$$

After the cancellation of the noise terms x, which appears within $u_0(x)$ and $u_1(x)$. Specifically, with the zeroth component $u_0(x)$, we obtain the exact solution

$$u(x) = x\sin x. \tag{4.47}$$

4.1.1.5 The variational iteration method [2,9,10]

The variational iteration method was applied to the second kind of Fredholm Integral Equation, and this was useful to provide us with an approximate solution. The technique of this method lies in the direct dependence on the construction of a convergent sequence of functions which approaches the exact solution with less iteration [12]. In order for this method to work effectively, it is necessary for the kernel $f(x, y)$ to be separable so it can be written in the form $f(x ,y) = g(x)v(t)$. The Fredholm Integral Equation should be converted to an identical Fredholm integro-differential equation by differentiating both sides. It is important to note that an integro-differential equation needs an initial condition to be defined. We will mention here that the study will only be in the cases where $g(x) = x_n$, $n \geq 1$.To illustrate this method, assume that we have the second kind of the Fredholm Integral Equation:

$$u(x) = h(x) + \int_a^b f(x, y)u(y)dy, \tag{4.48}$$

or equivalently

$$u(x) = h(x) + g(x)\int_a^b v(y)u(y)dy. \tag{4.49}$$

We can note that the integral within the right side is a constant value, and by differentiating both sides of (4.49) with respect to x, we obtain

$$u'(x) = h'(x) + g'(x)\int_a^b v(y)u(y)dy. \tag{4.50}$$

The correction functional for the integro-differential equation (4.50) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \mu(\xi)\left(u_n'(\xi) - f'(\xi) - g'(\xi)\int_a^b f(r)\tilde{u}_n(r)dr\right)d\xi. \quad (4.51)$$

Using the method of variational iteration requires applying two basic steps. It first requires identifying the Lagrange multiplier $\mu(\xi)$ that can be identified optimally by means of integration by parts and secondly by using a restricted variation. The Lagrange multiplier for first order integro-differential equations is$\mu(\xi) = -1$. As a result, we obtain the following iteration formula:

$$u_{n+1}(x) = u_n(x) - \int_0^x \lambda(\xi)\left(u_n'(\xi) - h'(\xi) - g'(\xi)\int_a^b f(r)u_n(r)dr\right)d\xi.$$

(4.52)

This formula is used to fix successive approximations $u_{n+1}(x)$, $n \geq 0$ of the solution $u(x)$. It is worth mentioning that we can take the zeroth approximation $u_0$of any selective function. For a selective zeroth approximation u_0,it is prefer able to use the given initial value $u(0)$. Therefore, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x). \quad (4.53)$$

By studying the following example, the variational iteration method will be illustrated.

Example 4.5

By using the variational iteration method, solve the Fredholm Integral Equation

$$u(x) = e^x - x + x\int_0^1 yu(y)dy. \quad (4.54)$$

First, we differentiate once from both sides of equation (4.54) with respect to x:

$$u(x) = e^x - 1 + \int_0^1 yu(y)dy. \quad (4.55)$$

By applying the variational iteration method to equation (4.55), the correct function can be written in the following form:

$$u_{n+1}(x) = u_n(x) - \int_0^x\left(u_n'(\xi) - e^{\xi} + 1 - \int_0^1 ru_n(r)dr\right)d\xi, \quad (4.56)$$

where we use $\mu = -1$ for first order integro-differential equations. We obtain an initial condition $u(0) = 1$ by substituting $x = 0$ into (4.54).Also, with this initial condition, we select $u_0(x)$ = $u(0)$ = 1. Consequently, using this selection in the correction functional gives the following successive approximations $u_0(x)$=1,

$$u_1(x) = u_0(x) - \int_0^x \left(u_0'(\xi) - e^{\xi} + 1 - \int_0^1 r u_0(r) dr \right) d\xi = e^x - \frac{1}{2}x,$$
$$u_2(x) = u_1(x) - \int_0^x \left(u_1'(\xi) - e^{\xi} + 1 - \int_0^1 r u_1(r) dr \right) d\xi = e^x - \frac{1}{2\times 3}x,$$
$$u_3(x) = u_2(x) - \int_0^x \left(u_2'(\xi) - e^{\xi} + 1 - \int_0^1 r u_2(r) dr \right) d\xi = e^x - \frac{1}{2\times 3^2}x,$$

$$u_4(x) = u_3(x) - \int_0^x \left(u_3'(\xi) - e^{\xi} + 1 - \int_0^1 r u_3(r) dr \right) d\xi = e^x - \frac{1}{2\times 3^3}x,$$
$$\dots\dots\dots = \dots\dots\dots$$
$$u_{n+1} = e^x - \frac{1}{2\times 3^n}x, n \geq 0.$$
(4.57)

Thus, we have

$$u(x) = \lim_{n\to\infty} u_n(x) = e^x \ . \tag{4.58}$$

4.1.1.6 The direct computation method [1,2]

This method can be applied to solve Fredholm Integral Equations effectively. With this method, the exact solution is determined in a closed form and not in a series form. It is worth mentioning that the direct computation method will be applied to the separable or degenerate kernels of the form

$$F(x,y) = \sum_{k=1}^{n} g_k(x) v_k(y). \tag{4.59}$$

This method can be applied as follows:

By substituting(4.59) into the Fredholm Integral Equation of the form

$$u(x) = h(x) + \int_a^b f(x,y)u(y)dy. \tag{4.60}$$

We get

$$u(x) = h(x) + g_1(x)\int_a^b v_1(y)u(y)dy + g_2(x)\int_a^b v_2(y)u(y)dy + \dots$$
$$+ g_n(x)\int_a^b v_n(y)u(y)dy. \tag{4.61}$$

In equation (4.61), each integral at the right side is equivalent to a constant and we can set

$$\alpha_i = \int_a^b v_i(y)u(y)dy,\ 1 \le i \le n. \tag{4.62}$$

According to equation (4.62), the equation (4.61) becomes

$$u(x) = h(x) + \mu\alpha_1 g_1(x) + \mu\alpha_2 g_2(x) + ... + \mu\alpha_n g_n(x), \tag{4.63}$$

We have a system of n algebraic equations when substituting (4.63) into (4.62). This can be solved to determine the constants $\alpha_i, 1 \le i \le n$. Then substituting (4.62) into (4.63) gives the exact solution.

To illustrate this method, we will study the following example:

Example 4.6

Solve the Fredholm Integral Equation

$$u(x) = 3x + 3x^2 + \frac{1}{2}\int_0^1 x^2 yu(y)dy. \tag{4.64}$$

using the direct computation method.

The kernel $p(x,y) = x^2 y$ is separable. Therefore, we can rewrite the above equation(4.64) as

$$u(x) = 3x + 3x^2 + \frac{1}{2}x^2\int_0^1 yu(y)dy. \tag{4.65}$$

We note that the integral within the right side relies only on the variable t with constant limits of integration for t, so it is equivalent to a constant. Hence, we rewrite (4.65) as

$$u(x) = 3x + 3x^2 + \frac{1}{2}\alpha x^2, \tag{4.66}$$

where

$$\alpha = \int_0^1 yu(y)dy. \tag{4.67}$$

To obtain the value of α, we substitute (4.66) into (4.67) to obtain

$$\alpha = \int_0^1 y\left(3y + 3y^2 + \frac{1}{2}\alpha y^2\right)dy. \tag{4.68}$$

Integrating the right side of (4.68), we get

$$\alpha = \frac{7}{4} + \frac{1}{8}\alpha, \tag{4.69}$$

Which gives

$$\alpha = 2. \tag{4.70}$$

Now, substituting (4.70) into (4.66) to find the exact solution

$$u(x) = 3x + 4x^2 \tag{4.71}$$

4.1.1.7 The series solution method [2]

The conceptual basis of the Series Solution Method comes basically from the Taylor series of analytical functions, where it should be noted that Taylor series need to have the derivatives of all orders, which is why we also calculate these. In addition, Taylor series at any point b in its domain converges to $h(x)$ in a neighborhood of b.

$$u(x) = \sum_{n=0}^{k} \frac{u^{(n)}(b)}{n!}(x-b)^n, \tag{4.72}$$

When x=0,theTaylor series can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{4.73}$$

Now suppose we have the Fredholm Integral Equation

$$u(x) = h(x) + \mu\int_a^b f(x,y)u(y)dy. \tag{4.74}$$

We will suppose that the solution of the Fredholm Integral Equation is analytic. To solve (4.74) by the Series solution method, we first substitute(4.73) into both sides of (4.74), which gives

$$\sum_{n=0}^{\infty} a_n x^n = T(h(x)) + \mu\int_a^b f(x,y)\left(\sum_{n=0}^{\infty} a_n y^n\right)dy, \tag{4.75}$$

Or we can simplify the previous form, to become

$$a_0 + a_1 x + a_2 x^2 + ... = T(h(x)) + \mu\int_a^b f(x,y)(a_0 + a_1 y + a_2 y^2 + ...)dy, \tag{4.76}$$

Where *T(h(x))* is the Taylor series for *h(x)*. In this manner, the integral equations are transformed into a traditional integral in (4.75) or (4.76), where instead of integrating the unknown function *u(x)*, terms of the form y^n, $n \geq 0$ will be integrated. It is important to note that the Taylor expansions for functions contributing to *h(x)* should be used if $h(x)$ contains elementary functions, such as exponential functions, trigonometric functions, etc. This method can be applied by using the following steps:

- Merge the right side of the integral in(4.75)or (4.76)to obtain a recurrence relation in a_j ,$j \geq 0$.
- We collect the coefficients of such as powers of x and equate these coefficients on both sides of the resulting equation.
- To complete the determination of the coefficients a_j, $j \geq 0$, we will seek to solve the recurrence relation.

After applying the above steps, we can note that the Series solution method is more effective at providing the exact solutions when the solution $u(x)$ is a polynomial. In other words, when the solution u(x) is not a polynomial such as *cos x*, *sin x*, e^x, etc., the exact solution may be obtained after rounding a few of the coefficients a_j, $j \geq 0$. To illustrate this method, we will study the following example:

Example 4.7

Solve the Fredholm Integral Equation

$$u(x) = -x^4 + \int_{-1}^{1} (xy^2 - x^2 y)u(y)dy. \tag{4.77}$$

By using the series solution method.

We first substitute $u(x)$ with the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{4.78}$$

into both sides of equation (4.77) and get

$$\sum_{n=0}^{\infty} a_n x^n = -x^4 + \int_{-1}^{1} \left((xy^2 - x^2 y) \sum_{n=0}^{\infty} a_n y^n \right) dy. \tag{4.79}$$

After simplification by using the three steps mentioned earlier in this method, we get

$a_0 = 0, a_3 = 0$

$$a_1 = \frac{-30}{133} \quad , \; a_4 \; = -1 \; n \geq 5.$$

(4.80)

$$a_2 = \frac{20}{133} \quad , \; a_n \; = 0.$$

As a result, the exact solution is

$$u(x) = -\frac{30}{133}x + \frac{20}{133}x^2 - x^4 \quad . \tag{4.81}$$

4.1.1.8 The homotopy perturbation method (HPM) [2,11]

This method has been provided and developed by Ji-Huan He in [11]. In fact, the homotopy perturbation method was used recently in order to solve linear and nonlinear problems. Moreover, this method works to combine a perturbation technique and a homotopy technique of topology. Additionally, a homotopy with an embedding parameter p ∈ [0, 1] is constructed, which is considered to be a small parameter [11]. Furthermore, the limitations of the traditional perturbation technique have been eliminated due to the coupling between the perturbation method and the homotopy method. Finally, HPM can deal with the first kind and the second kind of Fredholm Integral Equation. However, in this section, we will discuss a special case of the Fredholm Integral Equation of the second kind according to what will be explained as follows:

First of all, let us consider the Fredholm Integral Equations of the Second Kind

$$v(x) = h(x) + \mu\int_a^b f(x,y)v(y)dy \,. \tag{4.82}$$

Second, we define the operator

$$L(u) = u(x) - h(x) - \int_a^b f(x,y)u(y)dy = 0, \tag{4.83}$$

where $u(x)$ $=v(x)$. Then we define the homotopy $H(u, p)$, $p \in [0, 1]$ via :

$$H(u,0) = F(u) \text{ and } H(u,1) = L(u) \,. \tag{4.84}$$

$F(u)$ represents a functional operator. Now we establish a convex homotopy according to the following form:

$$H(u, p) = (1 - p)F(u) + p\, L(u) = 0 \tag{4.85}$$

This homotopy satisfies (4.84) for p = 0 and p = 1 respectively. By including the parameter p ,u increases monotonically from zero to one such as the trivial problem $L(u)$ = 0. Additionally, the HPM admits the use of the following expansion:

$$u = \sum_{n=0}^{\infty} p^n u_n \tag{4.86}$$

then

$$v = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n u_n \tag{4.87}$$

When the solution exists, it leads to the series (4.87), which converges to the exact solution. By substituting (4.86) into (4.85) and using the formula $F(x) = u(x)-h(x)$ and using mathematical operations of equating conditions with like powers of the embedding parameter p, we obtain the following recurrence relation:

$$\begin{aligned} &p^0 : u_0(x) = h(x), \\ &p^{n+1} : u_{n+1} = \int_a^b f(x,y)u_n(y)dy, n \geq 0. \end{aligned} \tag{4.88}$$

When we notice the recurrence relation above, we find that it is identical to the standard adomian decomposition method previously mentioned.

4.1.2 Methods to solve fredholm integral equation of the first kind [1-5,12]

As stated in Chapter two, we have defined the Fredholm Integral Equation of the first kind as the following form

$$h(x) = \mu \int_a^b f(x,y)u(y)dy, x \in \Omega, \tag{4.89}$$

where Ω is a bounded and closed set and h(x) is the data. The equation above (4.89) is determined by the occurrence of the unknown function *h(x)* only inside the integral sign and this causes special difficulties[2,12]. Fredholm integral equation of the first kind is considered ill-posed problem , postulated the following three properties [2]:

1. Existence of a solution.
2. Uniqueness of a solution.
3. Continuous dependence of the solution on the data *u(x).*

This property means that small errors in the data *u(x)* should cause small errors. Now we will offer some of the methods which will be used to find a solution to the Fredholm Integral Equation of the first kind.

4.1.2.1 The method of regularization [2,12,16]

Tikhonov and Phillips [2,12] established this method independently. This method transforms the linear Fredholm Integral Equation of the first kind

$$h(x) = \int_a^b f(x,y)u(y)dy, x \in \Omega. \tag{4.90}$$

To an approximation of a Fredholm Integral Equation of the second kind

$$\sigma u_{\sigma}(x) \ = h(x) - \int_a^b f(x,y) u_{\sigma}(y) dy, x \in \Omega \quad , \tag{4.91}$$

where σ is a small positive parameter. It can be also noted that the Fredholm Integral Equation of the second kind (4.91) maybe expressed as follows:

$$u_{\sigma}(x) = \frac{1}{\sigma} h(x) - \frac{1}{\sigma} \int_a^b f(x,y) u_{\sigma}(y) dy, x \in \Omega. \tag{4.92}$$

Furthermore, it was proved that the solution u_{σ} of the Fredholm Integral Equation of the second kind (4.92) converges to the solution $u(x)$ of the linear Fredholm Integral Equation of the first kind (4.90) as $\sigma \to 0$, which means that the exact solution $u(x)$may be obtained by

$$u(x) = \lim_{\sigma \to 0} u_{\sigma}(x) \tag{4.93}$$

4.1.2.2 The homotopy perturbation method (HPM) [2,16]

As previously stated in this chapter, we discussed this method and how to deal with the second type of Fredholm Integral Equation. In this section, we will discuss this method and how to deal with the first type of Fredholm Integral Equation. We first assume the following Fredholm integral equations of the first kind:

$$h(x) = \int_a^b f(x,y) v(y) dy. \tag{4.94}$$

Second, we define the operator

$$L(u) = h(x) - \int_a^b f(x,y) u(y) dy = 0. \tag{4.95}$$

Now, we establish a convex homotopy according to the following form:

$$H(u,p) = (1-p)u(x) + pL(u)(x) = 0. \tag{4.96}$$

To include the parameter p monotonically increasing from zero to one. The HPM admits the use of the following expansion:

$$u = \sum_{n=0}^{\infty} p^n u_n \quad . \tag{4.97}$$

Therefore,

$$v(x) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n \, . \tag{4.98}$$

When a solution exists, it leads to that series and (4.98) converges to the exact solution. Substituting (4.97) into (4.96) and by re-applying the same previous steps, we obtain the recurrence relation

$$\begin{aligned} &u_0(x) = 0, u_1(x) = h(x), \\ &u_{n+1}(x) = u_n(x) - \int_a^b f(x,y) u_n(y)dy, n \geq 1. \end{aligned} \tag{4.99}$$

Then the following condition must be a justification for convergence when the kernel is separable:

$$\left|1 - \int_a^b f(y,y)dy\right| < 1, \tag{4.100}$$

4.1.3 Methods to solve systems of fredholm integral equations [2,13,14]

There are two systems of Fredholm Integral Equation according to the following

The system of Fredholm Integral Equations of the first kind, as in the following formula

$$\begin{aligned} h_1(x) &= \int_a^b (F_1(x,y)u(y) + \breve{F}_1(x,y)v(y))dy, \\ h_2(x) &= \int_a^b (F_2(x,y)u(y) + \breve{F}_2(x,y)v(y))dy, \end{aligned} \tag{4.101}$$

Where a and b are constants and the functions $u(x)$ and $v(x)$ are the unknown, which will be here just under the integral sign [2].

The system of Fredholm Integral Equations of the second kind, as in the following formula

$$\begin{aligned} u(x) &= h_1(x) + \int_a^b (F_1(x,y)u(y) + \breve{F}_1(x,y)v(y) + ...)dy, \\ v(x) &= h_2(x) + \int_a^b (F_2(x,y)u(y) + \breve{F}_2(x,y)v(y) + ...)dy, \\ &\vdots \end{aligned} \tag{4.102}$$

where a and b are constants and the functions $u(x),v(x),...$ are the unknowns that will be determined. The function $h_i(x)$ and the kernels $F_i(x\ ,y), \breve{F}_1(x,y)$ are given a real-valued function. Now, we will apply some of the methods that were previously used in solving the Fredholm Integral Equations.

4.1.3.1 Adomian decomposition method [1,2]:

As presented before, this method offers the solution in a series form[1]. In fact, this method comprises decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series[2].

Suppose that we have a system of Fredholm Integral Equations of the second kind

$$u(x) = h_1(x) + \int_a^b (F_1(x,y)u(y) + \breve{F}_1(x,y)v(y) + ...)dy,$$
$$v(x) = h_2(x) + \int_a^b (F_2(x,y)u(y) + \breve{F}_2(x,y)v(y) + ...)dy. \qquad (4.103)$$

As explained previously, the Adomian decomposition method provides the solution in a series form, which means that

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \text{and } v(x) = \sum_{n=0}^{\infty} v_n(x). \qquad (4.104)$$

Where $u_n(x)$, $n \geq 0$ is the components of $u(x)$, and $v_n(x)$, $n \geq 0$ is the components of $v(x)$. By substituting (4.104) into (4.103) and using the modified decomposition method to obtain the recursive relation and set values $u_0(x)$, $u_1(x)$, $u_{k+1}(x)$ and $v_0(x)$, we have$v_1(x)$, $v_{k+1}(x)$ where $k \geq 1$. Consequently, we obtain the exact solutions by canceling the noise terms ∓ 2 from $u_0(x)$ and from $v_0(x)$.

4.1.3.2 The direct computation method [1,2]

The direct computation method was offered previously to solve Fredholm Integral Equations, now we will apply this method to solve systems of Fredholm Integral Equations of the second kind. It is important to note, the direct computation method will be applied for the separable or degenerate kernels, which means that

$$F_1(x,y) = \sum_{k=1}^{n} g_k(x)v_k(t), \qquad (4.105)$$

$$F_2(x,y) = \sum_{k=1}^{n} n_k(x)s_k(t).$$

This method can be applied as follows:

Substitute (4.105) into the system of Fredholm Integral Equations of the form

$$u(x) = h_1(x) + \int_a^b (F_1(x,y)u(y) + \breve{F}_1(x,y)\upsilon(y))dy,$$
$$\upsilon(x) = h_2(x) + \int_a^b (F_2(x,y)u(y) + \breve{F}_2(x,y)\upsilon(y))dy. \qquad (4.103)$$

This substitution gives

$$u(x) = h_1(x) + \sum_{k=1}^{n} g_k(x) \int_a^b v_k(y) u(y) dy + \sum_{k=1}^{n} \breve{g}_k(x) \int_a^b \breve{v}_k(y) \upsilon(y) dy,$$
$$\upsilon(x) = h_2(x) + \sum_{k=1}^{n} n_k(x) \int_a^b s_k(y) u(y) dy + \sum_{k=1}^{n} \breve{n}_k(x) \int_a^b \breve{s}_k(y) \upsilon(y) dy, \qquad (4.104)$$

Every integral on the right side is equal to a constant because of it relies on only the variable *y* with constant limits of integration for *y*. Accordingly, the equation (4.104) becomes

$$u(x) = h_1(x) + \psi_1 g_1(x) + ... + \psi_n g_n(x) + \delta_1 \breve{g}_1(x) + ... + \delta_n \breve{g}_n(x),$$
$$\upsilon(x) = h_1(x) + \beta_1 n_1(x) + ... + \beta_n n_n(x) + \varphi_1 \breve{n}_1(x) + ... + \varphi_n \breve{n}_n(x),$$

(4.105)

where

$$\psi_i = \int_a^b v_i(y) u(y) dy,$$
$$\delta_i = \int_a^b \breve{v}_i(y) \upsilon(y) dy,$$
$$\beta_i = \int_a^b s_i(y) u(y) dy,$$
$$\phi_i = \int_a^b \breve{s}_i(y) \upsilon(y) dy,$$

(4.106)

where$1 \le i \le n$. Finally, substituting (4.105) into (4.106) in turn gives a system of *n* algebraic equations which can be simplified and resolved to find the constants ψ_i, δ_i, β_i, ϕ_i.

4.2 Method to Solve Nonlinear Fredholm Integral Equations [1,2,5]

Previously in this chapter, linear Fredholm Integral Equations and methods of their solutions were presented in addition to systems of the linear Fredholm Integral Equations and methods of their solutions. Here, we study nonlinear Fredholm Integral Equations and systems of nonlinear Fredholm Integral Equations.

4.2.1 Method to solve nonlinear fredholm integral equations of the second kind

The general form of the nonlinear Fredholm Integral Equation of the second kind can be represented by the following formula

$$u(x) = h(x) + \mu\int_a^b F(x,y)G(u(y))dy, \tag{4.107}$$

Where $F(x, y)$ and $h(x)$are given real-valued functions. Moreover, $G(u(x))$ is a nonlinear function of $u(x)$and it is clear that the unknown function $u(x)$, which is to be determined, occurs inside and outside the integral sign. In the following, we shall present the various methods of solutions of the nonlinear Fredholm Integral Equation (4.107).

4.2.1.1 The direct computation method [1,2,5]

As we know, this method was applied before in this chapter to solve linear Fredholm Integral Equations. The Direct Computation Method is an efficient method which can be applied to solve nonlinear Fredholm Integral Equations. In this method, the exact solution is determined in a closed form and not in a series form[1,2]. It is worth mentioning that the direct computation method can be applied to separable or degenerate kernels of the form

$$F(x,y) = \sum_{k=1}^{n} g_k(x)v_k(y). \tag{4.108}$$

By substituting(4.108) into the nonlinear fredholm integral equation of the form

$$u(x) = h(x) + \mu\int_a^b F(x,y)G(u(y))dy, \tag{4.109}$$

we get

$$u(x) = h(x) + \mu g_1(x)\int_a^b v_1(y)G(u(y))dy + \mu g_2(x)\int_a^b v_2(y)G(u(y))dy + ...$$

$$+ \mu g_n(x)\int_a^b v_n(y)G(u(y))dy.$$

(4.110)

In equation(4.110), each integral on the right side is equivalent to a constant. Accordingly, equation (4.110) becomes

$$u(x) = h(x) + \mu\alpha_1 g_1(x) + \mu\alpha_2 g_2(x) + ... + \mu\alpha_n g_n(x), \tag{4.111}$$

where

$$\alpha_i = \int_a^b v_i(y)u(y)dy, 1 \le i \le n. \tag{4.112}$$

We get a system of n algebraic equations when substituting (4.111) into (4.112), which will be solved to determine the constants α_i, $1 \le i \le n$.
Therefore, when we obtain the values of α_i substituted into (4.111) to obtain on the solution $u(x)$ of the nonlinear Fredholm Integral Equation (4.109). It is important to note, the solution $u(x)$ of the nonlinear Fredholm Integral Equation is not necessarily unique because we may get more than one value for one or more of α_i, $1\le i \le n$.

4.2.1.2 The series solution method [2,5]

As mentioned previously, the main role in this method is the Taylor series that can be written at $x = 0$ as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{4.113}$$

Suppose we have the nonlinear Fredholm Integral Equations

$$u(x) = h(x) + \mu \int_0^1 F(x, y)G(u(y))dy, \tag{4.114}$$

Which has an analytic solution. Then, each coefficient a_n will be determined recurrently by using the form (4.113). Then substituting(4.113) into both sides of (4.114) gives

$$\sum_{n=0}^{\infty} a_n x^n = T(h(x)) + \int_0^1 F(x, y)G(\sum_{n=0}^{\infty} a_n y^n)dy, \tag{4.115}$$

Or we can simplify the previous form, to becomes

$$a_0 + a_1x + a_2x^2 + ... = T(h(x)) + \int_0^1 F(x, y)G(a_0 + a_1y + a_2y^2 + ...)dy, \tag{4.116}$$

Where $T(h(x))$ is the Taylor series for $h(x)$. In this manner, the integral equation(4.114) is transformed into a traditional integral in (4.115) or (4.116). Where in lieu of the integrating, the nonlinear term $F(u(x))$ terms of the form y_n, $n \ge 0$ will be integrated. It is important to note that, the Taylor expansions for functions contributory in $h(x)$ should be used if $h(x)$ contains elementary functions such as exponential functions, trigonometric functions, etc. This method can be applied by using the following steps:

The first step starts dealing with the right side of equation (4.115) or (4.116), where we work to integrate this side. In the second step, we work to collect the coefficients of like powers of x, and then process the equality for these coefficients on both sides to obtain a recurrence relation in a_j, $j \geq 0$. We then work on solving the recurrence relation to determine of the coefficients a_j, $j \geq 0$, directly followed by the series solution when substituting the derived coefficients into (4.113). Moreover, we can use the series which we obtained for numerical purposes if an exact solution is not possible to find. The series solution method gives exact solutions effectively when the solution $u(x)$ is a polynomial. However, if the solution $u(x)$ is not a polynomial, such as any other elementary function, by using the series solution method the exact solution is given after approximation of a few of the coefficients a_j, $j \geq 0$.

4.2.1.3 The adomian decomposition method [2,15]

As we know, in this chapter we presented this method previously, where it was applied to find a solution to the linear Fredholm Integral Equations. In this section, we will apply this method to the nonlinear Fredholm Integral Equation

$$u(x) = h(x) + \mu \int_a^b F(x,y) G(u(y)) dy. \tag{4.117}$$

We assume that $G(u(y))$ is a nonlinear function of $u(y)$. That means that the nonlinear Fredholm Integral Equation (4.117) contains the nonlinear function represented by $G(u(y))$ and the linear term is represented by $u(x)$. In fact, we can represent the linear term $u(x)$ with the following decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{4.118}$$

As explained previously, a recursive technique can easily compute the components $u_n(x)$, $n \geq 0$. Moreover, the so-called Adomian polynomials should be A_n, which represents the nonlinear $G(u(y))$ in equation(4.119). This process occurs by using the following algorithm

$$A_n = \frac{1}{n!} \frac{d^n}{d\mu^n} \left[G\left(\sum_{i=0}^{n} \mu^i u_i\right) \right]_{\mu=0}, n = 0,1,2,... \tag{4.119}$$

When substituting (4.118) and (4.119) into (15.84), we get

$$\sum_{n=0}^{\infty} u_n(x) = h(x) + \mu \int_a^b F(x,y) \left(\sum_{n=0}^{\infty} A_n(y)\right) dy. \tag{4.120}$$

Adomian's decomposition method uses the following recursive relations to determine the components $u_0(x)$, $u_1(x)$...

$$
\begin{aligned}
u_0(x) &= h(x) \\
u_1(x) &= \mu\int_a^b f(x,y)A_0(y)dy \\
u_2(x) &= \mu\int_a^b f(x,y)A_1(y)dy \\
u_3(x) &= \mu\int_a^b f(x,y)A_2(y)dy \\
\vdots &= \vdots \\
u_{n+1}(x) &= \mu\int_a^b f(x,y)A_n(y)dy,
\end{aligned}
\tag{4.121}
$$

where $n \geq 0$. With a reminder that the modified recurrence relation uses the formula $h(x) = h_1(x)+h_2(x)$ when $h(x)$ is decomposed into two components $h_1(x)$ and $h_2(x)$. This means the modified recurrence relation becomes

$$
\begin{aligned}
u_0(x) &= h_1(x), \\
u_1(x) &= h_2(x) + \mu\int_a^b f(x,y)A_0(y)dy, \\
u_2(x) &= \mu\int_a^b f(x,y)A_1(y)dy, \\
\vdots &= \vdots \\
u_{n+1}(x) &= \mu\int_a^b f(x,y)A_n(y)dy,
\end{aligned}
\tag{4.121}
$$

When a solution exists, the above series solution that we obtained may converge to the exact solution; otherwise, the above series solution can be used for numerical results.

4.2.1.4 The successive approximations method [2,5]

The method of successive approximations was applied previously in this chapter. In this section, it will be applied to the nonlinear Fredholm Integral Equation. In fact, this method grants a scheme that can be used to solve integral equations or initial value problems. Moreover, the technique of this method is to find successive approximations to the solution of the issue by starting with an initial guess, which is called a zeroth approximation. As a matter of fact, the zeroth approximation can be any real valued function $u_0(x)$, which will be used in a recurrence relation to determine the other approximations. Additionally, we can select any real-valued function for the zeroth approximation; the most frequently used values are 0, 1, or x. To illustrate how to apply this method, we consider the nonlinear Fredholm Integral Equation

$$u(x) = h(x) + \mu\int_a^b F(x,y)G(u(y))dy, \tag{4.122}$$

where $u(x)$ is the unknown function to be determined, $F(x\ ,y)$ is the kernel, $G(u(y))$ is a nonlinear function of $u(y)$, and μ is a parameter. As explained, the successive approximations method uses the following recurrence relation:

$u_0(x)$ = any selective real valued function

$$u_{n+1}(x) = h(x) + \mu\int_a^b F(x,y)u_n(y)dy\ , n \geq 0, \tag{4.123}$$

$$\mu < \frac{1}{f\,(b-1)} \quad , \tag{4.124}$$

where f is larger of the two numbers $F(1+\frac{R}{|\mu|F(b-a)})$and M. At the finish, we obtain the final solution by using the limit

$$u(x) = \lim_{n\to\infty} u_{n+1}(x)\,. \tag{4.125}$$

4.2.2 Method to solve a nonlinear fredholm integral equation of the first kind

The general form of the nonlinear Fredholm Integral Equation of the first kind maybe represented with the following formula:

$$h(x) = \int_a^b F(x,y)G(u(y))dy, \tag{4.126}$$

where the kernel $F(x\ ,y)$and $h(x)$are given real-valued functions and $G(u(x))$ is a nonlinear function of $u(x)$ [19] .When we want to determine a solution for this type of equation, equation (4.126) needs to be converted to a linear Fredholm Integral Equation of the first kind of the following form:

$$h(x) = \int_a^b F(x,y)v(y)dy, \tag{4.127}$$

by using the transformation

$$v(x) = G(u(x)). \tag{4.128}$$

On the assumption that *G(u(x))* is invertible, it can be set as

$$u(x) = G^{-1}(v(x)). \tag{4.129}$$

As we mentioned earlier, the linear Fredholm Integral Equation of the first kind tends to be an ill-posed problem because it does not satisfy certain properties and such a solution may not exist, or may not be unique even if it exists [19]. Now we will offer some of the methods which we will use to find a solution of the first kind of Fredholm Integral Equations.

4.2.2.1 The method of regularization [2,13,16]

This method was established independently by Tikhonov and Phillips [17,18]. The method transforms the linear Fredholm Integral Equation of the first kind

$$h(x) = \int_a^b f(x,y)u(y)dy \tag{4.130}$$

to the approximation Fredholm Integral Equation

$$\sigma u_\sigma(x) = h(x) - \int_a^b f(x,y)u_\sigma(y)dy \ , \tag{4.131}$$

where σ is a small positive parameter. It can also be noted that, the Fredholm Integral Equation of the second kind (4.131)can be represented as the following formula

$$u_\sigma(x) \ = \frac{1}{\sigma}h(x) - \frac{1}{\sigma}\int_a^b f(x,y)u_\sigma(y)dy \tag{4.132}$$

Furthermore, it was proved in [2] that the solution u_σ of the Fredholm Integral Equation of the second kind (4.132) converges to the solution $u(x)$ of the linear Fredholm Integral Equation of the first kind (4.130) as $\sigma \to 0$. This means that the exact solution $u(x)$ can be found by

$$u(x) = \lim_{\sigma \to 0} u_\sigma(x) \tag{4.133}$$

4.2.2.2 The homotopy perturbation method [2,19]

As previously stated, the method of homotopy perturbation in this chapter was discussed as to how this method can handle Fredholm Integral Equations of the second kind and linear Fredholm Integral Equations of the first kind. Now we will discuss this method and how to handle on linear Fredholm Integral Equations of the first kind. Firstly, assume we have the following nonlinear Fredholm Integral Equation of the first Kind:

$$h(x) = \int_a^b f(x, y)v(y)dy. \tag{4.134}$$

Second, we determine the operator

$$L(u) = h(x) - \int_a^b f(x, y)u(y)dy = 0, \tag{4.135}$$

Now we establish a convex homotopy according to the following form

$$h(u, p) = (1 - p)u(x) + pL(u)(x) = 0. \tag{4.136}$$

The embedding parameter p monotonically increases from zero to unity as the trivial problem $L(u) = 0$. The embedding parameter $p \in [0,1]$ can be considered to be an expanding parameter[22]. The HPM admits the use of the following expansion

$$u = \sum_{n=0}^{\infty} p^n u_n \ . \tag{4.137}$$

Then

$$v(x) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n(x) \ . \tag{4.138}$$

When the solution exists, it leads to the series (4.138) and converges to the exact solution. Substituting (4.137) into (4.136) and by re-applying the same previous steps, we get the recurrence relation

$$\begin{aligned} &u_0(x) = 0, u_1(x) = h(x), \\ &u_{n+1}(x) = u_n(x) - \int_a^b f(x, y)u_n(y)dy, n \geq 1. \end{aligned} \tag{4.139}$$

Then the following condition must be a justification for convergence when the kernel is separable:

$$\left| 1 - \int_a^b f(y, y)dy \right| < 1. \tag{4.140}$$

4.2.3 Method to solve systems of nonlinear fredholm integral equation [2,20]

In this section, our study will be limited to systems of nonlinear Fredholm Integral Equation of the second kind. These systems can be written in the following manner:

$$\begin{aligned} u(x) &= h_1(x) + \int_a^b (F_1(x, y)G_1(u(y)) + \breve{F}_1(x, y)G_1(v(y)))dy, \\ v(x) &= h_2(x) + \int_a^b (F_2(x, y)G_2(u(y)) + \breve{F}_2(x, y)G_2(v(y)))dy. \end{aligned}$$

(4.141)

Where a and b are constants, the functions $u(x), v(x)$ are the unknowns that will be determined. The function $h_i(x)$ and the kernels $F_i(x,y)$, $\breve{F}_1(x,y)$ are given a real valued function. Now, we will apply some of the methods that were previously used to solve systems of nonlinear Fredholm Integral Equations of the second kind.

4.2.3.1 The direct computation method [2]:

This method, as we presented previously, is an efficient method which can be applied to solve Fredholm Integral Equations. In this section, we will be applying this method to solve systems of nonlinear Fredholm Integral Equations of the second kind. In this method, the exact solution is determined in a closed form and not in a series form. It is worth mentioning that the direct computation method will be applied to the separable or degenerate kernels of the form

$$F_1(x,y)=\sum_{k=1}^{n} g_k(x)v_k(t),\ \breve{F}_1(x,y)=\sum_{k=1}^{n} \breve{g}_k(x)\breve{v}_k(t). \qquad (4.142)$$

$$F_2(x,y)=\sum_{k=1}^{n} n_k(x)s_k(t),\ \breve{F}_2(x,y)=\sum_{k=1}^{n} \breve{n}_k(x)\breve{s}_k(t).$$

This method can be applied as follows:

We substitute (4.142) into the systems of nonlinear Fredholm Integral Equations of the second kind (4.141)to obtain

$$\begin{aligned} u(x) &= h_1(x)+\sum_{k=1}^{n} g_k(x)\int_a^b v_k(y)G_1(u(y))dy+\sum_{k=1}^{n} \breve{g}_k(x)\int_a^b \breve{v}_k(y)\breve{G}_1(\upsilon(y))dy, \\ \upsilon(x) &= h_2(x)+\sum_{k=1}^{n} n_k(x)\int_a^b s_k(y)G_2(u(y))dy+\sum_{k=1}^{n} \breve{n}_k(x)\int_a^b \breve{s}_k(y)G_2(\upsilon(y))dy. \end{aligned} \qquad (4.143)$$

Every integral within the right side is equal to a constant because relies on only the variable y with constant limits of integration for y. Accordingly, the equation (4.143) becomes

$$\begin{aligned} u(x) &= h_1(x)+\psi_1 g_1(x)+\ldots+\psi_n g_n(x)+\delta_1\breve{g}_1(x)+\ldots+\delta_n\breve{g}_n(x), \\ \upsilon(x) &= h_1(x)+\beta_1 n_1(x)+\ldots+\beta_n n_n(x)+\varphi_1\breve{n}_1(x)+\ldots+\varphi_n\breve{n}_n(x), \end{aligned} \qquad (4.144)$$

where

$$\begin{aligned}\psi_i &= \int_a^b v_i(y)G_1(u(y))dy,\\ \delta_i &= \int_a^b \breve{v}_i(y)\breve{G}_1(\upsilon(y))dy,\\ \beta_i &= \int_a^b s_i(y)G_2(u(y))dy,\\ \phi_i &= \int_a^b \breve{s}_i(y)\breve{G}_2(\upsilon(y))dy,\end{aligned} \tag{4.145}$$

where $1 \leq i \leq n$. Finally, substituting (4.144) into (4.145) in turn gives a system of n algebraic equations which can be simplified and resolved to find the constants ψ_i, δ_i, β_i, ϕ_i and by substituting these constants into (4.144), we obtain the exact solutions.

4.2.3.2 The modified adomian decomposition method [2,21]

This method has been used previously in this chapter, where it was applied to find a solution to the linear Fredholm Integral Equations of the second kind. Let us assume that we have a system of nonlinear Fredholm Integral Equations of the second kind as follows:

$$\begin{aligned}u(x) &= h_1(x) + \int_a^b (F_1(x,y)G_1(u(y)) + \breve{F}_1(x,y)G_1(v(y)))dy,\\ v(x) &= h_2(x) + \int_a^b (F_2(x,y)G_2(u(y)) + \breve{F}_2(x,y)G_2(v(y)))dy.\end{aligned} \tag{4.146}$$

By this method, the above linear terms $u(x)$ and $v(x)$ will be decomposed by a certain number of components in the following form:

$$u(x) = \sum_{n=0}^{\infty} u_n(x),\ v(x) = \sum_{n=0}^{\infty} v_n(x). \tag{4.147}$$

By recurring, the components $v_n(x)$, $u_n(x)$ will be determined. Moreover, the modified adomian decomposition method can be used combined with the noise terms phenomenon or used independently. In the systems of nonlinear Fredholm Integral Equations of the second kind (4.146), the nonlinear functions G_i and $\breve{G}_i$, i=1, 2, should be replaced by A_n, where A_n represents the Adomian polynomials defined by

$$A_n = \frac{1}{n!}\frac{dn}{d\alpha}\left[G\left(\sum_{i=0}^{n}\alpha^i u_i\right)\right]_{\alpha=0}, n = 0,1,2,.... \tag{4.148}$$

Before using the recurrence relations, we should substitute the above suppositions into the systems of nonlinear Fredholm Integral Equations of the second kind (4.146) in order to determine the components $u_n(x)$ and $v_n(x)$. These procedures lead us to obtaining the exact solutions.

CHAPTER 5

NUMERICAL SOLUTION FOR SPECIAL NON-LINEAR FREDHOLM INTEGRAL EQUATION

In this chapter we will apply the Newton-Kantorovich method to compute a numerical solution for a special non-linear integral equation of the Fredholm type. Then we will compare between the results which we obtain by the numerical solution method and the results of the exact solution. Firstly, let us know the form of the special non-linear integral equation of the Fredholm type:

$$u(x) = h(x) + \mu\int_a^b f(x,y)(u(y))^t dy, x \in [a,b], t \geq 2, \tag{5.1}$$

where $h(x)$ is a given continuous function defined in $[a\ ,b]$, $f(x\ ,y)$ is a continuous function in $[a, b] \times [a\ ,b]$ and μ is a real number.

5.1 Method of Solution

As we previously stated, the nonlinear Fredholm Integral Equation of second kind is as follows:

$$u(x) = h(x) + \int_a^b F(x,y,u(y))dy,\ x \in [a,b], \tag{5.2}$$

where $u(y)$ is an unknown function, $F(x,y,u(y))$ is the kernel of the integral equations, with the assumption that all functions in (5.2) are continuous on $[a\ ,b]\times[a\ ,b]$, In the following, we will discuss the method of the solution.

5.1.1 The newton – kantorovich method [21,24]

This method is applied to find a solution for nonlinear Fredholm Integral Equation of second kind (5.2) using an iteration process as follows:

$$u_t(x) = u_{t-1} + \psi_{t-1}(x),\ \text{t=1,2,3,... .} \tag{5.3}$$

$$\psi_{t-1}(x) = \alpha_{t-1}(x) + \int_a^b F_t'(x, y, u(y))\psi_{t-1}(y)dy, \tag{5.4}$$

$$\alpha_{t-1} = h(x) + \int_a^b F(x, y, u_{t-1}(y))dy - u_{t-1}(x). \tag{5.5}$$

This means that from above equations (5.3),(5.4) and (5.5), we have

$$\psi_{t-1}(x) = h(x) + \int_a^b F(x, y, u_{t-1}(y))dy - u_{t-1}(x) + \int_a^b F_t'(x, y, u(y))\psi_{t-1}(y)dy. \tag{5.6}$$

Moreover, in this method, we need to know the concept of the quadrature method, which will be applied in order to solve the linear integral equation in equation (5.4) so that we can find an approximation of $\psi_{t-1}(x)$.

5.1.2 The quadrature method

In order to construct an approximate solution of an integral equation, the quadrature method has been used. This method is based on finding an alternative to integrals, where it works to replace the integrals by finite sums depending on the quadrature formula. In general, we have the following form:

$$\int_a^b \psi(x)dx = \sum_{i=1}^{n} A_i \psi(x_i), i = 1,...,n, \tag{5.7}$$

where $A_i \geq 0$, (i=1,...,n) are numerical coefficients specific to the choice of the function $\psi(x)$,x_i (i=1,...,n)which are the coordinates of the partition affiliate points to the integration interval [a ,b]. Simpson's Rule is the most flexible in terms of practice, where it is simplest and makes it the most common. The use of this method is as follows:

$$\int_a^b \psi(x)dx = \frac{b-a}{6}(\psi(a) + 4\psi(\frac{a+b}{2}) + \psi(b)). \tag{5.8}$$

As for the convergence of the algorithm and its existence, it is according to [22] that we can see that all conditions relating to establishing the convergence of the algorithm and its existence; for more details, see [22].

5.2 Numerical Solutions Using The Newton-Kantorovich Method

In this part, we will apply Newton-Kantorovich method to a special non-linear integral equation of the Fredholm type of the form (5.1) in order to find a numerical solution. To illustrate this, we consider the following example:

Example 5.1

Solve the following non-linear Fredholm Integral Equation by applying the Newton-Kantorovich method:

$$\psi(p) = \sin(\pi p) + \frac{1}{5}\int_0^1 \cos(\pi p)\sin(\pi y)(\psi(y))^3 dy,\ p \in [0,1], \tag{5.9}$$

whose exact solution is $\psi(p) = \sin(\pi p) + \frac{20-\sqrt{391}}{3}\cos(\pi p)$ [23].

Now first of all, for the initial approximation we take

$$u_0(p) = \sin(\pi p)\ . \tag{5.10}$$

According to equation (5.5), we find the residual

$$\alpha_0(p) = \sin(\pi p) + \frac{1}{5}\int_0^1 \cos(\pi p)\sin(\pi y)u_0^{\ 3}(y)dy - u_0(p)\,. \tag{5.11}$$

From (5.10), we have

$$u_o(y) = \sin(\pi y) \Rightarrow u_0^{\ 3}(y) \ = \sin^3(\pi y)\ . \tag{5.12}$$

Substituting (5.12) into (5.11), we get

$$\alpha_0(p) = \sin(\pi p) + \frac{1}{5}\int_0^1 \cos(\pi p)\sin(\pi y)\sin^3(\pi y)dy - \sin(\pi p) \tag{5.13}$$

$$= \frac{1}{5}\cos(\pi p)\int_0^1 \ \sin^4(\pi y) \tag{5.14}$$

$$= \frac{1}{5}\cos(\pi p)\frac{3}{8} \tag{5.15}$$

$$= \frac{3\cos(\pi p)}{40}. \tag{5.16}$$

we need to calculate the kernel $F'(p,y,u)$. Therefore, $F'(p,y,u)$ has the form

$$F'(p,y,u) = 3\sin(\pi p)\cos(\pi y)u^2(y). \tag{5.17}$$

According to equation (5.4), we form the following equation for $\psi_0(p)$,so

$$\psi_0(p) = \frac{3\cos(\pi p)}{40} + \frac{1}{5}\int_0^1 3\sin(\pi y)\cos(\pi p)u_0^{\ 2}(y)\psi_0(y)dy\,. \tag{5.18}$$

By applying the quadrature method to (5.8), we can obtain the solution for equation (5.18) such that

$$\psi_0(p)=\frac{3\cos(\pi p)}{40}+\frac{\cos(\pi p)}{10}\Big[\big(\sin(\pi 0)u_0{}^2(0)\psi(0)+4(\sin(\frac{\pi}{2})u_0{}^2(\frac{1}{2})\psi_0(\frac{1}{2}))\qquad(5.19)$$
$$+\sin(\pi)u_o{}^2(1)\psi_0(1)\Big].$$

After simplification of (5.19), we get

$$\psi_0(p)=\frac{3\cos(\pi p)}{40}+\frac{\cos(\pi p)}{10}\left[4u_0{}^2(\frac{1}{2})\psi_0(\frac{1}{2})\right].\qquad(5.20)$$

By computing $\psi_0(\frac{1}{2})$ in (5.19), we get $\psi_0(\frac{1}{2})=0$, and by substituting this result into (5.20), we get

$$\psi_0(p)=\frac{3\cos(\pi p)}{40}.\qquad(5.21)$$

Now we can define the first approximation to become the following function:

$$u_1(p)=u_0(p)+\psi_0(p)=\sin(\pi p)+\frac{3}{40}\cos(\pi p).\qquad(5.22)$$

By repeating the same previous processes, we get $\psi_1(p),\psi_2(p)$ as follows:

$$\alpha_1(p)=\sin(\pi p)+\frac{1}{5}\int_0^1\cos(\pi p)\sin(\pi y)u_1(y)^3\,dy-u_1(p).\qquad(5.23)$$

Also, by substituting (5.12) into (5.22) and integrating, we get

$$\alpha_1(p)=\frac{27}{6400}\cos(\pi p),\qquad(5.24)$$

and

$$\psi_1(p)=\frac{27}{64000}\cos(\pi p)+\frac{\cos(\pi p)}{5}\int_0^1 3\sin(\pi y)u_1{}^2(y)\psi_1(y)dy.\qquad(5.25)$$

By similar above computation, we get

$$\psi_1(p)=\frac{27}{64000}\cos(\pi p).\qquad(5.26)$$

Then we can define the second approximation to become the following function:

$$u_2(p)=u_1(p)+\psi_1(p)=\sin(\pi p)+\frac{3}{40}\cos(\pi p)+\frac{27}{64000}\cos(\pi p)$$
$$=\sin(\pi p)+\frac{4827}{64000}\cos(\pi p)\qquad(5.27)$$

To find $u_3(p)$ (final iteration), again we compute $\alpha_2(p),\psi_2(p)$

$$\alpha_2(p)=\sin(\pi p)+\frac{1}{5}\int_0^1\cos(\pi p)\sin(\pi y)u_2(y)^3\,dy-u_2(p).\qquad(5.28)$$

Also, by the same previous computational procedures, we get

$$\alpha_2(p) = \frac{779787}{163840000000}\cos(\pi p), \tag{5.29}$$

and

$$\psi_2(p) = \frac{779787}{163840000000}\cos(\pi p) + \frac{\cos(\pi p)}{5}\int_0^1 3\sin(\pi y)u_1^{\,2}(y)\psi_2(y)dy. \tag{5.30}$$

Again in the same previous manner, we solve this integral equation to obtain

$$\psi_2(p) = \frac{779787}{163840000000}\cos(\pi p), \tag{5.31}$$

Then we can define the third approximation as the following function:

$$u_3(p) = u_2(p) + \psi_2(p) = \sin(\pi p) + \frac{4827}{64000}\cos(\pi p) + \frac{779787}{163840000000}\cos(\pi p)$$

$$= \sin(\pi p) + \frac{12357899787}{163840000000}\cos(\pi p) \tag{5.32}$$

We stop the iteration at the third step. The table below shows the approximate solutions obtained by applying the Newton-Kantorovich method for three iterations according value of *p*, which is confined between zero and one. We compared these results with the results which were obtained by the exact solution for this example, where the results are very close. Moreover, we can see the convergence of solutions in Figure 5.1,which is shown clearly.

p	The exact solution	Newton–Kantorovich solution	Error=1.0e-007 *
0	0.075426688904937	0.075426634442139	0.544627982018708
0.1	0.380752038360555	0.380751986563356	0.517971990854349
0.2	0.648806725445999	0.648806681384670	0.440613293628545
0.3	0.853351689742522	0.853351657730092	0.320124295960511
0.4	0.974364644996211	0.974364628166281	0.168299302272246
0.5	1	1	0

0.6	0.927748387594096	0.927748404424026	0.168299302272246
0.7	0.764682299007373	0.764682331019803	0.320124295960511
0.8	0.526763779138947	0.526763823200276	0.440613293628545
0.9	0.237281950389340	0.237282002186539	0.517971991131905
1	-0.075426688904937	-0.075426634442139	0.544627982018708

Table (5.1) Convergence of Solutions

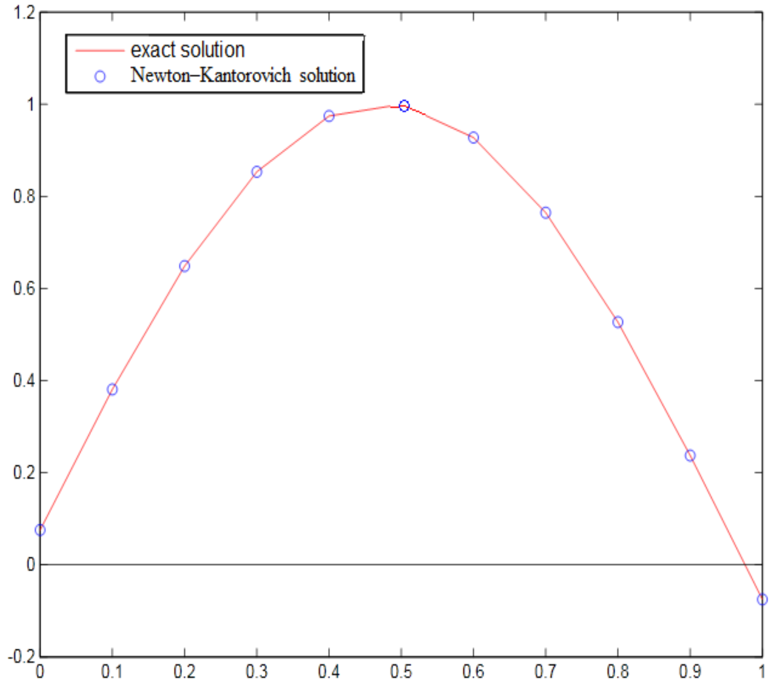

Figure 5.1 (Convergence of Solutions)

CHAPTER 6

CONCLUSION

No one disagrees on the importance of integral equations in our practical and scientific life. In fact, with these equations, we can identify various natural phenomena which may confront us. Moreover, from a scientific perspective, there are many physical, engineering and electrical phenomena the meaning and concept of which may be found with these integral equations, which represent the main players in these fields. A typical example of such equation is the Fredholm Integral Equation.

In this thesis after that we presented a detailed study about Fredholm Integral Equations. In particular, we identified the form of this equation and the definition of their types. In addition, we presented effective methods to find the exact solutions for each type of equation. Finally, we presented a special case of a Fredholm Integral Equation and applied the Newton-Kantorovich method to find a numerical solution which was then compared with the numerical results obtained by the exact solution method. We have shown that the solutions are convergent and closed. From this, it can be concluded that the Newton-Kantorovich method can be applied to find the exact solutions due to the method's high accuracy and convergence to the results of the exact solution. Computations and drawings of the functions were carried out with the MATLAB R2010a program.

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APPENDICES A

CURRICULUM VITAE

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EDUCATION

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WORK EXPERIENCE

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