Journal of Mathematical Physics

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Cite as: J. Math. Phys. **55**, 034101 (2014); https://doi.org/10.1063/1.4868479
Submitted: 30 April 2013 . Accepted: 01 March 2014 . Published Online: 13 March 2014

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JOURNAL OF MATHEMATICAL PHYSICS **55**, 034101 (2014)

Comment on "Maxwell's equations and electromagnetic Lagrangian density in fractional form" [J. Math. Phys. 53, 033505 (2012)]

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(Received 30 April 2013; accepted 1 March 2014; published online 13 March 2014)

In a recent paper, Jaradat *et al.* [J. Math. Phys. **53**, 033505 (2012)] have presented the fractional form of the electromagnetic Lagrangian density within the Riemann-Liouville fractional derivative. They claimed that the Agrawal procedure [O. P. Agrawal, J. Math. Anal. Appl. **272**, 368 (2002)] is used to obtain Maxwell's equations in the fractional form, and the Hamilton's equations of motion together with the conserved quantities obtained from fractional Noether's theorem are reported. In this comment, we draw the attention that there are some serious steps of the procedure used in their work are not applicable even though their final results are correct. Their work should have been done based on a formulation as reported by Baleanu and Muslih [Phys. Scr. **72**, 119 (2005)]. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4868479]

Jaradat *et al.*[1] presented a fractional form of the electromagnetic Lagrangian density by using the Riemann-Liouville fractional derivatives. They reported that they have used the Agrawal approach presented in Ref. 2 to obtain the fractional Euler-Lagrange equations for the constructed fractional electromagnetic Lagrangian density. Although they obtained the correct results, they did not use the correct formulations as well as the previous works relevant for their problem. The purpose of this comment is to present the correct procedure which should be used.

We start by reviewing the definition of partial fractional derivative for a function of several variables. The partial left Riemann-Liouville fractional derivative of order α_k is defined as[3]

$$({}_{a_k}\mathbf{D}^{\alpha_k}_{x_k} f)(x) = \frac{1}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int_{a_k}^{x_k} \frac{f(x_1, ..., x_{k-1}, u, x_{k+1}, ...x_n)}{(x_k - u)^{\alpha_k}} du, \tag{1}$$

while the partial right Riemann-Liouville fractional derivative of order α_k is of the form[3]

$$({}_{x_k}\mathbf{D}^{\alpha_k}_{b_k} f)(x) = \frac{1}{\Gamma(1-\alpha_k)} \frac{\partial}{\partial x_k} \int_{x_k}^{b_k} \frac{f(x_1, ..., x_{k-1}, u, x_{k+1}, ...x_n)}{(x_k - u)^{\alpha_k}} du, \tag{2}$$

where $0 < \alpha_k < 1$. Note that we have used bold $\mathbf{D}$ to represent partial fractional derivatives. If we have the Lagrangian for the field system in the form $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$, then the corresponding Euler-Lagrange equation becomes

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\mathcal{L}}{\partial(\partial_\mu \phi)} = 0, \tag{3}$$

where ϕ is the field variable.

Let us consider the action function of fractional classical fields of the form

$$J = \int \mathcal{L}(\phi_\rho, {}_{a_\mu}\mathbf{D}^{\alpha}_{x_\mu}\phi_\rho, {}_{x_\mu}\mathbf{D}^{\beta}_{b_\mu}\phi_\rho, x) d^3x dt, \tag{4}$$

which is the corrected form of Eq. (9) in Ref. 1. Following Ref. 3, in the case of the continuous field fractional derivatives, the Euler-Lagrange equations for this action are given as

$$\frac{\partial \mathcal{L}}{\partial \phi_\rho} + [{}_{x_\mu}\mathbf{D}^{\alpha}_{\mathbf{b}_\mu}(\frac{\partial \mathcal{L}}{\partial_{a_\mu}\mathbf{D}^{\alpha}_{x_\mu}\phi_\rho}) + {}_{a_\mu}\mathbf{D}^{\beta}_{x_\mu}(\frac{\partial \mathcal{L}}{\partial_{x_\mu}\mathbf{D}^{\beta}_{b_\mu}\phi_\rho})] = 0, \tag{5}$$

where a and b in Eq. (10) in Ref. 1 are replaced by a_μ and b_μ.

At this point we notice that Jaradat *et al.*[1] have done some technical errors in their calculations. They used ordinary fractional derivatives instead of the partial fractional derivatives. Also, they have fixed the limits of the integration for all fractional derivatives. They should use Eqs. (1) and (2) instead of Eqs. (1) and (2) in their paper. Therefore, Eq. (10) in Ref. 1 should be replaced by Eq. (5). Also the left and right partial fractional derivatives in their work should be written as ${}_{a_k}\mathbf{D}^{\alpha_k}_{x_k}$ and ${}_{x_k}\mathbf{D}^{\alpha_k}_{b_k}$, respectively.

In addition, they claimed that Eq. (10) in Ref. 1 has been obtained by Agrawal.[2] We recall that Agrawal discussed in that paper only the discrete systems but Eq. (10) is related to the continuous systems, so the authors should have used the generalization of Agrawal work which is given by Baleanu and Muslih in Ref. 3. We believe that all equations should be corrected according to the notes discussed above. For example, Eq. (21) in Ref. 1 should be corrected as

$$\frac{4\pi}{c} j_i = {}_{a_t}\mathbf{D}^{\alpha}_{t}(-{}_{a_t}\mathbf{D}^{\alpha}_{t}A^j - {}_{a_i}\mathbf{D}^{\alpha}_{x^i}\phi) + {}_{a_i}\mathbf{D}^{\alpha}_{x^i}({}_{a_i}\mathbf{D}^{\alpha}_{x^i}A^j - {}_{a_j}\mathbf{D}^{\alpha}_{x^j}A^i). \tag{6}$$

Another technical error they used is the traditional method in obtaining the Hamiltonian Eq. (27) in Ref. 1, while they should have used the theory of constrained systems given by Dirac[4] because one of the conjugate momenta (namely, $\pi_{\alpha_\phi} = 0$) is equal to zero.

In addition to this, the fractional Lagrangian Eq. (26) in Ref. 1 has no obvious form. We are unable to figure out the meaning of its arguments, e.g., what is the meaning of A^i and A^j and what is the difference between them.

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