VOLTERRA TYPE INTEGRAL EQUATIONS

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VOLTERRA TYPE INTEGRAL EQUATIONS

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ABSTRACT

VOLTERRA TYPE INTEGRAL EQUATIONS

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It is truism that, Volterra integral equations have many applications in various disciplines of sciences. Therefor, these equations have been attracted the attention of a huge number of mathematicians and scientists who work in the areas in which these equations appear. Hence, it is worth of note to study these equations from theoretical and computational points of views. Because of the significance and broadly usage of these equations in numerous fields of mathematics.

So, in this thesis, the first and the second kind of Volterra integral equations have been defined. The uniqueness theorems of these equations are discussed by using the fixed point theory. Many methods to solve linear as well as nonlinear Volterra integral equations are being considered. Then, methods of solving systems of such equations are also mentioned. In addition to the above mentioned, an attention is paid to the singular Abel Volterra integral equation.

Keywords: Volterra Integral Equations, First Kind, Second Kind, Linear, Nonlinear.

ÖZ

VOLTERRA TİPİNDEN İNTEGRAL DENKLEMLERİ

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Volterra integral denklemleri, çeşitli bilim dallarında birçok uygulama alanına sahip olduğundan, çok sayıda matematikçi ve bilim adamının dikkatini çekmektedir.
Volterra integral denklemlerinin önemi ve birçok alandaki yaygın kullanımları göz önüne alındığında, bu denklemleri hem teorik hemde hesaplama açısından incelemek önemlidir.
Bu tezde, hem birinci hem de ikinci tipten Volterra integral denklemleri tanımlanır. Sabit nokta teorisi kullanılarak, bu denklemler için teklik teoremleri tartışılır. Doğrusal olmayan Volterra integral denklemlerinin yanı sıra doğrusal olan Volterra integral denklemlerini çözmek içinde birçok yöntem ele alınır. Bu tip denklemleri içeren sistemleri çözme yöntemlerinden de bahsedilir. Ayrıca, tekil Volterra integral denklemleride dikkate alınır.

Anahtar Kelimeler: Volterra İntegral Denklemleri, Birinci Tip, İkinci Tip, Doğrusal, Doğrusal Olmayan.

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CHAPTER 1

INTRODUCTION

1.1 Background

It is clear that, if someone works in a scientific discipline, this field will enhance us to understand the world we are living in. However, one should encounter differential equations, integral equations and integro-differential equations [1]. Among these equations, integral equations occur naturally in many fields of science such as elasticity, heat transfer, fluid dynamics, game theory, etc… [2].

Here, J. Fourier in this sense, set out to initial the theory of integral equations when he worked on the well known Fourier transform and its inversion formula. Abel in accordance with this, devoted his time to the solution of the so called "Abel's integral equation". Abel's efforts encouraged many scientists in this area like Rouche', Sonine, Bois Reymond, Goursat, Tamalin and Tonellito who worked on integral equations. They completed the work which has been done by Abel who considers the various types of integral equations in the late years of the 19^{th}. century and the first years of the 20^{th}. century [3].

Accordinglly, Liouville was also one of the mathematicians who worked on integral equations. He discovered in 1840's that a certain type of second order linear ordinary differential equation under some initial conditions is equivalent to an integral equation known later as a Volterra-type integral equation of the second type [3].

However, in 1895 a new era for the theory of integral equations started under the term of Volterra, not like other mathematicians who tried to formulize the solutions of integral equations or worked on special cases. Thus, they paid efforts on the type of equations named later by Lalesco and picard as Volterra equations. Volterra studied these equations from a functional analytic point of view. So, he was interested in the existence of the solutions of these equations. Moreover to that, he was also interested in the applications of these equations. One of the most interesting applications is 'hereditarary mechanics' which was observed when Volterra was inspecting a population growth model [4].

Later in the early years of the last century, it was found that Volterra integral equations have many applications. Since that period of time, the theory of Volterra-type equations have called attention of many scientists from different countries. These scientists have been contributing to the development of the theory of these equations from all aspects until this moment of time [5].

In this thesis, the researcher has studied linear and nonlinear Volterra type equations and presented their uniqueness theorems and some methods of solutions of these equations.

1.2 Organization of the Thesis

In this current study, there are six independent chapters which include the sufficient and major relevant information to the topic (Volterra Type Integral Equations).
The researcher covers the kinds, properties, the possible solutions, and the set of theorms which proved the uniqueness of solution. Therefore, this thesis is organized as follows:
Chapter one is dealing with the background of the Volterra Integral Equations affairs.
In chapter two, the study conductor focuses attention on some important definitions which are related to a scope of understanding the properties of integral equations and Volterra integral equations.
In chapter three, the researcher presents a set of theorms that confirm the uniqueness of solution to the Integral Equations in general and to the Volterra Type of Integral Equations in particular.

In chapter four, the study conductor employs the possible efforts to produce variable authentic methods to solve the targeted Equation. These methods will deal with these Equation (linear and nonlinear), whether it was isolated or as a system.

Chapter five, is designed to deal with Singular Volterra Integral Equation as well as their features, trying to find a possible solution in terms of dealing with such Equations.

Finally, chapter six is assigned to conclusion part of what related to the study concerns.

CHAPTER 2

DEFINITIONS

In this chapter, we will present some definitions in order to help us to understand the content of this thesis.

2.1 Basic Definitions

Definition 1. [4-6] An integral equation is an equation that involves the unknown function $u(x)$ that appears inside of an integral sign. The most standard type of an integral equation in $u(x)$ is of the form

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt \tag{2.1}$$

Definition 2. [4-5] If the exponent of the unknown function $u(x)$ inside the integral sign in (2.1) is one, the integral equation is called linear. If the unknown function $u(x)$ has exponent other than one, or if the equation contains nonlinear functions of $u(x)$, the integral equation is called nonlinear.

Definition 3. If the function $f(x) = 0$ in equation (2.1), then equation (2.1) is called homogeneous. Otherwise it is called inhomogeneous [7].

Definition 4. [4-6] Equation (2.1) is called singular if one of the limits of integration $g(x)$, $h(x)$, or both are infinite, or if the kernel $K(x,t)$ becomes unbounded at one or more points in the interval of integration.

Definition 5. [8] If at least one limit of the integral in equation (2.1) is a variable so, is called a Volterra integral equation.

These types of equations are classified into two types, the general form of Volterra integral equation of the first kind is

$$f(x) = \lambda \int_0^x K(x,t)u(t)dt \tag{2.2}$$

where, the unknown function $u(x)$ appears inside the integral sign.

The second kind given by

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt \tag{2.3}$$

where, the unknown function $u(x)$ appears inside and outside the integral sign.

Generally, it is quite hard to transact with Volterra equations of the first kind [9]. So, we will convert Volterra integral equations of the first kind to Volterra integral equations of the second kind by two methods according to the following section of this thesis.

Here, in the first method, we assume that $K(x,t)$ and $f(x)$ are sufficiently differentiable in (2.2) [2], [4-5]. Then by differentiating both sides of (2.2) with respect to x one finds

$$f'(x) = K(x,x)u(x) + \int_0^x K_x(x,t)u(t)dt \tag{2.4}$$

the equation (2.4) can be reduced to an equation of the second kind if $K(x,x) \neq 0$

$$\frac{f'(x)}{K(x,x)} = u(x) - \int_0^x \frac{K_x(x,t)}{K(x,x)} u(t)dt.$$

In the second method, equation (2.2) can be reduced to an equation of the second type by assuming that

$$\Phi(x) = \int_0^x \Phi(t)dt$$

and execute an integration by parts in (2.2) [2,4].

Then

$$f(x) = K(x,x)\Phi(x) - \int_0^x K_x(x,t)\Phi(t)dt$$

thus, if

$$K(x,x) \neq 0,$$

then

$$\frac{f(x)}{K(x,x)} = \Phi(x) - \int_0^x \frac{K_x(x,t)}{K(x,x)} \Phi(t)dt,$$

the last equation is Volterra integral equations of the second kind.

CHAPTER 3

UNIQUENESS THEOREMS OF SOLUTIONS OF INTEGRAL EQUATIONS

Before we proceed thinking about the current methods to solve Volterra integral equations, we must contemplate of whether the solution exists and whether or not it is a single solution.
Henceforth, in this chapter, we will submit a set of theorems that ensure that we have the solution and guarantee its uniqueness.

3.1 Fixed Point Theorems

We will show in this section some properties of a class of alleged contraction operators. These properties enable us to get a number of existence and uniqueness theorems for the solution of integral equation [9].

Definition 1.

Let B be a Hilbert space and N a limited operator on B. N may not be a linear operator. If there exists a positive constant $\omega < 1$
such that

$$\|Nf_1 - Nf_2\| \leq \omega \|f_1 - f_2\| \tag{3.1}$$

for all f_1, f_2 in B, then N is called a contraction operator [9].

Theorem 1.

The equation

$$Nf = f \tag{3.2}$$

has a unique solution f in B if N a contraction operator on B. Such a solution is said to be a fixed point of N [9].

Proof. Let f and h be a fixed points so that

$$Nf = f$$

$$Nh = h$$

Then

$$\| f - h\| = \|Nf - Nh\| \le \omega \| f - h\|$$

and

$$(1-\omega)\| f - h\| \le 0$$

because $\| f - h\|$ must be non-negative, then

$$\| f - h\| = 0$$

thus

$$f = h.$$

It is being concluded that if (3.2) does have a solution it should be unique. To show that (3.2) has a solution so that, we shall constitute an iteration procedure. Choose any f_0 and then set up a sequence $\{f_m\}$ defined by

$$f_{m+1} = Nf_m, \quad m = 0,1,2,...$$

First we shall show that this sequence is a Cauchy sequence, and then that its limit is actually a solution of (3.2). That it has a limit to be followed from the certainty that a Cauchy sequence essential has a unique limit in a Hilbert space. The limit will separate of the initial select f_0, because it will be a solution of (3.2), which is necessary to be unique.

We note first that

$$\begin{aligned}\| f_{m+1} - f_m\| &= \|Nf_m - Nf_{m-1}\| \\ &\le \omega \| f_m - f_{m-1}\|\end{aligned}$$

By the above successive application, we get

$$\begin{aligned}\| f_{m+1} - f_m\| &\le \omega \| f_m - f_{m-1}\| \\ &\le \omega^2 \| f_{m-1} - f_{m-2}\| \\ &\le \ldots\end{aligned}$$

$$\leq \omega^n \| f_1 - f_0 \|$$

So we have in general, if $m > r$,

$$\| f_m - f_r \| = \| (f_m - f_{m-1}) + (f_{m-1} - f_{m-2}) + \ldots + (f_{r+1} - f_r) \|$$

$$\leq \| (f_m - f_{m-1}) + (f_{m-1} - f_{m-2}) + \ldots + (f_{r+1} - f_r) \|$$

$$\leq (\omega_{m-1} + \omega_{m-2} + \ldots + \omega_r) \| (f_1 - f_0) \|$$

$$\leq (\omega_r + \omega_{r+2} + \ldots) \| (f_1 - f_0) \|$$

$$= \frac{\omega^r}{1-\omega} \| f_1 - f_0 \|$$

so that

$$\lim_{m,r \to \infty} \| (f_{m+1} - f_r) \| = 0$$

we obtained that, $\{f_m\}$ is a Cauchy sequence, and we denote its limit by f.

We shall have to proof that the limit f is a solution of (3.2). In opinion of the fact that N is a continuous operator so, we have

$$Nf = N(\lim f_m)$$

$$= \lim Nf_m$$

$$= \lim f_{m+1}$$

$$= f$$

thus

$$Nf = f$$

it follows that

$$f = \lim_{m\to\infty} N^m f_0$$

There is a generalization of the previous theorem that will show to be specially favorable for Volterra operators.

Theorem 2.

If N^m is a contraction operator where N an operator on B and m is the nth power of N. Then the equation

$$Nf = f \tag{3.3}$$

has a unique solution f in B [9].

Proof. We can assure that, the equation

$$N^m f = f$$

has a unique solution by the preceding theorem. We can get the solution by finding

$$\lim_{k\to\infty} N^{km} f_0 = f$$

for an arbitrary initial function f_0.

In special, we see that, by allowing

$$f_0 = Nf$$

$$\lim_{k\to\infty} N^{km} Nf = f.$$

But we have

$$N^m f = f$$

and

$$N^{km} f = f$$

hence

$$\begin{aligned} \lim_{k\to\infty} N^{km} Nf &= \lim_{k\to\infty} NN^{km} f \\ &= \lim_{k\to\infty} Nf \\ &= Nf \end{aligned}$$

thus

$$Nf = f.$$

To prove that, this solution is unique so, we note that if

$$Nf = f, \qquad Nh = h$$

then also we have

$$N^m f = f, \qquad N^m h = h$$

hence N^m is a contraction operator with a unique fixed point

$$f = h.$$

3.2 Uniqueness Theorems for Volterra Integral Equations

The results in the previous section can be applied to Volterra integral equations as well [9].

Theorem 3.

Let $f(x) \in L_2[0,1]$ (we consider a finite interval let it be $[0,1]$ and without loss of generality) and assume that $K(x,t)$ is continuous for $x,t \in [0,1]$ and consequently uniformly bounded, say $|K(x,t)| \le A$. Then the equation

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt \tag{3.4}$$

for all λ and $f(x)$ in $L_2[0,1]$, has a unique solution $u(x)$ [6].

Proof. We consider the operator

$$Nu = f(x) + \lambda \int_0^x K(x,t)u(t)dt$$

If Nu has a fixed point, such a fixed point should be a solution of (3.4). And to prove that, such a fixed point exists we will prove that N^m is a contraction operator for some m. Then N will have a unique fixed point by theorem 2.

So now

$$N^m u = f + \lambda Kf + ... + \lambda^{m-1}K^{m-1} f + \lambda^m K^m u$$

where

$$N^m u = \int_0^x K_m(x,t)u(t)dt$$

Thus

$$\left\|T^m u_1 - T^m u_2\right\| = |\lambda|^n \left\|\int_0^x K_m(x,t)(u_1(t) - u_2(t))dt\right\|$$

We can use the equation

$$K_m(x,t) = \int_0^x K(x,z)K_{m-1}(z,t)dz, \quad m = 2,3,...$$

$$K_1(x,t) = K(x,t)$$

to determine $K_m(x,t)$

By hypothesis $\left|K_1(x,t)\right| \le A$ and one can then prove inductively that

$$\left|K_n(x,t)\right| \le \frac{A^m (x-t)^{m-1}}{(m-1)!}, \quad 0 \le t \le x$$

For $m=1$, the above is clearly true. If it is true for m, then

$$\left|K_{m+1}(x,t)\right| \le \int_t^x \left|K(x,z)\right|\left|K_m(z,t)\right| dz$$

$$\le \frac{A^{m+1}}{(m-1)!}\int_t^x (z-t)^{m-1} dz$$

$$= \frac{A^{m+1}(x-t)^m}{m!}.$$

Thus

$$\left\|T^m u_1 - T^m u_2\right\| \le \frac{\left|\lambda\right|^m A^m}{(n-1)!}\left\|\int_0^x \left(u_1(t)-u_2(t)\right)dt\right\|$$

$$\le \frac{\left|\lambda\right|^m A^m}{(n-1)!}\left\|u_1 - u_2\right\|.$$

For m sufficiently large

$$\frac{\left|\lambda\right|^m A^m}{(n-1)!} < 1,$$

so that N^m is a contraction operator, and therefore (3.4) has a unique solution.

Theorem 4.

Let $f(x) \in L_2[0,1]$, and assume that $K(x,t)$ is such that

$$\int_0^1\int_0^1 \left|K(x,t)\right|^2 dxdt < \infty.$$

Then the equation

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt \tag{3.5}$$

for all λ, in $L_2[0,1]$ has a unique solution [9].

Proof. We assume

$$C^2(x) = \int_0^x |K(x,t)|^2 dt, \qquad D^2(x) = \int_t^1 |K(x,t)|^2 dx$$

and both $C^2(x)$ and $D^2(t)$ are integrable by hypothesis.

Let E be such that

$$\int_0^1 C^2(x)dx \le E,$$

$$\int_0^1 D^2(t)dt \le E.$$

In addition, we define the function $q(x)$ by

$$q(x) = \int_0^x C^2(t)dt, \quad \text{where } q(1) \le N$$

As in the proof of theorem 3 we consider alternatively of (3.5) the equivalent equation

$$u(x) = f(x) + \lambda K f + \lambda^2 K^2 f ... + \lambda^{m-1} K^{m-1} f + \lambda^m K^m u \tag{3.6}$$

where

$$K^m u = \int_0^x K_m(x,t)u(t)dt$$

To estimate $\|K^m\|$ we examine $K_m(x,t)$.

Now

$$K_2(x,t) = \int_t^x K(x,z)K(z,t)dz$$

and by use inequality of the Cauchy-Schwarz

$$|K_2(x,t)|^2 \le \int_t^x |K(x,z)|^2 dz \int_t^x |K(x,t)|^2 dt$$

$$\le C^2(x)D^2(t).$$

Similarly,

$$K_3(x,t) = \int_t^x K(x,z)K_2(z,t)dt$$

So that

$$\left|K_3(x,t)\right|^2 \leq \int_t^x \left|K(x,z)\right|^2 dz \int_t^x \left|K_2(z,t)\right|^2 dz$$

$$\leq C^2(x)D^2(t)\int_t^x C^2(z)dz$$

$$= C^2(z)dz$$

$$= C^2(x)D^2(t)\left[q(x)-q(t)\right]$$

It is easy to carry through an inductive argument to prove that

$$\left|K_m(x,t)\right|^2 \leq C^2(x)D^2(t)\frac{\left[q(x)-q(t)\right]^{n-2}}{(n-2)!}, \quad n \geq 2.$$

Thus, (3.6) can be written as

$$u = N^m u$$

where

$$Nu = f + \lambda K u$$

$$\left|N^m u_1 - N^m u_2\right|^2 = \left|\int_0^x K_m(x,t)\left[u_1(t)-u_2(t)\right]dt\right|^2$$

$$\leq \int_0^x \frac{C^2(x)D^2(t)\left[q(x)-q(t)\right]^{m-2}}{(m-2)!}dt$$

$$\times \int_0^x \left|u_1(t)-u_2(t)\right|^2 dy$$

$$\leq \frac{C^2(x)\left[q(x)\right]^{m-2}}{(m-2)!}\int_0^1 D^2(t)dt\left\|u_1-u_2\right\|^2$$

By an integration we then find

$$\left\|N^m u_1 - N^m u_2\right\|^2 \leq \frac{\left[q(1)\right]^{m-1}E}{(m-1)!}\left\|u_1-u_2\right\|^2$$

$$\leq \frac{E^m}{(m-1)!}\left\|u_1-u_2\right\|^2,$$

so that if

$$\frac{E^m}{(m-1)!} < 1,$$

then N is a contraction operator.

For large m that will be the case so that (3.6) and therefore (3.5) as well will have a unique solution in $L_2[0,1]$.

CHAPTER 4

METHODS OF SOLUTIONS

In this chapter, we are going to consider some methods and techniques for solving Volterra integral equations.

4.1 Methods to Solve Linear Volterra Integral Equations

There are many methods to solve linear Volterra integral equations. Some of them are old and the others are new. As aresult, we will explain the most important of these methods in this section.

4.1.1 Methods to solve the second kind of the Volterra integral equations

4.1.1.1 The Adomian decomposition method

The Adomian decomposition method (ADM) was developed and introduced by Adomian in 1990. This method arises to work for linear, nonlinear integral equations, differential equations and integro-differential equations [4,10].

We shall explain the technique of this method by expressing $u(x)$ in the form of a series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{4.1}$$

or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \ldots \tag{4.2}$$

and the value of $u_0(x)$ as the term outside the integral sign of equation (2.3),

hence,

$$u_0(x) = f(x)$$

To establish the recurrence relation, we substitute (4.1) into the Volterra integral equation (2.3) to obtain:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x K(x,t) \left(\sum_{n=0}^{\infty} u_n(x) \right) dt$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + ... = f(x) + \lambda \int_0^x K(x,t) [u_0(t) + u_1(t) + ...] dt$$

We can get the value of the components $u_0(x),\ u_1(x),\ u_2(x),\ ...,\ u_n(x),\ ...$ of the unknown function $u(x)$ as follows

$$u_0(x) = f(x)$$

$$u_1(x) = \lambda \int_0^x K(x,t) u_0(t) dt$$

$$u_2(x) = \lambda \int_0^x K(x,t) u_1(t) dt$$

$$u_3(x) = \lambda \int_0^x K(x,t) u_2(t) dt$$

$$u_{n+1}(x) = \lambda \int_0^x K(x,t) u_n(t) dt, \quad n \geq 0. \tag{4.3}$$

Then, we can get the solution $u(x)$ by

$$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + ...$$

that converges to a closed form solution.

In the next example, we will explain the technique of this method. Let us suppose that, we have the following equation

$$u(x) = 1 + \int_0^x u(t) dt. \tag{4.4}$$

Hence,

$$f(x) = 1, \qquad \lambda = 1, \qquad K(x,t) = 1.$$

By substituting (4.2) into both sides of (4.4) gives

$$u_0(x) + u_1(x) + u_2(x) + ... = 1 + \int_0^x [u_0(t) + u_1(t) + ...] dt$$

then we set

$$u_0(x) = f(x) = 1$$

to get the value of the other components, we will apply the recurrence relation (4.3)

$$u_1(x) = \int_0^x u_0(t)dt = \int_0^x 1dt = x$$

$$u_2(x) = \int_0^x u_1(t)dt = \int_0^x tdt = \frac{1}{2!}x^2$$

$$u_3(x) = \int_0^x u_2(t)dt = \int_0^x \frac{1}{2!}t^2 dt = \frac{1}{3!}x^3$$

and so on.

Thus, by using (4.2)

$$u(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

the series solution $u(x)$ converges to the closed form solution

$$u(x) = e^x$$

4.1.1.2 The modified decomposition method

If the function $f(x)$ consists of a mixture of two or more of trigonometric functions, hyperbolic functions, polynomials, and others. The evaluation of the components $u_j, j \geq 0$ requires long time and difficult work [5].

We can set the function $f(x)$ as the sum of two partial functions, such as $f_1(x)$ and $f_2(x)$. In other words, we can set

$$f(x) = f_1(x) + f_2(x)$$

we identify the component $u_0(x)$ by one part of $f(x)$ to minimize the size of calculations. We will use the other part of $f(x)$ to find the value of the component $u_1(x)$. In other words, the modified decomposition method introduces the modified recurrence relation

$$u_0(x) = f_1(x)$$

$$u_1(x) = f_2 + \lambda \int_0^x K(x,t) u_0(t) dt$$

$$u_{k+1}(x) = \lambda \int_0^x K(x,t) u_k(t) dt. \qquad (4.5)$$

We can get the exact solution $u(x)$ by correct selection of the functions $f_1(x)$ and $f_2(x)$ and by using very few iterations, and may be by evaluating only two or three components. The success of this method depends only on the correct choice of $f_1(x)$ and $f_2(x)$, and this can be made through experience only. A rule that may help for the correct choice of $f_1(x)$ and $f_2(x)$ could not be found until now.

We can not use this method if $f(x)$ consists of one term only, in this case the standard decomposition method can be used.

The following example will illustrate the technique of this method. Let us suppose that, we have the following equation

$$u(x) = 2x - (1 - e^{-x^2}) + \int_0^x e^{-x^2+t^2} u(t) dt.$$

Hence,

$$f(x) = 2x - (1 - e^{-x^2})$$

we set

$$f_1(x) = 2x,$$

$$f_2(x) = -(1 - e^{-x^2}),$$

thus

$$u_0(x) = f_1(x) = 2x,$$

using (4.5) gives

$$u_1(x) = -(1 - e^{-x^2}) + \int_0^x e^{-x^2+t^2} 2t dt$$

$$= -1 + e^{-x^2} + [e^{-x^2+t^2}]_0^x$$

$$= -1 + e^{-x^2} + [e^0 - e^{-x^2}]$$

$$= -1 + e^{-x^2} + 1 - e^{-x^2}$$

$$= 0$$

$$u_{k+1}(x) = \lambda \int_0^x K(x,t) u_k(t) dt = 0, \quad k \geq 1.$$

Thus,

$$u(x) = 2x.$$

4.1.1.3 The noise terms phenomenon method

The congruent terms with opposite signs that appear in the components $u_0(x)$ and $u_1(x)$ in Adomian decomposition method are called the noise terms. Among others components may arise other noise terms. These noise terms may exist for some equations, and may not arise for other equations [5,11].

The technique of this method is done by canceling the noise terms between $u_0(x)$ and $u_1(x)$, the non-canceled remaining terms of $u_0(x)$, after processing the cancellations may give the exact solution of the integral equation. The appearance of the noise terms between $u_0(x)$ and $u_1(x)$ is not permanently adequate to get the exact solution by canceling these noise terms. So, it is needful to show that the non-canceled remaining terms of $u_0(x)$ satisfy the given integral equation.

It was officially proved that, for specific cases of inhomogeneous integral equations the noise terms arise, while homogeneous integral equations do not give rise to the noise terms. The conclusion about the self-canceling noise terms was based on solving several specific integral models.

It was officially proved that necessary condition is governed the appearance of the noise terms. The component $u_0(x)$ should contain the exact solution $u(x)$ among other components. Moreover, it was proved that, the inhomogeneity condition of the equation does not always guarantee the appearance of the noise terms.

In the next example, we will explain the technique of this method. Let us suppose that, we have the following equation

$$u(x) = 6x + 2x^3 - \int_0^x tu(t) dt.$$

Hence

$$f(x) = 6x + 2x^3$$

$$u_0(x) = f(x) = 6x + 2x^3$$

$$u_1(x) = -\int_0^x t[6t + 2t^3]dt$$

$$= -\left[2t^3 + \frac{2}{5}t^5\right]_0^x$$

$$= -2x^3 - \frac{2}{5}x^5,$$

the noise terms $\mp 2x^3$ arise in $u_0(x)$ and $u_1(x)$.

Thus, by canceling these terms from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = 6x$$

4.1.1.4 The successive approximations method

In the successive approximations method, we substitute any selective real-valued continuous function $u_0(x)$, called the zeroth approximation, instead of the unknown function $u(x)$ under the integral sign of the Volterra equation (2.3). The most usually selected function for $u_0(x)$ are $0, 1,$ and x [2,4], [12].

We will obtain the first approximation $u_1(x)$ by this substitution

$$u_1(x) = f(x) + \lambda\int_0^x K(x,t)u_0(t)dt \tag{4.6}$$

It is evident that, $u_1(x)$ is continuous if $f(x)$, $K(x,t)$, and $u_0(x)$ are continuous.

The second approximation $u_2(x)$ can be obtained similarly by replacing $u_0(x)$ in equation (4.6) by $u_1(x)$ obtained above. And we find

$$u_2(x) = f(x) + \lambda\int_0^x K(x,t)u_1(t)dt$$

we obtain an infinite sequence of functions $u_0(x),\ u_1(x),\ u_2(x),\ ...,\ u_n(x),\ ...$ by continuing in this technique which satisfies the recurrence relation

$$u_n(x) = f(x) + \lambda\int_0^x K(x,t)u_{n-1}(t)dt, \quad n = 1,2,3,... \tag{4.7}$$

The solution $u(x)$ is obtained of the equation (2.3) as

$$u(x) = \lim_{n\to\infty} u_n(x)$$

The following example will illustrate the technique of this method. Let us suppose that, we have the following equation

$$u(x) = x + \int_0^x u(t)dt \tag{4.8}$$

first,we set

$$u_0(x) = x$$

then apply (4.7) gives

$$u_1(x) = x + \int_0^x tdt = x + \frac{1}{2!}x^2$$

$$u_2(x) = x + \int_0^x (t + \frac{1}{2!}t^2)dt = x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3,$$

and so on.

The solution $u(x)$ of (4.8) is given by

$$u(x) = \lim_{n\to\infty} u_n(x)$$

$$= e^x - 1$$

4.1.1.5 The method of successive substitutions

In this method, we substitute successively for $u(x)$ its value as given by equation (2.3) [4,13].We find that

$$u(x) = f(x) + \lambda \int_0^x K(x,t) \left\{ f(t) + \lambda \int_0^t K(t,t_1) u(t_1) dt_1 \right\} dt$$

$$= f(x) + \lambda \int_0^x K(x,t) f(t) dt + \lambda^2 \int_0^x K(x,t) \int_0^t K(t,t_1) u(t_1) dt_1 dt$$

$$= f(x) + \lambda \int_0^x K(x,t) f(t) dt + \lambda^2 \int_0^x K(x,t) \int_0^t K(t,t_1) f(t_1) dt_1 dt$$

$$+ \ldots + \lambda^n \int_0^x K(x,t) \int_0^t K(t,t_1) \ldots$$

$$\times \int_0^{t_{n-2}} K(t_{n-2}, t_{n-1}) f(t_{n-1}) dt_{n-1} \dots dt_1 dt + R_{n+1}(x),$$

where,

$$R_{n+1} = \lambda^{n+1} \int_0^x K(x,t) \int_0^t K(t,t_1) \dots \int_0^{t_{n-1}} K(t_{n-1}, t_n) u(t_n) dt_n \dots dt_1 dt$$

is the remainder after n terms. It can be easily shown that $\lim_{n\to\infty} R_{n+1} = 0$ [14].

Thus, the general series for $u(x)$ can be written as

$$u(x) = f(x) + \lambda \int_0^x K(x,t) f(t) dt$$

$$+ \lambda^2 \int_0^x \int_0^t K(x,t) K(t,t_1) f(t_1) dt_1 dt$$

$$+ \lambda^3 \int_0^x \int_0^t \int_0^{t_1} K(x,t) K(t,t_1) K(t_1,t_2) f(t_2) dt_2 dt_1 dt + \dots$$

It is being noted that here, the unknown function $u(x)$ in this method is substituted by the given function $f(x)$ that makes the assessment of the multiple integrals easily computable.

In the next example, we will explain the technique of this method. Let us suppose that we have the following equation

$$u(x) = 1 + \int_0^x u(t) dt, \tag{4.9}$$

by substitute successively for $u(x)$ its value as given by equation (4.9), we find that

$$u(x) = 1 + \int_0^x dt + \int_0^x \int_0^x dt^2 + \int_0^x \int_0^x \int_0^x dt^3 + \dots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= e^x.$$

4.1.1.6 The Laplace transform method

The general form of the Volterra integral equation of convolution type is

$$u(x) = f(x) + \lambda \int_0^x K(x-t)u(t)dt \tag{4.10}$$

where the kernel $K(x,t)$ is of convolution type. $\cos(x-t), (x-t), e^{x-t}$ are examples of such kernel [2,4], [15].

So, we can easily solve (4.10) by using the Laplace transform method.

The first step is assigned to begin the solution process, we determine the Laplace transform of $u(x)$

$$L\{u(x)\} = \int_0^\infty e^{-sx} u(x)dx$$

By using the Laplace transform of the convolution integral, we get

$$L\left\{\int_0^x K(x-t)u(t)dt\right\} = L\{K(x)\}L\{u(x)\}$$

So, taking the Laplace transform of equation (4.10), we get

$$L\{u(x)\} = L\{f(x)\} + \lambda L\{K(x)\}L\{u(x)\}$$

and the solution for $L\{u(x)\}$ is given by

$$L\{u(x)\} = \frac{L\{f(x)\}}{1-\lambda L\{K(x)\}} \tag{4.11}$$

by taking the inverse Laplace transform of both sides of (4.11) we will get the solution $u(x)$, where

$$u(x) = \int_0^x \varphi(x-t) f(t)dt \tag{4.12}$$

where it is presumed that

$$L^{-1}\left\{\frac{1}{1-\lambda L\{K(x)\}}\right\} = \varphi(x).$$

The equation (4.12) is the solution of the second kind Volterra integral equation of convolution type. The following example will illustrate the technique of this method.

Let us suppose that we have the following equation

$$u(x) = 1 - \int_0^x (x-t)u(t)dt. \tag{4.13}$$

Taking Laplace transform of both sides (4.13) gives

$$L\{u(x)\} = L\{1\} - L\{(x-t)\}L\{u(x)\},$$

thus

$$L\{u(x)\} = \frac{1}{s} - \frac{1}{s^2} L\{u(x)\},$$

the solution for $L\{u(x)\}$ is given by

$$L\{u(x)\} = \frac{s}{s^2+1}. \tag{4.14}$$

By taking the inverse Laplace transform of both sides of (4.14), we will get the exact solution

$$u(x) = \cos x$$

4.1.1.7 The series solution method

We apply this method if $u(x)$ is an analytic function, i.e. $u(x)$ has a Taylor's expansion around $x = 0$ [4].

Accordingly, we can express the function $u(x)$ by a series expansion given by

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{4.15}$$

where the coefficients x and a are constants that are desired to be determined.

Substitution of equation (4.15) into (2.3) we get

$$\sum_{n=0}^{\infty} a_n x^n = f(x) + \lambda \int_0^x K(x,t) \sum_{n=0}^{\infty} a_n t^n dt$$

by using a few terms of the expansion in both sides, we get

$$\begin{aligned} &a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \\ &= f(x) + \lambda \int_0^x K(x,t) a_0 dt + \lambda \int_0^x K(x,t) a_1 t dt \\ &\quad + \lambda \int_0^x K(x,t) a_2 t^2 dt + \ldots + \lambda \int_0^x K(x,t) a_n t^n dt + \ldots. \end{aligned} \tag{4.16}$$

The integral equation (2.3) will be converted to a traditional integral in (4.16) where terms of the form $t^n, n \geq 0$ will be integrated alternatively of integrating the unknown function $u(x)$ [5].

Notice that, if $f(x)$ includes elementary functions such as exponential functions, trigonometric functions, etc., then Taylor expansions for functions involved in $f(x)$

must be used because we are seeking series solution. In the integral equation (4.16) we first integrate the right side and then collect the coefficients of like powers of x. Then, we equate the coefficients of like powers of x in both sides of the resulting equation to get a recurrence relation in $a_n, n \geq 0$. This will determine the unknown coefficients $a_0, a_1, a_2, ..., a_n, ...$ [4-5]. Accordingly, substituting these coefficients $a_n, n \geq 0$, which are determined in equation (4.16), will get us the solution as a series form.

We will solve the following example to illustrate the technique of this method. Let us suppose that we have the following equation

$$u(x) = 1 - x\sin x + \int_0^x t u(t) dt.$$

We will write the solutio $u(x)$ and $x\sin x$ in the form of Taylor series to find

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... = 1 - x\left(x - \frac{x^3}{3!} + ...\right) + \int_0^x t(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ...)dt.$$

Thus, integrating the right side and collecting the like terms of x we find

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... = 1 + (\frac{1}{2}a_0 - 1)x^2 + \frac{1}{3}a_1 x^3 + (\frac{1}{6} + \frac{1}{4}a_2)x^4 + ...$$

Comparing the coefficients of the same power of x gives the following set of values

$$a_0 = 1,$$

$$a_1 = 0,$$

$$a_2 = \frac{1}{2}a_0 - 1 = -\frac{1}{2!},$$

$$a_3 = \frac{1}{3}a_1 = 0,$$

$$a_4 = \frac{1}{6} + \frac{1}{4}a_2 = \frac{1}{4!},$$

and so on, generally

$$a_{2n+1} = 0,$$

$$a_{2n} = \frac{(-1)^n}{(2n)!}, \quad n \geq 0.$$

Hence, the solution is given by

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

thus, the exact solution is

$$u(x) = \cos x.$$

4.1.1.8 A special case of Volterra integral equation

In the second kind, Volterra equation (2.3) if the kernel $K(x,t)$ can be written as [4]

$$K(x,t) = \frac{A(x)}{A(t)}$$

such that the equation takes the form

$$u(x) = f(x) + \lambda \int_0^x \frac{A(x)}{A(t)} u(t)dt$$

and upon dividing throughout by $A(x)$ yields

$$\left\{\frac{u(x)}{A(x)}\right\} = \left\{\frac{f(x)}{A(x)}\right\} + \lambda \int_0^x \left\{\frac{u(t)}{A(t)}\right\} dt \tag{4.17}$$

Now define

$$\frac{u(x)}{A(x)} = u_1(x)$$

and

$$\frac{f(x)}{A(x)} = f_1(x)$$

then equation (4.17) can be written as

$$u_1(x) = f_1(x) + \lambda \int_0^x u_1(t)dt \tag{4.18}$$

Assuming that

$$u_2(x) = \int_0^x u_1(t)dt,$$

equation (4.18) can be reduced to an ordinary differential equation

$$\frac{du_2}{dx} - \lambda u_2 = f_1(x)$$

the general solution of which can be obtained as

$$u_2(x) = \int_0^x e^{\lambda(x-t)} f_1(t)dt + C_1. \qquad (4.19)$$

Using the initial condition $u_2(0) = 0$ at $x = 0,$ the equation (4.19) reduces to

$$u_2(x) = \int_0^x e^{\lambda(x-t)} f_1(t)dt.$$

But,

$$u_1(x) = \frac{du_2}{dx}$$

and so the above equation can be reduced to an integral equation in terms of u_1 by differentiating according to the Leibnitz rule to yield

$$u_1(x) = \lambda\int_0^x e^{\lambda(x-t)} f_1(t)dt + f_1(t)$$

Hence, the solution to the original problem can be obtained multiplying throughout by $A(x)$

$$u(x) = f(x) + \lambda\int_0^x e^{\lambda(x-t)} f(t)dt.$$

Obviously, this formula can also be obtained by the previous method of successive approximation.

4.1.2 Methods to solve the first kind of the Volterra integral equations

Before we explain methods to solve the first kind, we have to recall that the unknown function $u(x)$ arises only inside the integral sign for the first kind of the Volterra integral equations. The usefulness in this section is to point out that, we have two methods to convert Volterra integral equations of the first kind to the second kind. So, after the conversion process, we can use the methods in which we presented previously to solve Volterra integral equations of the second kind. In the next pages of our research, we will explain two principle methods that are usually used for solving the first kind of the Volterra integral equations [5].

4.1.2.1 The series solution method

If the solution $u(x)$ has derivatives of all orders, we will consider it to be analytic, and it has Taylor series at $x=0$ [5]. So, the solution $u(x)$ can be written as follows

$$u(x)=\sum_{n=0}^{\infty} a_n x^n. \tag{4.20}$$

By substituting (4.20) into (2.2) will be obtained

$$T\big(f(x)\big)=\int_0^x K(x,t)(a_0+a_1t+a_2t^2+...)dt \tag{4.21}$$

where the Taylor series for $f(x)$ is $T\big(f(x)\big)$.

The idea in this method is the integral equation (2.2) which will be changed to an imitative integral in (4.21), where terms of the form $t^n, n\geq 0$ will be integrated alternatively of integrating the unknown function $u(x)$.

We have to note that, if $f(x)$ contains elementary functions such as exponential functions, trigonometric functions, etc., then Taylor expansions for functions involved in $f(x)$ must be used because we are seeking series solution.

In integral equation (4.21), we firstly integrate the right side and then collect the coefficients of like powers of x. And the next step is to equate the coefficients of like powers of x in both sides of the resulting equation to get a recurrence relation in $a_n, n\geq 0.$ This will determine the unknown coefficients $a_0, a_1, a_2, ..., a_n, ...$ Accordingly, substituting these coefficients $a_n, n\geq 0,$ which are determined in equation (4.21), get us the solution as a series form.

4.1.2.2 The Laplace transform method

If the kernel $K(x,t)$ is of convolution type, in other words, the the kernel $K(x,t)$ in the form of $K(x-t),$ then Volterra integral equation of the first kind can be written as follows [5]

$$f(x)=\int_0^x K(x-t)u(t)dt. \tag{4.22}$$

The first step in this method is taking the Laplace transform of both sides of (4.22) we get

$$F(s) = K(s)U(s), \tag{4.23}$$

where

$$U(s) = L\{u(x)\},$$
$$K(s) = L\{K(x)\},$$
$$F(s) = L\{f(x)\}.$$

Solving (4.23) for U(s) gives

$$U(s) = \frac{F(s)}{K(s)}, \quad K(s) \neq 0. \tag{4.24}$$

By taking the inverse Laplace transform of both sides of (4.24), we will get the solution $u(x)$ where

$$u(x) = L^{-1}\left\{\frac{F(s)}{K(s)}\right\}.$$

4.1.3 Systems of Volterra integral equations

It is well known that the systems of integral equations (nonlinear or linear) arise in the scientific applications in populations growth models, chemistry, physics, and engineering.

The study of system of integral equation has attracted the attention of many researchers in applied sciences. The general ideas and the fundamental attributes of these systems are of extensive applicability [5].

4.1.3.1 Systems of Volterra integral equations of the second kind

The general form of Systems of Volterra Integral Equations of the Second Kind is given by [5]

$$u(x) = f_1(x) + \int_0^x \left(K_1(x,t)u(t) + \widetilde{K}_1(x,t)v(t) + ...\right)dt$$

$$v(x) = f_2(x) + \int_0^x \left(K_2(x,t)u(t) + \widetilde{K}_2(x,t)v(t) + ...\right)dt \tag{4.25}$$

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And we will use two methods to solve this systems.

4.1.3.1.1 The Adomian decomposition method

The technique for this method boils down to making each solution in the form of an infinite sum of the components as follows [5]

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \qquad v(x) = \sum_{n=0}^{\infty} v_n(x), \tag{4.26}$$

where $u(x)$ and $v(x)$ are determined recurrently.

By substitute (4.26) into (4.25) we get

$$\sum_{n=0}^{\infty} u_n(x) = f_1(x) + \int_0^x \left(K_1(x,t)\sum_{n=0}^{\infty} u_n(x) + \widetilde{K}_1(x,t)\sum_{n=0}^{\infty} v_n(x) \right) dt$$

$$\sum_{n=0}^{\infty} v_n(x) = f_2(x) + \int_0^x \left(K_2(x,t)\sum_{n=0}^{\infty} u_n(x) + \widetilde{K}_2(x,t)\sum_{n=0}^{\infty} v_n(x) \right) dt$$

the value of the zeroth components $u_0(x)$ and $v_0(x)$ are terms that are not included under the integral sign

$$u_0(x) = f_1(x)$$

$$v_0(x) = f_2(x)$$

we can get the value of the other components by the following recursive relations

$$u_{k+1}(x) = f_1(x) + \int_0^x \left(K_1(x,t)u_k(t) + \widetilde{K}_1(x,t)v_k(t)\right)dt$$

$$v_{k+1}(x) = f_2(x) + \int_0^x \left(K_2(x,t)u_k(t) + \widetilde{K}_2(x,t)v_k(t)\right)dt$$

thus, after determining the value of the components $u_n(x)$ and $v_n(x)$, get a solution through the equation (4.26). This method can be used in its standard form, or combined with the noise terms phenomenon. Moreover, the modified decomposition method will be used wherever it is appropriate.

4.1.3.1.2 The Laplace transform method

If the kernels $K_m(x,t)$ and $\widetilde{K}_m(x,t)$, $m=1,2$ is of convolution type then, the system (4.25) will be written in the following form [5]

$$u(x)=f_1(x)+\int_0^x \left(K_1(x-t)u(t)+\widetilde{K}_1(x-t)v(t)+\ldots\right)dt$$

$$\tag{4.27}$$

$$v(x)=f_2(x)+\int_0^x \left(K_2(x-t)u(t)+\widetilde{K}_2(x-t)v(t)+\ldots\right)dt.$$

By taking Laplace transform of both sides of each equation in (4.27) get us

$$U(s)=F_1(s)+K_1(s)U(s)+\widetilde{K}_1(s)V(s)$$

$$V(s)=F_2(s)+K_2(s)U(s)+\widetilde{K}_2(s)V(s),$$

where

$$U(s)=L\{u(x)\},\quad F_1(s)=L\{f(x)\},\quad K_1(s)=L\{K_1(x)\},$$

$$\widetilde{K}_1(s)=L\{\widetilde{K}_1(x)\}, V(s)=L\{v(x)\},$$

$$F_2(s)=L\{f_2(x)\},\quad K_2(s)=L\{K_2(x)\},\ \widetilde{K}_2(s)=L\{\widetilde{K}_2(x)\}.$$

Thus

$$(1-K_1(s))U(s)=F_1(s)+\widetilde{K}_1(s)V(s)$$

$$\tag{4.28}$$

$$(1-\widetilde{K}_2(s))V(x)=F_2(s)+K_2(s)U(s).$$

Solving system (4.28) for $U(s)$ and $V(s)$ gives

$$U(s) = \frac{F_1(s) + \widetilde{K}_1(s)V(s)}{1 - K_1(s)}$$

$$V(s) = \frac{F_2(s) + K_2(s)U(s)}{1 - \widetilde{K}_2(s)} \tag{4.29}$$

The final step to get the solutions is done by taking the inverse Laplace transform of both sides of each equation in (4.29)

$$u(x) = L^{-1}\left\{\frac{F_1(s) + \widetilde{K}_1(s)V(s)}{1 - K_1(s)}\right\}$$

$$v(x) = L^{-1}\left\{\frac{F_2(s) + K_2(s)U(s)}{1 - \widetilde{K}_2(s)}\right\}.$$

Thus, the exact solutions are given by

$$\big(u(x), v(x)\big) = \left(L^{-1}\left\{\frac{F_1(s) + \widetilde{K}_1(s)V(s)}{1 - K_1(s)}\right\}, L^{-1}\left\{\frac{F_2(s) + K_2(s)U(s)}{1 - \widetilde{K}_2(s)}\right\}\right).$$

4.1.3.2 Systems of Volterra integral equations of the first kind

The general form of the systems of Volterra integral equations of the first kind can be written as follows

$$f_1(x) = \int_0^x \big(K_1(x,t)u(t) + \widetilde{K}_1(x,t)v(t) + \ldots\big)dt$$

$$f_2(x) = \int_0^x \big(K_2(x,t)u(t) + \widetilde{K}_2(x,t)v(t) + \ldots\big)dt \tag{4.30}$$

.

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where the kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$, and the functions $f_i(x)$ are given real-valued functions in advance, and $u(x), v(x),\ldots$ are functions that seek to be determined. We have provided previously that, the unknown functions arise just inside the integral sign for the Volterra integral equations of the first kind [5].

4.1.3.2.1 Conversion to a Volterra system of the second kind

We have provided in chapter (2) that, there are two methods to convert Volterra integral equations of the first kind to Volterra integral equations of the second kind. We will use the first technique to convert system of Volterra integral equations of the first kind to system of Volterra integral equations of the second kind.

By differentiating both sides of each equation in (4.30), and using Leibnitz rule, we get

$$\begin{aligned} f_1'(x) &= K_1(x,x)u(x) + \widetilde{K}_1(x,x)v(x) \\ &\quad + \int_0^x \left(K_{1_x}(x,t)u(t) + \widetilde{K}_{1_x}(x,t)v(t) + \ldots \right) dt \\ f_2'(x) &= K_2(x,x)u(x) + \widetilde{K}_2(x,x)v(x) \\ &\quad + \int_0^x \left(K_{2_x}(x,t)u(t) + \widetilde{K}_{2_x}(x,t)v(t) + \ldots \right) dt. \end{aligned}$$

That can be rewritten as

$$\begin{aligned} u(x) &= \frac{f_1'(x) - \widetilde{K}_1(x,x)v(x)}{K_1(x,x)} \\ &\quad - \frac{1}{K_1(x,x)} \int_0^x \left(K_{1_x}(x,t)u(t) + \widetilde{K}_{1_x}(x,t)v(t) + \ldots \right) dt \\ v(x) &= \frac{f_2'(x) - K_2(x,x)u(x)}{\widetilde{K}_2(x,x)} \\ &\quad - \frac{1}{\widetilde{K}_2(x,x)} \int_0^x \left(K_{2_x}(x,t)u(t) + \widetilde{K}_{2_x}(x,t)v(t) + \ldots \right) dt. \end{aligned}$$

The last system is a system of Volterra integral equations of the second kind. This system can be handled by many different methods such as, Laplace transform method as we have explained previously. Here it is useful and necessary to recall that which must be $K_1(x,x) \neq 0$ and $\widetilde{K}_2(x,x) \neq 0$ for the system to be converted to a system of Volterra integral equations of the second kind. But if $K_1(x,x) = 0$ and $\widetilde{K}_2(x,x) = 0,$ we can do the derivation process again [5].

4.2 Methods to Solve Nonlinear Volterra Integral Equations

Our aim in this section is to study the nonlinear Volterra integral equations of the first and the second kind. The unknown function $u(x)$ arises outside and inside the integral sign in the nonlinear Volterra integral equations of the second kind. The general form of nonlinear Volterra integral equation of the second kind is

$$u(x) = f(x) + \lambda \int_0^x K(x,t) F(u(t)) dt. \tag{4.31}$$

While, the nonlinear function $F(u(x))$ arises only inside the integral sign in the nonlinear Volterra integral equations of the first kind. The general form of nonlinear Volterra integral equation of the first kind is

$$f(x) = \lambda \int_0^x K(x,t) F(u(t)) dt. \tag{4.32}$$

The function $f(x)$ and the kernel $K(x,t),$ in these two kinds of equations are given real-valued functions and $F(u(x))$ is a nonlinear function of $u(x)$ such as, $\sin(u(x))$, $e^{u(x)}$ and $u^2(x)$ [5].

4.2.1 Nonlinear Volterra integral equations of the second kind

In this section, we will use three different methods to solve the nonlinear Volterra itegral equation (4.31). These methods are the successive approximations method, the series solution method, and the Adomian decomposition method (ADM).

4.2.1.1 The successive approximations method

Technique of this method is to start with an initial guess which is called zeroth approximation to finding successive approximations to the solution $u(x)$. Any real-valued function to suffice for the zeroth approximation that will be used in a recurrence relation to get the other approximations. Suppose that, we have the following nonlinear Volterra integral equation of the second kind (4.31), the recurrence relation [5]

$$u_{n+1}(x) = f(x) + \lambda \int_0^x K(x,t) F(u_n(t)) dt \tag{4.33}$$

will be used to get the unknown function $u(x)$. Where, any real- valued function to suffice for the zeroth approximation $u_0(x)$, often we use x, 0, or 1 for $u_0(x)$. By using this value of $u_0(x)$ into (4.33), several successive approximations $u_n, n \geq 1$ will be determined in the following

$$u_1(x) = f(x) + \lambda \int_0^x K(x,t)F(u_0(t))dt$$

$$u_2(x) = f(x) + \lambda \int_0^x K(x,t)F(u_1(t))dt$$

$$u_3(x) = f(x) + \lambda \int_0^x K(x,t)F(u_2(t))dt$$

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$$u_{n+1}(x) = f(x) + \lambda \int_0^x K(x,t)F(u_n(t))dt$$

by using

$$u(x) = \lim_{n \to \infty} u_{n+1}(x)$$

the solution $u(x)$ is obtained.

4.2.1.2 The series solution method

We apply this method if $u(x)$ is an analytic function, i.e. $u(x)$ has a Taylor's expansion around $x = 0$ [5].

Accordingly, we can express the function $u(x)$ by a series expansion given by

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \tag{4.34}$$

where the coefficients a_n will be determined recurrently. By substituting (4.34) into both sides of (4.31) we get

$$a_0 + a_1 x + a_2 x^2 + ... = T(f(x)) + \int_0^x K(x,t)\left(F\left(a_0 + a_1 t + a_2 t^2 + ...\right)\right)dt \tag{4.35}$$

where the Taylor series for $f(x)$ is $T(f(x))$.

The integral equation (4.31) will be changed to an imitative integral in (4.35) where terms of the form $t^n, n \geq 0$ will be integrated alternatively of integrating the unknown function $F(u(x))$.

We have to note that, if $f(x)$ contains elementary functions such as exponential functions, trigonometric functions, etc., then, Taylor's expansions for functions involved in $f(x)$ must be used because we are seeking series solution.

In itegral equation (4.35) we firstly integrate the right side and then collect the coefficients of like powers of x. And the next step is to equate the coefficients of like powers of x in both sides of the resulting equation to get a recurrence relation in $a_j, j \geq 0$. This will give us a complete determination of the unknown coefficients $a_0, a_1, a_2, \ldots, a_n, \ldots$. Accordingly, substituting these coefficients $a_n, n \geq 0$ which are determined in equation (4.35), that gets us the solution as a series form.

4.2.1.3 The Adomian decomposition method

In this method, the unknown function $u(x)$ is commonly decomposes into an infinite sum of components that will be determined recursively by iterations as debate before. The nonlinear terms such as $u^5, u^6, \cos u, e^u$, etc. that arise in the equation must be expressed by a particular representation which is named the Adomian polynomials $D_n, n \geq 0$. Adomian gives us a formal algorithm to establish an authoritative representation for all nonlinear terms forms. The Adomian technique remains the usually used one to evaluate Adomian polynomials in spite of the development of other techniques.

To handle the nonlinear integral equations in an authoritative way, we will use the Adomian algorithm to evaluate Adomian polynomials [5].

To explain the technique for this method, suppose that, we have the nonlinear Volterra integral equations of the second kind (4.31) and we assumed that, the function $f(x)$ and the kernel $K(x,t)$ are analytical functions. The standard technique for this method is started by decomposing $u(x)$ in (4.31) into $u_0, u_1, u_2, u_3, \ldots$, and assuming that

$$\lim_{m\to\infty}\left\{\sum_{j}^{m} u_j(x)\right\} = u(x)$$

and we set

$$F(u) = \sum_{m=0}^{\infty} D_m$$

for the non-linear function $F(u)$, where $D_m, (m \geq 0)$ are particular polynomials known as Adomian polynomials [16]. In [5], close formulas of these polynomials, for any non-linear function $F(u)$ introduced as follow

$$D_m = \frac{1}{m!}\frac{dm}{d\lambda^m}\left[F\left(\sum_{j=0}^{m}\lambda^j u_j\right)\right]_{\lambda=0}, m = 0,1,2,...$$

in other words, the Adomian polynomials D_n is defined by

$$D_0 = F(u_0)$$

$$D_1 = u_1 F'(u_0)$$

$$D_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0)$$

$$D_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0)$$

$$D_4 = u_4 F'(u_0) + \left(\frac{1}{2!}u_2^2 + u_1 u_3\right)F''(u_0) + \frac{1}{2!}u_1^2 u_2 F'''(u_0) + \frac{1}{4!}u_1^4 F^{(iv)}(u_0).$$

Thus, equation (4.31) is read as follows

$$\sum_{m=0}^{\infty} u_m(x) = f(x) + \int_0^x \left(K(x,t)\sum_{m=0}^{\infty} D_m(u_0,u_1,u_2,u_3,...,u_m)\right)dt$$

now let

$$u_0(x) = f(x)$$

and we can get the $u_j(x), j = 1,2,...$ by using the recurrent formula

$$u_{j+1}(x) = \int_0^x K(x,t)D_j\left(u_0(t),u_1(t),u_2(t),...,u_j(t)\right)dt. \tag{4.36}$$

Therefore, the solution of the integral equation (4.31) in the series form that can be determined forthwith by using (4.36).

As for the modified Adomian method, there are two techniques [16], in the first modified technique we assume that, the function $f(x)$ can be splitted as follows

$$f(x) = f_1(x) + f_2(x)$$

then, we calculate the value of $u_0(x), u_1(x)$ as following

$$u_0(x) = f_1(x)$$

$$u_1(x) = f_2(x) + \int_0^x K(x,t) D_0(t)dt$$

$$u_{m+1}(x) = \int_0^x K(x,t) D_m(t)dt, m \geq 1.$$

The idea of the second modified technique is replacing the function $f(x)$ by a series of infinite components. It is obvious to express that, in [17] the function $f(x)$ in terms of the Taylor series and introduces the recursive formula

$$u_0(x) = f_0(x)$$

$$u_{m+1}(x) = f_{m+1} + \int_0^x K(x,t) D_m(t)dt, m \geq 0$$

where $f_j(x), (j = 0,1,2,...,m)$ represents the Taylor series components of $f(x)$.

In [17-18] the modified technique reduces the computational difficulties and accelerates the convergence which is better than the standard Adomian technique.

4.2.2 Nonlinear Volterra integral equations of the first kind

The first stept to solve equation (4.32) is done by converting it to a linear Volterra integral equation of the first kind of the form

$$f(x) = \lambda \int_0^x K(x,t) v(t) dt \tag{4.37}$$

by using the transformation

$$v(t) = F(u(t))$$

Consequently, this means

$$u(t) = F^{-1}(v(t))$$

And we can use any method that we studied to solve the Volterra integral equation of the first kind (4.37) [5].

4.2.2.1 The Laplace transform method

The technique of this method is the same technique that has been used in the method of Laplace transform for solving Volterra integral equations of the second and the first kinds. We presume that, the kernel $K(x,t)$ is a difference kernel [5].

We will get

$$L\{f(x)\} = L\{K(x-t)\} \times L\{v(x)\}$$

By taking the Laplace transforms of both sides of equation (4.37).

Thus

$$V(s) = \frac{F(s)}{K(s)} \tag{4.38}$$

where

$$V(s) = L\{v(x)\},$$
$$F(s) = L\{f(x)\},$$
$$K(s) = L\{K(x)\}.$$

To get $v(x)$ we taking the inverse Laplace transform of both sides of (4.38)

$$v(x) = L^{-1}\left\{\frac{F(s)}{K(s)}\right\}$$

thus, by using

$$u(t) = F^{-1}(v(t))$$

The solution $u(x)$ is obtained.

4.2.2.2 Conversion to a Volterra equation of the second kind

There are two steps to convert the nonlinear Volterra integral equation of the first kind (4.32) to a Volterra integral equations of the second kind. Where the first step is to convert the nonlinear Volterra integral equation of the first kind to a linear Volterra integral equation of the first kind as explained in the previous section. The second step is to convert a linear Volterra integral equation of the first kind to a linear Volterra integral equation of the second kind as explained in chapter (2).

After completion of the conversion process we can use one of the methods that have been presented in this research to solve linear Volterra integral equation of the second kind [5].

4.2.3 Systems of nonlinear Volterra integral equations

In this section we will study the systems of nonlinear Volterra integral equations of the second kind and the first kind. Commonly to handle the two kinds of systems many analytical and numerical methods are used. However, to handle the nonlinear Volterra integral equations of the second kind and the first kind, we will use the successive approximations method and the Adomian decomposition method [5].

4.2.3.1 Systems of nonlinear Volterra integral equations of the second kind

The general form of system of nonlinear Volterra integral equations of the second kind given by

$$\begin{aligned} u(x) &= f_1(x) + \int_0^x \left(K_1(x,t)F_1(u(t)) + \widetilde{K}_1(x,t)\widetilde{F}_1(v(t))\right)dt \\ v(x) &= f_2(x) + \int_0^x \left(K_2(x,t)F_2(u(t)) + \widetilde{K}_2(x,t)\widetilde{F}_2(v(t))\right)dt \end{aligned} \tag{4.39}$$

The functions F_i and $\widetilde{F}_i$ for $i=1,2$ are nonlinear functions of $u(x)$ and $v(x)$. The function $f_i(x)$ and the kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$ are given real-valued functions, for $i=1,2$. The unknown functions $u(x)$ and $v(x)$, that will be determined, arise outside and inside the integral sign [5].

Every solution decomposes as an infinite sum of components as presented before. In Adomian decomposition method where these components are determined recurrently. This method can be used in its standard form or combined with the noise terms phenomenon. Moreover, the modified decomposition method will be used wherever it is appropriate.

The technique of this method to solve system (4.39) as follows

first, we express $u(x)$ and $v(x)$ in the form of a series

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

and

$$v(x) = \sum_{n=0}^{\infty} v_n(x)$$

For the non linear functions $F(u)$ and $F(v)$ we set

$$F(u) = \sum_{n=0}^{\infty} C_n$$

$$F(v) = \sum_{n=0}^{\infty} D_n$$

where C_n, D_n are the Adomian polynomials which we have explained in a previous section. Thus system (4.39) becomes

$$u_0(x) + u_1(x) + ... = f_1(x) + \int_0^x \left(K_1(x,t) \sum_{n=0}^{\infty} C_n(t) + \widetilde{K}_1(x,t) \sum_{n=0}^{\infty} D_n(t) \right) dt$$

$$v_0(x) + v_1(x) + ... = f_2(x) + \int_0^x \left(K_2(x,t) \sum_{n=0}^{\infty} C_n(t) + \widetilde{K}_2(x,t) \sum_{n=0}^{\infty} D_n(t) \right) dt$$

then we set

$$u_0(x) = f_1(x)$$

to obtain the other components of the solution $u_n(x)$, we will use the following recursive relations

$$u_{m+1}(x) = \int_0^x \left(K_1(x,t) C_m(t) + \widetilde{K}_1(x,t) D_m(t) \right) dt$$

similarly

$$v_0(x) = f_2(x)$$

$$v_{m+1}(x) = \int_0^x \left(K_1(x,t) C_m(t) + \widetilde{K}_1(x,t) D_m(t) \right) dt$$

the solution $u(x)$ is obtained by

$$\lim_{n \to \infty} \left\{ \sum_{m}^{n} u_m(x) \right\}$$

Similarly, the solution $v(x)$ is obtained by

$$\lim_{n \to \infty} \left\{ \sum_{m}^{n} v_m(x) \right\}$$

4.2.3.2 Systems of nonlinear Volterra integral equations of the first kind

The general form of the systems of nonlinear Volterra integral equations of the first kind is given by

$$f_1(x) = \int_0^x \left(K_1(x,t)F_1(u(t)) + \widetilde{K}_1(x,t)\widetilde{F}_1(v(t))\right)dt$$

$$f_2(x) = \int_0^x \left(K_2(x,t)F_2(u(t)) + \widetilde{K}_2(x,t)\widetilde{F}_2(v(t))\right)dt \tag{4.40}$$

where the functions $f_i(x)$ and the kernels $K_i(x,t)$ and $\widetilde{K}_i(x,t)$ are given real-valued functions. So, the unknown functions $u(x)$ and $v(x)$, that will be determined which arise just inside the integral sign for the Volterra integral equations of the first kind. We need firstly to convert this system to a system of nonlinear Volterra integral equation of the second kind. The conversion process is done by differentiating both sides of each portion of the system and using Leibnitz rule [5].

Differentiating both sides of each equation in (4.40), and using Leibnitz rule, we get

$$f_1'(x) = K_1(x,x)u(x) + \widetilde{K}_1(x,x)\widetilde{F}_1(v(x)) + \int_0^x \left(K_{1_x}(x,t)u(t) + \widetilde{K}_{1_x}(x,t)\widetilde{F}_1(v(t))\right)dt$$

$$f_2'(x) = K_2(x,x)F_2(u(x)) + \widetilde{K}_2(x,x)v(x) + \int_0^x \left(K_{2_x}(x,t)F_2(u(t)) + \widetilde{K}_{2_x}(x,t)v(t)\right)dt$$

That can be rewritten as

$$u(x) = \frac{f_1'(x) - \widetilde{K}_1(x,x)\widetilde{F}_1(v(x))}{K_1(x,x)} - \frac{1}{K_1(x,x)}\int_0^x \left(K_{1_x}(x,t)u(t) + \widetilde{K}_{1_x}(x,t)\widetilde{F}_1(v(t))\right)dt$$

$$v(x) = \frac{f_2'(x) - K_2(x,x)F_2(u(x))}{\widetilde{K}_2(x,x)} - \frac{1}{\widetilde{K}_2(x,x)}\int_0^x \left(K_{2_x}(x,t)F_2(u(t)) + \widetilde{K}_{2_x}(x,t)v(t)\right)dt.$$

The last system is a system of nonlinear Volterra integral equations of the second kind. This system can be handled by many different methods.

Here are useful and necessary to recall that which is must be $K_1(x,x) \neq 0$ and $\widetilde{K}_2(x,x) \neq 0$ for the system to be converted to a system of Volterra integral equations of the second kind.

But if $K_1(x,x) = 0$ and $\widetilde{K}_2(x,x) = 0,$ we can do the derivation process again [5].

CHAPTER 5

THE SINGULAR VOLTERRA INTEGRAL EQUATIONS

It is evident that, there are variables of scientific applications in order to deal with such brand of equations such as radar ranging, bio-mechanics, satellite photometry and seismology [19-20].

In the integral equation of the first kind

$$f(x) = +\lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt, \tag{5.1}$$

or the integral equation of the second kind

$$u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} K(x,t)u(t)dt, \tag{5.2}$$

if the kernel $K(x,t)$ becomes infinite at one or more points in the domain of integration then, the equations (5.1) and (5.2) is called singular. In addition, if $\beta(x)$ or $\alpha(x)$, or both limits of integration are infinite then the equations (5.1) and (5.2) are also called singular equations.

The following are examples of the first type and second type of singular integral equations, respectively [4-5]

$$u(x) = e^x + \int_0^\infty K(x,t)u(t)dt$$

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt.$$

5.1 Abel's Problem

The general form of Abel's integral equations is given by [5,21]

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt. \tag{5.3}$$

At the same time, the equation (5.3) is Volterra integral equation of the first kind with

$$K(x,t) = \frac{1}{\sqrt{x-t}},$$

where

$$K(x,t) \to \infty, \quad \text{as} \quad t \to x.$$

To get the solution of equation (5.3), we will use the Laplace transform method [4]. Taking the Laplace transform of both sides of the above equation which yields

$$L\{f(x)\} = L\left\{\int_0^x \frac{u(t)}{\sqrt{x-t}}\,dt\right\}$$

Using the convolution theorem and after a little reduction, the transformed equation can be written in a simple form

$$L\{u(x)\} = \frac{\sqrt{s}}{\sqrt{\pi}}\,L\{f(x)\}$$

Here, we have used the result of

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

The above transform cannot be inverted as it stands now.We rewrite the equation as follows:

$$L\{u(x)\} = \frac{s}{\sqrt{\pi}}\left[\frac{1}{\sqrt{s}}\,L\{f(x)\}\right]$$

Using the convolution theorem, it can be at once inverted to yield

$$u(x) = \frac{1}{\sqrt{\pi}}\,L^{-1}\left\{s\left[\frac{1}{\sqrt{s}}\,L\{f(x)\}\right]\right\}$$

$$= \frac{1}{\sqrt{\pi}}\frac{d}{dx}\int_0^x \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{x-t}}\,f(t)dt$$

$$= \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{\sqrt{x-t}}dt$$

Note that, the Leibnitz rule of differentiation cannot be used in the above integral.

So, we have to integrate the integral firstly and then to take the derivative with respect to x.

Then this gives

$$u(x)=\frac{1}{\pi}\frac{d}{dx}\left\{-2\left(\sqrt{x-t}\right)f(t)\Big|_0^x+2\int_0^x\sqrt{x-t}\,f'(t)dt\right\}$$

$$=\frac{1}{\pi}\left\{\frac{f(0)}{\sqrt{x}}+\int_0^x\frac{f'(t)}{\sqrt{x-t}}dt\right\}$$

This is the desired solution of Abel's problem [4].

5.2 The Generalized Abel's Integral Equation of the First Kind

The integral equation is given by [2-4], [5,21]

$$\int_0^x\frac{u(t)dt}{(x-t)^{\alpha}}=f(x),\qquad 0<\alpha<1$$

Taking the Laplace transform of both sides with the help of convolution theorem, we obtain

$$L\{x^{-\alpha}\}L\{u(x)\}=L\{f(x)\}$$

or

$$\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}L\{u(x)\}=L\{f(x)\}$$

Thus, to rearranging the terms we have

$$L\{u(x)\}=\frac{1}{\Gamma(1-\alpha)}s\left\{\frac{1}{s^{\alpha}}L\{f(x)\}\right\}. \tag{5.4}$$

By using the convolution theorem of Laplace transform, the equation (5.4) can be obtained as

$$u(x)=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dx}\left\{\int_0^x(x-t)^{\alpha}f(t)dt\right\}$$

$$=\frac{\sin(\pi\alpha)}{\pi}\frac{d}{dx}\left\{\frac{(x-t)^{\alpha}}{-\alpha}f(t)\Big|_0^x+\frac{1}{\alpha}\int_0^x(x-t)^{\alpha}f'(t)dt\right\}$$

$$=\frac{\sin(\pi\alpha)}{\pi}\frac{d}{dx}\left\{\frac{(x)^{\alpha}}{\alpha}f(0)+\frac{1}{\alpha}\int_0^x(x-t)^{\alpha}f'(t)dt\right\}$$

$$=\frac{\sin(\pi\alpha)}{\pi}\left\{\frac{f(0)}{x^{1-\alpha}}+\int_0^x\frac{f'(t)dt}{(x-t)^{1-\alpha}}\right\}$$

This is the desired solution of the integral equation.

Here, it is to be noted that,

$$\Gamma(\alpha)\Gamma(1-\alpha)=\frac{\sin(\pi\alpha)}{\pi}.$$

The definition of Gamma function is [4]

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx.$$

5.3 Abel's Problem of the Second Kind Integral Equation

The second kind Volterra equation in terms of Abel's integral equation is written as

$$\begin{aligned} u(x) &= f(x)+\int_0^x K(x,t)u(t)dt, \\ &= f(x)+\int_0^x \frac{u(t)}{\sqrt{x-t}}dt \end{aligned} \tag{5.5}$$

The solution of this integral is attributed by the convolution theorem of Laplace transform [4].

Taking the Laplace transform of both sides of (5.5)

$$\begin{aligned} L\{u(x)\} &= L\{f(x)\}+L\left\{\frac{1}{\sqrt{x}}\right\}L\{u(x)\} \\ &= L\{f(x)\}+\frac{\sqrt{\pi}}{\sqrt{s}}L\{u(x)\} \end{aligned}$$

and after reduction, this can be expressed as

$$\begin{aligned} L\{u(x)\} &= \left\{\frac{\sqrt{\pi}}{\sqrt{s}-\sqrt{\pi}}\right\}L\{f(x)\} \\ &= L\{f(x)\}+\left\{\frac{\sqrt{\pi}}{\sqrt{s}-\sqrt{\pi}}\right\}L\{f(x)\} \end{aligned} \tag{5.6}$$

The inversion of equation (5.6) is given by

$$u(x)= f(x)+\int_0^x g(t)f(x-t)dt, \tag{5.7}$$

where,

$$g(x)=L^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{s}-\sqrt{\pi}}\right\}.$$

In reference [22], the Laplace inverse of

$$\left\{\frac{\sqrt{\pi}}{\sqrt{s}-\sqrt{\pi}}\right\},$$

can be obtained from the formula

$$L^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{s}-\sqrt{\pi}}\right\} = e^{ax}\left\{\frac{1}{\sqrt{\pi x}} + be^{b^2 x} erfc(-b\sqrt{x}\right\}.$$

In our problem, $a = 0, b = \sqrt{\pi}$ and so

$$L^{-1}\left\{\frac{\sqrt{1}}{\sqrt{s-a}-b}\right\} = \sqrt{\pi}\left\{\frac{1}{\sqrt{\pi x}} + \sqrt{\pi} e^{\pi x} erfc(-\sqrt{\lambda x})\right\}$$

here, it is noted that,

$$erfc(-\sqrt{\lambda x}) = (\sqrt{\pi x}).$$

Hence,

$$g(x) = L^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{s}-\sqrt{\pi}}\right\} = \sqrt{\pi}\left\{\frac{1}{\sqrt{\pi x}} + \sqrt{\pi} e^{\pi x} erfc(-\sqrt{\lambda x})\right\}$$

Thus, the solution of the problem is given by equation (5.7) [4].

5.4 The Weakly-Singular Volterra Equation

The general form of weakly-singular Volterra-type integral equations of the second kind, is [4-5]

$$u(x) = f(x) + \int_0^x \frac{\lambda}{\sqrt{x-t}} u(t)dt \tag{5.8}$$

arises frequently in many applications of chemistry and mathematical physics like electrochemistry, heat conduction, and crystal growth. In (5.8) λ is a constant parameter. We will ensure access to a unique solution for equation (5.8), because we assume that, the function $f(x)$ is sufficiently smooth.

The kernel

$$K(x,t) = \frac{1}{\sqrt{x-t}}$$

is a singular kernel.

We shall use the decomposition method to evaluate this integral equation. To determine the solution, we usually adopt the decomposition in the series form

$$u(x) = \sum_{0}^{\infty} u_n(x), \tag{5.9}$$

into both sides of equation (5.7) to get

$$\sum_{0}^{\infty} u_n(x) = f(x) \int_0^x \frac{\lambda}{\sqrt{x-t}} \left(\sum_{0}^{\infty} u_n(t) \right) dt.$$

The components $u_1, u_1, u_2, ...$ are immediately determined upon applying the following recurrence relations

$$u_0(x) = f(x),$$

$$u_1(x) = \int_0^x \frac{\lambda}{\sqrt{x-t}} u_0(t) dt$$

$$u_2(x) = \int_0^x \frac{\lambda}{\sqrt{x-t}} u_1(t) dt$$

$$...... = ...$$

$$u_n(x) = \int_0^x \frac{\lambda}{\sqrt{x-t}} u_{n-1}(t) dt.$$

It is easy to get into the solution $u(x)$ of equation (5.8) when we determine the components $u_0, u_1, u_1, u_2, ...,$ etc. in the form of a rapid convergence power series by substituting the derived components in equation (5.9).

It is important to note that, the noise terms may appear between various components $u(x)$, and by canceling these noise terms between the components $u_0(x)$ and $u_1(x)$ which may give the exact solution that should be justified through substitution. Commonly, the appearance of these terms minimizes the size of the computational work and speeds the convergence of the solution. The use of modified decomposition method would be appropriate for us sometimes [4-5].

CHAPTER 6

CONCLUSION

It is worth of note that, differential equations and integral equations are the most significant types of mathematical equation that occur naturally in the sciences to help us to understand the natural phenomena and modeling them, mathematically. It turned out that, numerous significant phenomena in electrical engineering, dynamical systems of economics, and biology can be modeled by Volterr type integral equations.

Since Volterra started his work on integral equations, the theory of Volterra type integral equations have attracted the attentions of many scientists. These scientists worked in terms of the development of this theory and its applications in numerous disciplines.

In this thesis, the researcher studied the existence and uniqueness theorems of Volterra type equations and discussed some methods that solve linear and nonlinear Volterra integral equations. Systems of these equations are discussed as well. So, this thesis can be considered as a survey on methods of solutions of Volterra type equations and systems of both linear and nonlinear.

The study conductor hopes in this research is to enlighten the students in this huge fields to work forwardly on Volterra type equations.

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APPENDICES A

CURRICULUM VITAE

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1. Antegral Equations and Their Applications by Rahman M. UK, WIT press, 2007, 385 p., ISBN: 978-1-84564-101-6.

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